Triangle Centres in an Isosceles Triangle

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Every triangle has certain lines associated with it. The most prominent among them are the perpendicular bisectors of the sides, the bisectors of the angles, the altitudes, and the medians. Figure 1 represents a scalene triangle ABC, with $AB < AC$. Also shown are the altitude AD from A to BC, the bisector AE of angle A, the median AF where F is the midpoint of BC, and the perpendicular bisector of BC.

We must justify the order in which these lines appear in the figure: the altitude is the closest to AB (the shorter of the sides AB and AC), then the angle bisector, followed by the median, and the perpendicular bisector is closest to side AC. It is of interest to see whether this ordering can be justified using the regular results of Euclidean geometry. Indeed it can, and here's how:

- $\triangle ADB$ and $\triangle ADC$ are right angled. Also, $\measuredangle ABC > \measuredangle ACB$, hence $\measuredangle BAD < \measuredangle CAD$ and therefore $\measuredangle BAD < \frac{1}{2}\measuredangle BAC$,

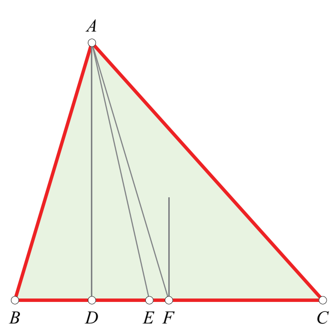

Figure 1. Four significant lines: altitude, angle bisector, median, perpendicular bisector

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i.e., $\measuredangle BAD < \measuredangle BAE$. Therefore the altitude lies between AB and AE. Hence D lies between B and E.

- The angle bisector theorem tells us that angle bisector AE divides the base BC in the ratio $AB : AC = c : b$. Since $AB < AC$, it follows that $BE < EC$ and therefore that $BE < \frac{1}{2}BC$. Hence E lies between B and F.
- In $\triangle ABF$ and $\triangle ACF$ we have: $BF = FC$, and AF is a shared ('common') side. Since $AB < AC$, it follows that $\measuredangle AFB < \measuredangle AFC$. Therefore the perpendicular to BC at F lies to the right of median AF.

Note: We are using here the "inequality form of the SAS congruence theorem." We state it with reference to two triangles PQR and LMN (see Figure 2).

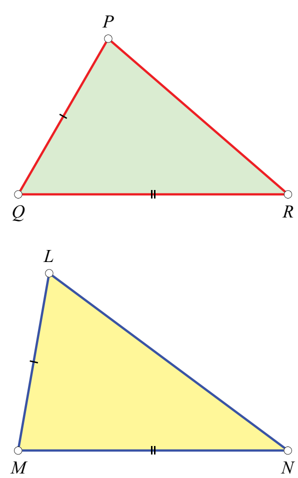


Figure 2. Inequality form of the SAS congruence theorem

Suppose that $PQ = LM$ and $QR = MN$. Then we have the following:

⋆ if $\measuredangle Q < \measuredangle M$, then $PR < LN$;

⋆ if $PR < LN$, then $\measuredangle Q < \measuredangle M$.

Note that the second part is the converse of the first part.

If $AB = AC$, the four lines discussed above (altitude, angle bisector, median and perpendicular bisector of side) merge into a single line of symmetry of the triangle, with

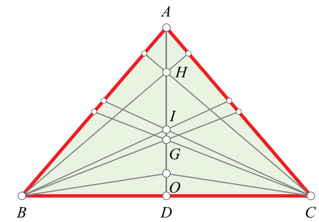


Figure 3. The case of an isosceles triangle

D, E, F coinciding (see Figure 3). That is, these three points are either all distinct or all coincident. The corresponding lines associated with the other two sides of the triangle continue to be distinct unless $AB = AC = BC$.

