Azim Premji University

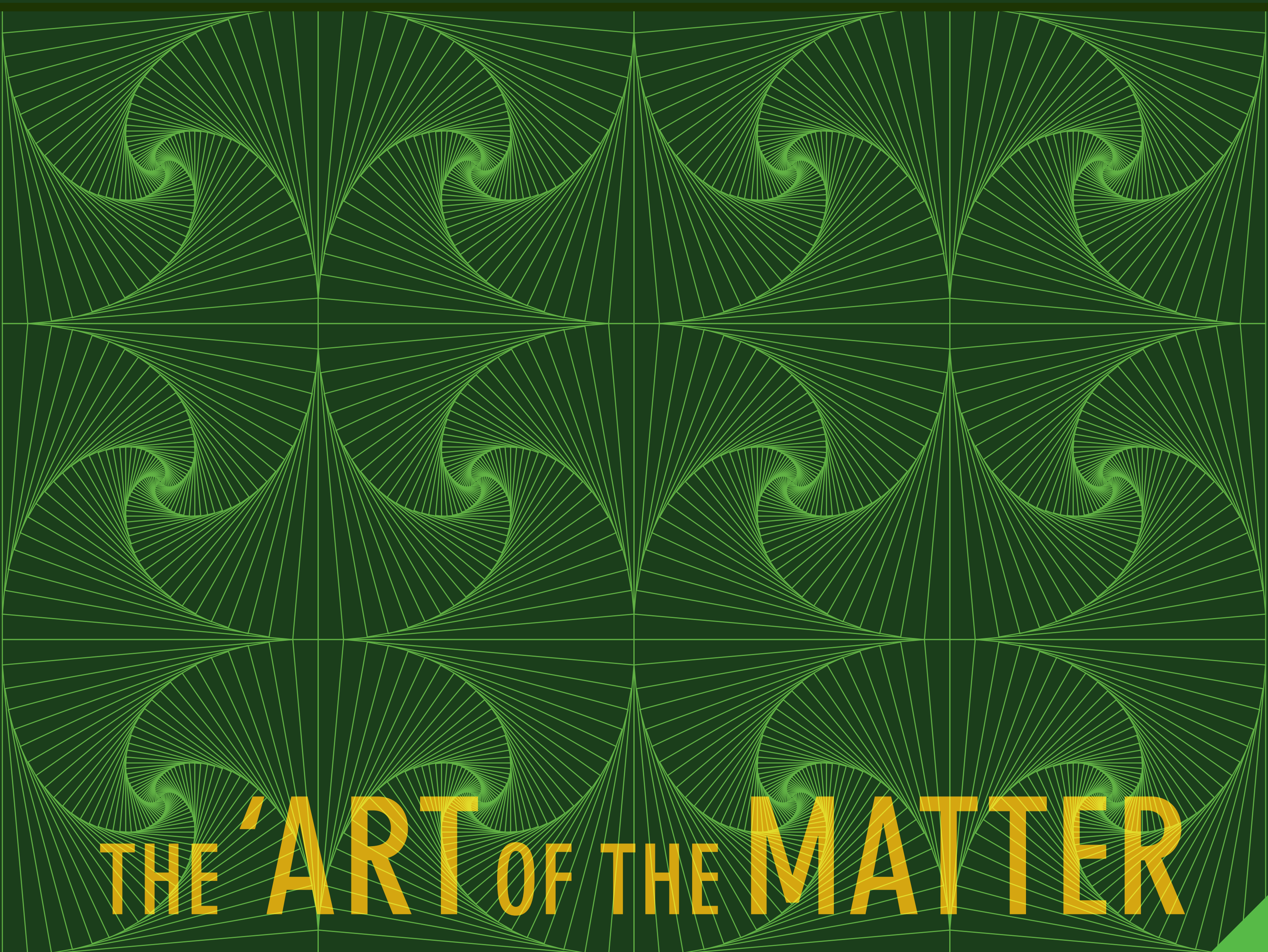
A publication of Azim Premji University together with Community Mathematics Centre, Rishi Valley