It is well known that the altitudes of a triangle are concurrent at the ***orthocentre*** (generally denoted by the letter H), the angle bisectors at the ***incentre*** (I), the medians at the ***centroid*** (G) and the perpendicular bisectors of the sides at the ***circumcentre*** (O). These four "triangle centres" are distinct points in a scalene triangle. (It will be a nice exercise for you to prove that if any two of the points I, O, G, H coincide, then they all coincide.)

In any triangle the points H, G, O are collinear, as shown by Leonhard Euler in 1765. The line of collinearity is called the ***Euler line*** of the triangle, and G lies between H and O on this line, dividing HO in the ratio $2 : 1$. (The point I in general does not lie on the Euler line, unless the triangle is isosceles.) In an equilateral triangle the three points merge into a single point. In an isosceles triangle such as $\triangle ABC$, with $AB = AC \neq BC$ (Figure 3), they are distinct and lie on the line of symmetry AD, which is also the Euler line for the triangle.

From this point on we shall confine our discussion to the case of an isosceles triangle ABC in which $AB = AC$. Let D be the midpoint of BC; then AD is a line of symmetry for the triangle. The claim that the points O, G, I, H all lie on AD is easy to justify, using the well-known theorems of congruence. We claim that the points always occur in the order O, G, I, H on the line. For the proof, use will be made of the fact that in any triangle, the centroid G lies $2/3$ the way along each of the medians. So the ratio $AG : AD$ equals $2 : 3$, regardless of the shape of the triangle. (See page 52 of the

November 2013 issue of *At Right Angles* for a proof of this assertion.)

More specifically, we claim the following:

- If $\measuredangle A < 60°$, then $BC < AB = AC$, so H lies closest to BC, followed by I, G and O, in that order.
- If $\measuredangle A > 60°$, the order gets reversed, since now $BC > AB = AC$. (Of course, when $\measuredangle A = 60°$, the four points are coincident.)
- If $\measuredangle A = 90°$, then H coincides with A, while O coincides with the midpoint D of BC. Observe that in this configuration the fact (Euler's theorem) that the ratio $HG : GO$ equals $2 : 1$ reduces to the known fact that the centroid lies 2/3 the way along a median.
- If $\measuredangle A > 90°$, then both H and O lie outside the triangle.

To justify the first two claims, we derive expressions for the distances AH, AI and AO, as fractions of the altitude AD, as $\measuredangle A$ varies.

With reference to Figures 4 and 5, we have:

$$\frac{HD}{BD} = \tan\frac{A}{2}, \quad \frac{BD}{AD} = \tan\frac{A}{2}, \tag{1}$$

hence:

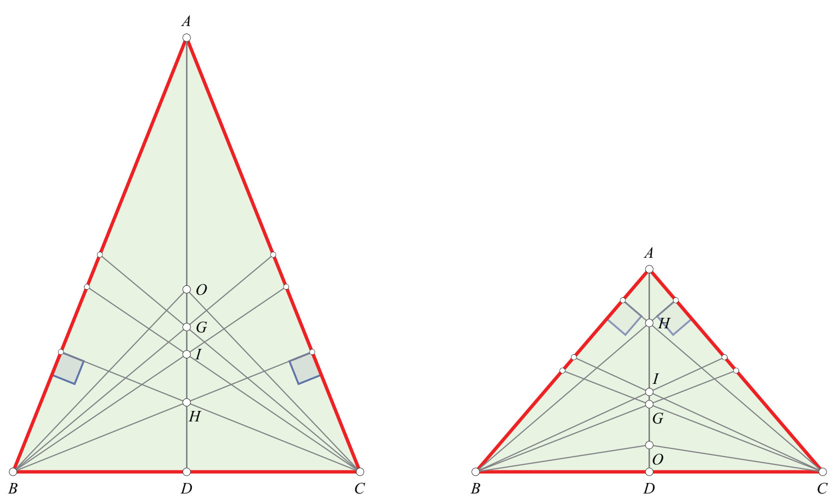
$$\frac{HD}{AD} = \tan^2\frac{A}{2}. \tag{2}$$