COMMUNITY MATH CENTRE RISHI VALLEY SCHOOL (KFI)

At Right Angles

A RESOURCE FOR SCHOOL MATHEMATICS

Volume 6, No.1
March 2017

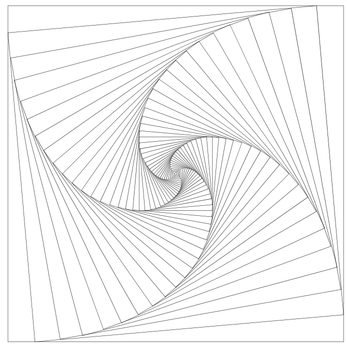


THE 'ART OF THE MATTER

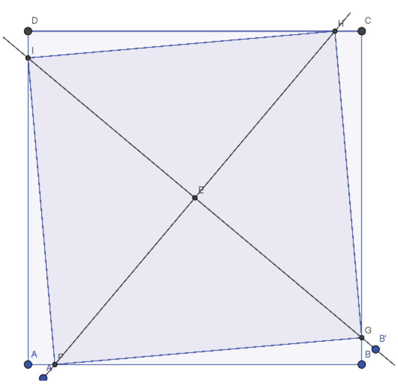
PULLOUT
TEACHING TIME

The analysis of patterns in mathematics, like the study of genetics in medicine is an extremely powerful tool- one that can unlock the key to what at first glance seems beautiful, mysterious and unpredictable. Trying to go to the heart of the matter is key to the study of mathematics and we thought that the pattern on the cover was an intriguing study. To be fair, we analyzed a simpler pattern.

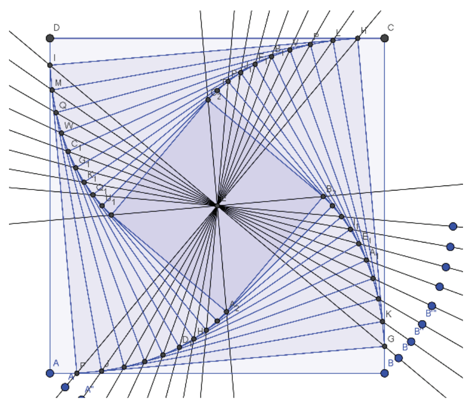
It all began, when Avita Chauhan, an artist and resource person in mathematics in Udham Singh Nagar District Institute of Azim Premji Foundation asked us how this pattern was generated. We zoomed in on the centre of the pattern and then...



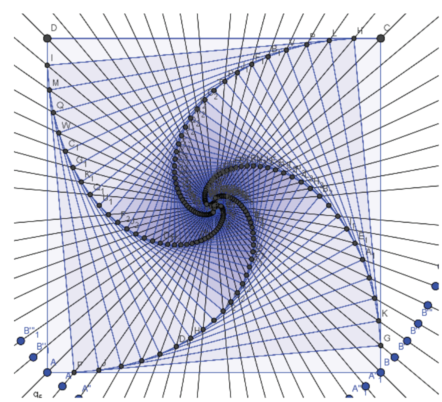
this is how we started…



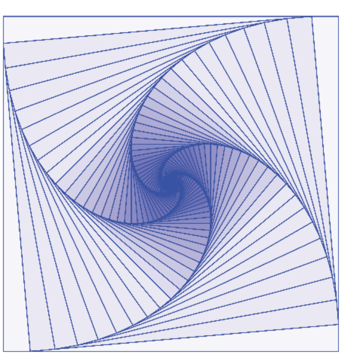
we repeated the first rotation several times



slowly the pattern emerged

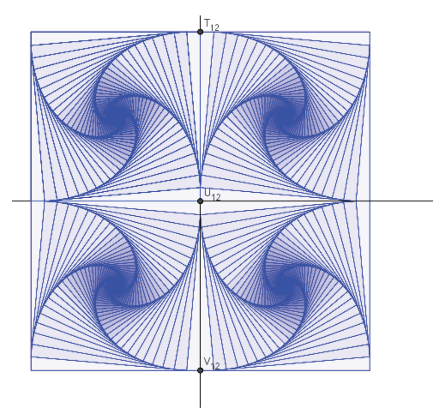


A little cleaning up



And voila! we had replicated the pattern, (we used GeoGebra and created a tool that allowed us to repeat this a large number of times).