Next:

$$\frac{ID}{BD} = \tan\frac{B}{2} = \tan\left(45° - \frac{A}{4}\right), \tag{3}$$

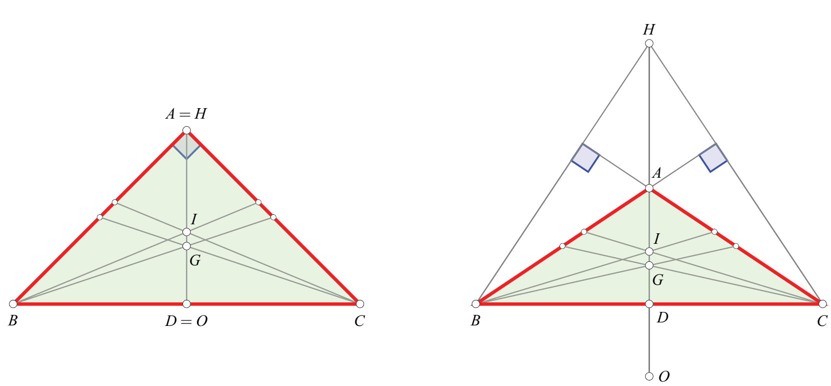
so:

$$\frac{ID}{AD} = \frac{ID}{BD}\cdot\frac{BD}{AD} = \tan\frac{A}{2}\cdot\tan\left(45° - \frac{A}{4}\right). \tag{4}$$



When $\measuredangle A < 60°$ When $\measuredangle A > 60°$

Figure 4. Isosceles triangles with different apex angles



When $\measuredangle A = 90°$ When $\measuredangle A > 90°$

Figure 5. Isosceles triangles with different apex angles

The ratio for G is easy:

$$\frac{GD}{AD} = \frac{1}{3}. \tag{5}$$

Finally, for O we have: $\measuredangle BOC = 2\measuredangle A$, therefore $\measuredangle OBD = 90° - A$. This yields:

$$\frac{OD}{BD} = \tan(90° - A) = \frac{1}{\tan A},$$

hence:

$$\frac{OD}{AD} = \frac{\tan \frac{1}{2}A}{\tan A}. \tag{6}$$

Using the double-angle formula to express $\tan A$ in terms of $\tan \frac{1}{2}A$, the above expression for the ratio $OD : AD$ may be written more usefully as:

$$\frac{OD}{AD} = \frac{1 - \tan^2 \frac{1}{2}A}{2}. \tag{7}$$

From the above relations we see that

$$\frac{HD}{AD} + 2 \cdot \frac{OD}{AD} = 1,$$

and hence:

$$\boxed{\frac{1}{3} \cdot \frac{HD}{AD} + \frac{2}{3} \cdot \frac{OD}{AD} = \frac{GD}{AD}}. \tag{8}$$

This directly shows that G lies between O and H and divides segment OH in the ratio $1 : 2$.

It is an easy exercise to verify that if $\measuredangle A = 60°$ then

$$\frac{HD}{AD} = \frac{ID}{AD} = \frac{GD}{AD} = \frac{OD}{AD} = \frac{1}{3}.$$

The relative order of the four points H, I, G, O on AD. Observe that if $A < 60°$ then $\frac{3}{4}A < 45°$ and so $\frac{1}{2}A < 45° - \frac{1}{4}A$. It follows that if $A < 60°$ then $\tan \frac{1}{2}A < \tan\left(45° - \frac{1}{4}A\right)$ and hence from relations (2) and (4) that

$$\boxed{\frac{HD}{AD} < \frac{ID}{AD}}.$$

The inequality is reversed when $A > 60°$.