At this stage, we did something that was easy as pie in GeoGebra and which is again another mathematical skill, we used this pattern to generate another pattern, simply by repeatedly reflecting the original pattern!



And then we knew that we had our cover for the March issue!

From the Editor's Desk . . .

The '*Art of the Matter* – Can a picture be a powerful pedagogical tool? *At Right Angles* believes it can, and if you read the write up on the facing page, I'm sure you will agree with us. Going to the heart of the art is key to mathematics and there's more of the same inside this issue. We begin with Ramya Ramalingam, a sixteen year old school girl, unraveling the mysteries of **Knot Theory** for us. And Haneet Gandhi picks up where her series on **Tessellations** stopped, with a fascinating article on tiling and the pictures we can create with different combinations and permutations of polygons.

In the ClassRoom section, Khushboo Awasthi opens up the **Square Root Spiral** with a series of investigative questions; Ujjwal Rane proves **Fagnano's Theorem** in several innovative ways. CoMaC describes an unusual way **to bisect an angle** and also manages to pull yet another **3-4-5 triangle** out of a picture problem from Dan Meyer's blog. The **golden ratio** which has long connected math with art pops up in Kepler's triangle – read more about it in Marcus Bizony's article. And in **How To Prove It**, Shailesh Shirali uses Ptolemy's theorem to reveal all kinds of fascinating relationships in cyclic quadrilaterals.

TechSpace features the first part of a two part series on **constructive definitions;** Michael de Villiers shows you how to do so with a GeoGebra activity centred on the golden rhombus. Truly cutting edge math pedagogy!

Our Review this time is by Kamala Mukunda who shares her views on Liping Ma's classic **Knowing and Teaching Elementary Mathematics**. This is a must-have for every school library and a must-read for mathematics teachers of all classes.

Finally, it's **Time** – this PullOut by Padmapriya Shirali will give you several new ideas to introduce this all important concept and help students quantify something which impinges on their consciousness long before they come to school.

We hope you enjoy this issue – do share your thoughts with us on AtRia.editor@apu.edu.in or our FaceBook page AtRiUM.

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At Right Angles is a publication of Azim Premji University together with Community Mathematics Centre, Rishi Valley School and Sahyadri School (KFI). It aims to reach out to teachers, teacher educators, students & those who are passionate about mathematics. It provides a platform for the expression of varied opinions & perspectives and encourages new and informed positions, thought-provoking points of view and stories of innovation. The approach is a balance between being an 'academic' and 'practitioner' oriented magazine.

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Contents

Features

Our leading section has articles which are focused on mathematical content in both pure and applied mathematics. The themes vary: from little known proofs of well-known theorems to proofs without words; from the mathematics concealed in paper folding to the significance of mathematics in the world we live in; from historical perspectives to current developments in the field of mathematics. Written by practising mathematicians, the common thread is the joy of sharing discoveries and the investigative approaches leading to them.

ClassRoom

This section gives you a 'fly on the wall' classroom experience. With articles that deal with issues of pedagogy, teaching methodology and classroom teaching, it takes you to the hot seat of mathematics education. ClassRoom is meant for practising teachers and teacher educators. Articles are sometimes anecdotal; or about how to teach a topic or concept in a different way. They often take a new look at assessment or at projects; discuss how to anchor a math club or math expo; offer insights into remedial teaching etc.

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TechSpace

'This section includes articles which emphasise the use of technology for exploring and visualizing a wide range of mathematical ideas and concepts. The thrust is on presenting materials and activities which will empower the teacher to enhance instruction through technology as well enable the student to use the possibilities offered by technology to develop mathematical thinking. The content of the section is generally based on mathematical software such dynamic geometry software (DGS), computer algebra systems (CAS), spreadsheets, calculators as well as open source online resources. Written by practising mathematicians and teachers, the focus is on technology enabled explorations which can be easily integrated in the classroom.

Problem Corner

Review

We are fortunate that there are excellent books available that attempt to convey the power and beauty of mathematics to a lay audience. We hope in this section to review a variety of books: classic texts in school mathematics, biographies, historical accounts of mathematics, popular expositions. We will also review books on mathematics education, how best to teach mathematics, material on recreational mathematics, interesting websites and educational software. The idea is for reviewers to open up the multidimensional world of mathematics for students and teachers, while at the same time bringing their own knowledge and understanding to bear on the theme.

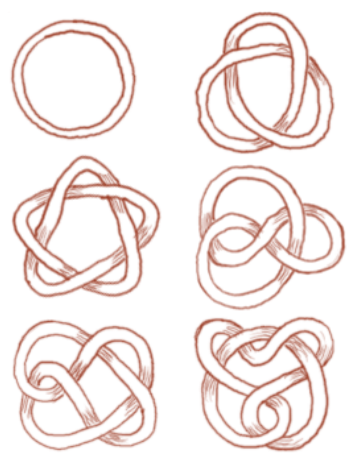
PullOut

The PullOut is the part of the magazine that is aimed at the primary school teacher. It takes a hands-on, activity-based approach to the teaching of the basic concepts in mathematics. This section deals with common misconceptions and how to address them, manipulatives and how to use them to maximize student understanding and mathematical skill development; and, best of all, how to incorporate writing and documentation skills into activity-based learning. The PullOut is theme-based and, as its name suggests, can be used separately from the main magazine in a different section of the school.

Tying it up …
KNOT THEORY
… without loose ends

RAMYA RAMALINGAM

Knot theory is an important sub-field of **topology** that studies the properties of different kinds of knots. This subject is a fairly new and still developing branch of mathematics. Interestingly, the roots of this subject originate in physics, not math. Physicists at one time believed that atoms and molecules were configurations of knotted thread. Although later models of the atom abandoned this postulate, knot theory came into its own in mathematics and in other branches of science.



Knot theory has applications in other mathematical branches as well – it is used substantially in graph theory, which in turn has implications in computer science while studying networks, data organization and computational flow. Knot theory also has uses in biology – it turns out that in certain organisms, DNA often twists itself up into knots which results in a host of different properties and sometimes problems for the organism. Knowledge of the properties of knots can be indispensable in studying this.

Keywords: Topology, knot, string, projection, tri-colourable, Reidemeister, unknot, trefoil, twist, poke, slide

What is Topology?

In Euclidean geometry we have the notion of congruence of triangles. One way of approaching this topic is through the study of ***functions*** or ***mappings*** or ***transformations*** from the plane into itself. To start with, let us consider only functions which have the property that *the distance between any pair of points remains unaltered as a result of the mapping*. Such a mapping is known as an ***isometry*** ('iso' = same, 'metre' = distance). Under such mapping, any figure is mapped to a figure which is congruent to itself. Here the word 'congruence' is used in its usual sense. But we can turn the definition around and instead define congruence in terms of the mapping. That is, if a figure A can be mapped to a figure B using such a mapping, we say that B is congruent to A, and the study of all properties of these figures which remain unchanged as a result of these mappings is what we call ***Euclidean geometry***.

Note that the class of mappings allowed is of critical importance. If we enlarge the class, the notion of congruence changes accordingly.

The class of isometric mappings is a highly restricted one; so let us replace it with something larger. If we consider instead, functions which have the following property: *for any three points A, B, C which are mapped to the points A', B', C' respectively, angle ABC must be equal to angle A'B'C'*, then we get what is ordinarily called ***similarity geometry***.

In Topology, the functions permitted belong to a much bigger class. They are what are known as ***Homeomorphisms***. Rather than give a technical definition, let us say simply that they refer to ***continuous deformations***. Thus, we allow stretching, contracting, twisting, shearing and so on; but we do not allow ruptures or tears. If an object A in 3-D space can be transformed into another object B using continuous deformations, and B in turn can be transformed into A using continuous deformations, we say that A and B are ***topologically indistinguishable***.