Now let us compare the relative positions of I and G. This involves more manipulations than the other cases. We have:

$$\begin{aligned}\frac{ID}{AD} &= \tan \frac{A}{2} \cdot \tan\left(45° - \frac{A}{4}\right) \\ &= \frac{2\tan \frac{1}{4}A}{1 - \tan^2 \frac{1}{4}A} \cdot \frac{1 - \tan \frac{1}{4}A}{1 + \tan \frac{1}{4}A} = \frac{2\tan \frac{1}{4}A}{\left(1 + \tan \frac{1}{4}A\right)^2} \\ &= \frac{2t}{(1+t)^2}, \quad \text{where } t = \tan \frac{A}{4}.\end{aligned}$$

Since $0° < A < 180°$, it must be that $0 < \tan \frac{1}{4}A < 1$, i.e., $0 < t < 1$. Differentiation yields:

$$\frac{d}{dt}\left(\frac{2t}{(1+t)^2}\right) = \frac{2(1-t)}{(1+t)^3},$$

which is positive for $0 < t < 1$. Hence the expression $2t/(1+t)^2$ steadily increases as t goes from 0 to 1. Therefore the quantity

$$\frac{ID}{AD} = \frac{2\tan \frac{1}{4}A}{\left(1 + \tan \frac{1}{4}A\right)^2}$$

steadily increases as A rises from $0°$ to $180°$. Simple computation shows that the above fraction equals $1/3$ when $A = 60°$. (We need the identity $\tan 15° = 2 - \sqrt{3}$.) It follows that

$$\frac{ID}{AD} < \frac{1}{3} \quad \text{if } \measuredangle A < 60°,$$

$$\frac{ID}{AD} > \frac{1}{3} \quad \text{if } \measuredangle A > 60°.$$

We conclude from this that I always lies between H and G for an isosceles triangle, and the four points H, I, G, O occur in that order always, with O being closer to vertex A when $\measuredangle A < 60°$, and H being closer to vertex A when $\measuredangle A > 60°$.

The various possibilities which may exist for the order of the points O, G, H, I on the Euler line in the case of an isosceles triangle ABC with $AB = AC$ are summarised in Figure 6.

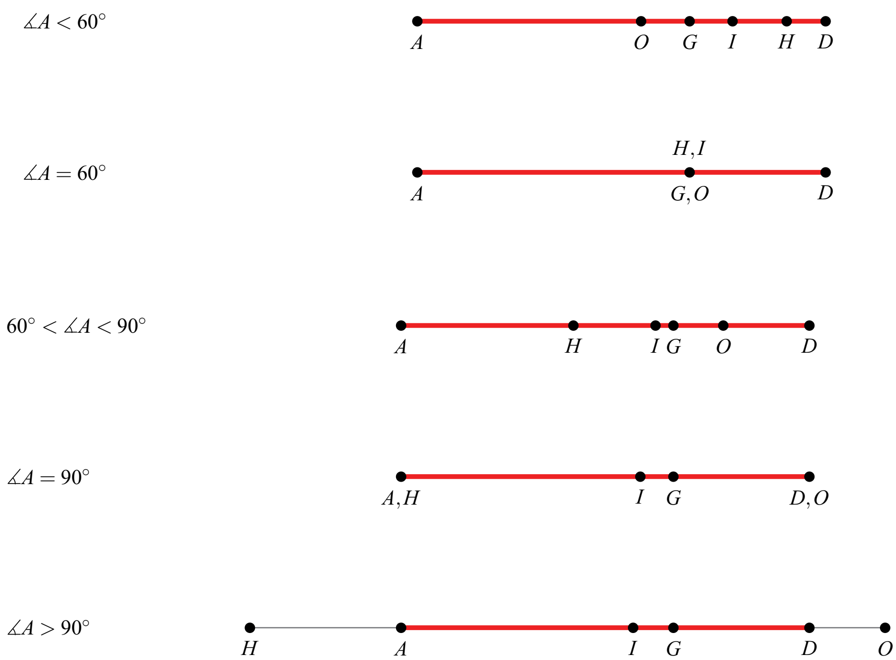
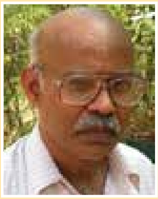


Figure 6.



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