Using functions from within this class, it is easy to see that:

- A circle of any size is topologically indistinguishable from an ellipse of any size and any eccentricity.
- A circle of any size is topologically indistinguishable from a rectangle of any size and any shape.
- A line segment of any length is topologically indistinguishable from a planar arc of any shape and any length, provided the arc does not intersect itself at any point.
- A doughnut is topologically indistinguishable from a coffee cup (assuming that the cup is of the usual kind, with a single handle!), and both these shapes are topologically distinct from a saucer.

The study of which shapes are topologically indistinguishable from which other shapes is an informal way of understanding the term ***Topology***.

Organic chemistry can also use knot theory in differentiating mirror image molecules (which can have astoundingly different properties) from each other.

What is a Knot?

We start with the most basic question about this topic: ***What exactly is a knot?*** In simple terms, a knot can be thought of as a piece of string crossed in a certain manner, the ends being tied together. The tying of ends is particularly important to the definition as any pattern of an open-ended string can be manipulated into any other pattern through continuous deformation. Thus, without closing the ends, it becomes impossible to differentiate between knots, an essential requirement in their study.

Some Basic Knots

1. **The Unknot**: This is also known as the trivial knot and is the simplest knot of all. It is simply a loop; the simplest way to look at it is this way – connect two ends of a string together without knotting the string or crossing it over itself at any point.

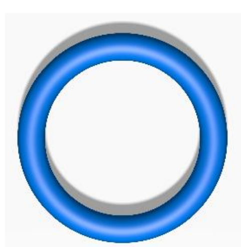


Figure 1

2. **The Trefoil**: The next simplest knot surprisingly shows up in a number of places in nature and microscopic organisms. Its simplest picture has 3 crossings. It is also called the overhead knot, and is the closed-end version of the simplest knot most people use to tie string.

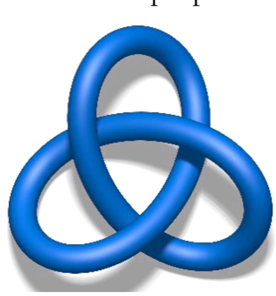


Figure 2

3. **Figure-8 knot**: The simplest viewing of this knot has 4 places where the string crosses itself. It is called so because one of its most common projections looks like the digit eight.

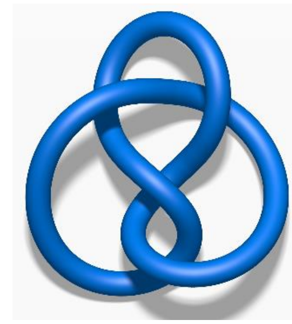


Figure 3

Experiment 1: Take a long piece of string; it should be sturdy but deformable. First, create the unknot by connecting the two ends of the string together. Next, deform it by passing the string over itself and creating different crossings and loops – but do not untie the string or cut it at any place. After you have made a number of crossings, place the knot on a table. Simply by looking at it, can you tell that it is the unknot, or does it look different?

Experiment 2: Now take another piece of string and create the trefoil knot by using Figure 2. Does it look like it could be the unknot? Try moving the knot around and deforming it. Is it possible to make the trefoil look like the simple version of the unknot? You could try to make the Figure-8 knot as well and see if you can transform it into either the trefoil or the unknot.

Equivalent Knots

Two knots are considered to be the same, or equivalent to one another, if you can 'deform' one into the other without breaking the knots open. While this is not a formal definition, it is intuitively understandable. If you imagine yourself to be holding the knot, and can somehow twist it or turn it to look like another knot, clearly the two must be the same. But there is still the need for a more official (rigorous?) definition, which leads to the questions:

1. How do we prove that two knots are the same?
2. How do we prove that two knots are different?

At first glance the two problems seem to be the same, but the second question is significantly more difficult to answer. We will try to provide a few things that might help.

Definitions

A ***projection*** is a 2-dimensional picture of a 3-dimensional knot.

The places where two strands of a projection meet is called a ***crossing***. The strand that goes over is called an *over-crossing* and the strand that goes under is called an *under-crossing*.

A ***strand*** of a projection is a piece of the knot cut off on both ends by a crossing. Basically, while drawing a knot projection, each strand corresponds to the longest piece of the knot you can draw at that point without lifting your pencil off the paper.

Figure 4

All of the differently coloured pieces of the knot represent strands.

A ***link*** is a collection (or a union) of multiple knots, possibly linked or knotted together. An *n-component link* is a link that consists of *n* individual knots.

A ***planar isotopy*** of a projection is a manipulation of a part of the knot in 2D space (by shrinking, straightening, enlarging) that does *not* change the number of crossings of the projection.

An ***ambient isotopy*** of a projection is a manipulation of the knot in 3D space that can change crossings. The only restriction is that the knot may not be cut anywhere.

Note: A knot can have a large number of differing projections.

The Reidemeister Theorem

The Reidemeister Theorem (named after the German mathematician Kurt Reidemeister, 1893 – 1971) states that *two projections are of the same knot if and only if either of the projections can be transformed into the other using a series of planar isotopies and/or Reidemeister moves*. We now explain what this means, but we shall not try to prove the theorem here.

We first need to explain the term ***'Reidemeister move'***. It turns out that there are three types of moves that encompass the ways you can manipulate a knot, provided you do not cut it anywhere. The three moves are called:

i. **The Twist (RM 1)**: If you have a straight piece of the knot, and twist it once so as to create a single crossing, you have performed Reidemeister Move 1; see Figure 5.

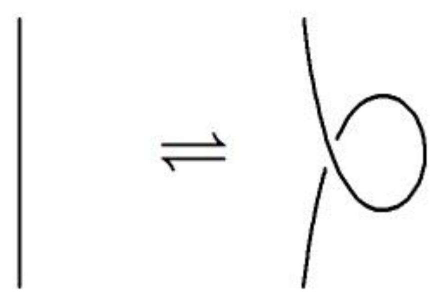


Figure 5

ii. **The Poke (RM 2)**: To perform Reidemeister Move 2, you need to push one part of the knot under (or over) another part of the knot so you get two under-crossings (or over-crossings) beside each other; see Figure 6.

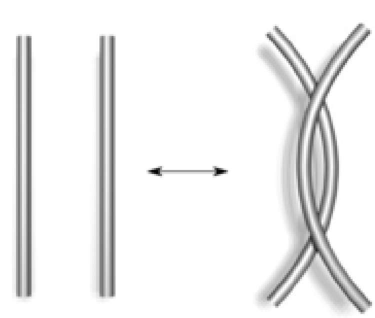


Figure 6

iii. **The Slide (RM 3)**: If a part of the knot goes under (or over) two other pieces which cross each other, that part can be slid under (or over) the crossing to the other side – this is Reidemeister Move 3; see Figure 7.

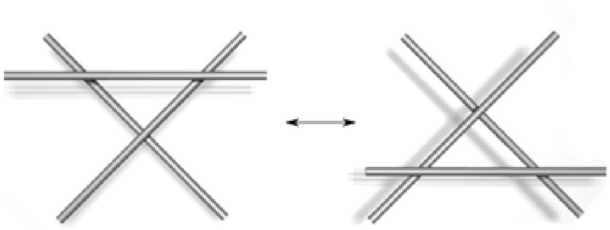
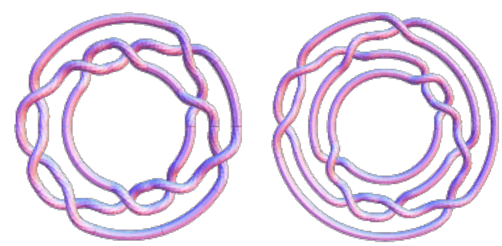


Figure 7

If two knot diagrams are projections of the same knot, the Reidemeister theorem clearly makes it easier to establish this fact. If we can use the Reidemeister moves to make one projection resemble the other, this theorem tells us they are the same knot.

But if we find ourselves unable to do this, we cannot assume that the two knot projections are different; perhaps we just made the wrong set of Reidemeister moves! In order to actually differentiate between knots, we need an *invariant*; something that does not change for a knot, no matter what projection of the knot is used. Then, if two different projections have different invariants, we can show that the two knots are different. For this purpose, something called the *tri-colourability* of a knot was defined.



What is an invariant?

Invariance is a powerful tool in mathematics. It is often used when we are trying to prove that something is not possible – which can sometimes be far more difficult to do than to prove that something is possible. The key step here is to find some quantity which does not change as we apply various transformations to the configuration. Many of the famous impossibility results in mathematics are proved via identification and skillful use of a suitable invariant. For example, it may be that there is a certain well-defined quantity whose parity does not change as a result of the permitted transformations, but such that the parities of the quantities associated with two given states are different. In such a case, it should be obvious that there can be no sequence of transformations which will take you from one state to the other state. A well-known problem of this genre is that of finding a method using Euclidean geometry tools (i.e., compass and unmarked straight edge) to trisect an arbitrary angle. The ancient Greek geometers struggled with this problem but did not make any headway. A full two millennia later, in the first half of the nineteenth century, it was shown that no such procedure can exist. This is an example of an impossibility result. In our context we may use invariance to help us with the second question – namely, showing that two knots are not the same.

Invariants are not used only for proving impossibility results. Many beautiful results relating to triangles and the conic sections can be proved using notions of invariance. Likewise, certain results in elementary number theory relating to divisibility can be proved using such notions.

Tri-colourability

A knot is defined as ***tri-colourable*** if one can colour the strands of the projection, using only 2 or 3 colours, such that at each crossing, either all strands are the same colour or all strands are of different colours.

Tri-colourability is useful as it turns out to be an invariant for any given knot – that is, if one projection of a knot is tri-colourable, then all projections of that knot are tri-colourable.

Proof

To prove that tri-colourability does not change with the projection of a knot, all we need to do is show that whether or not a projection is tri-colourable is not affected by any Reidemeister moves. This should be evident because any projection of a knot can be transformed into any other projection of the same knot using only Reidemeister moves.

RM-1

When we go from an untwisted strand to a twist (in which we can see two strands), it is clear from the diagram that the twisted projection can be coloured in such a way that preserves tri-colourability. When we look at the reverse direction (twisted to untwisted), we can see there is only one way to colour Strand 1 and Strand 2 such that the diagram is tri-colourable; that is, both strands have to be the same colour. In this case as well, we can see tri-colourability is preserved by the untwist. Thus, tri-colourability is preserved by Reidemeister Move 1.

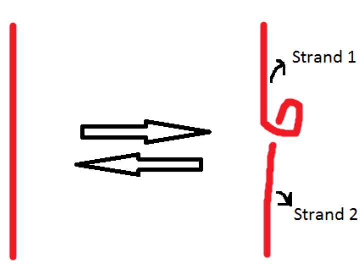


Figure 8

RM-2

In Reidemeister Move 2, there are two possibilities: the two strands, one of which 'pokes' over the other, may be of the same colour (Figure 9), or they may be of two different colours (Figure 10). In both cases, we can see that tri-colourability can be preserved.

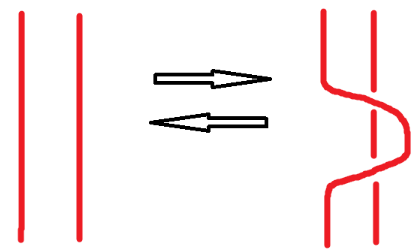


Figure 9 - Case 1

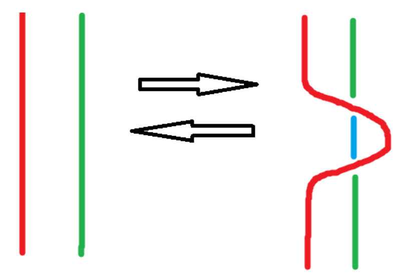


Figure 10 - Case 2

RM-3

There are 3 ways to colour the strands of Reidemeister Move 3 such that all the crossings satisfy the conditions of tri-colourability. In each of these 3 cases, it is possible to recolour the new crossings after the move to preserve tri-colourability.

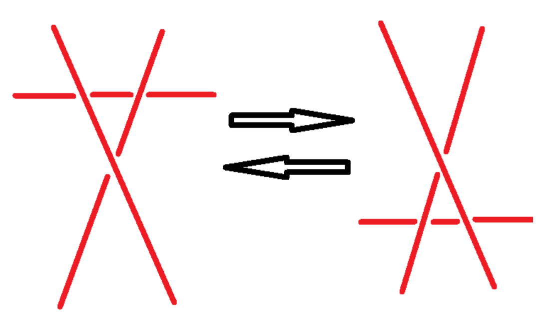


Figure 11 - Case 1

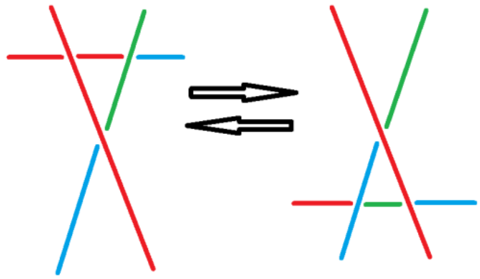


Figure 12 - Case 2

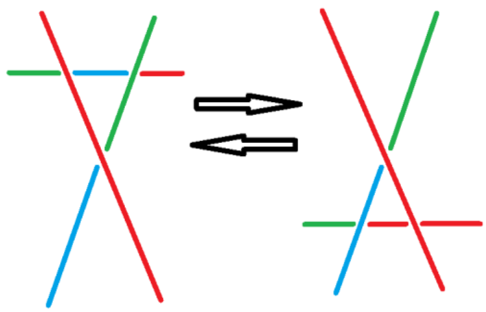


Figure 13 - Case 2

As we see from the figures, tri-colourability is always preserved after making any Reidemeister moves. Hence, it follows that if one projection of a knot is tri-colourable, all projections of that knot are tri-colourable. Therefore, tri-colourability is an invariant for any given knot; a knot is either tri-colourable or not.

Note: Another way to phrase the above is that if even one projection of a knot is not tri-colourable, no projection of the knot is tri-colourable. This should be clear since Reidemeister moves preserve tri-colourability.

Differentiating the unknot from the Trefoil

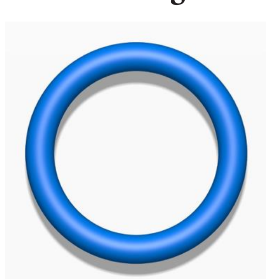


Figure 14

Until tri-colourability we had no definite way to prove that the two arguably simplest knots – the unknot and the trefoil – are actually different. When we look at them and try to manipulate one into the other, it seems intuitively clear that they are not the same; but that is not a rigorous proof. But we now have a tool – an invariant – that we can use. In the projection of the unknot above, there is only one strand – so we cannot colour it using two or three colours, which is a necessary condition for tri-colourability. Hence the unknot is not tri-colourable. In the picture of the trefoil above (on the right), we can see that it is tri-colourable. Hence, we have a difference between the two knots. Since the trefoil knot is tri-colourable whereas the unknot is not, it follows that the two knots are different.

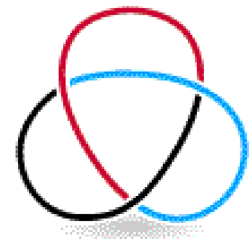


Figure 15

Is It Enough?

Obviously tri-colourability has its uses, since it allows us to definitively differentiate between two knots, something we haven't been able to do until now. But if we dig a little deeper, we see it doesn't help us much more. Tri-colourability is a Boolean invariant – a knot either has the property or it doesn't. Unfortunately, this means there aren't a lot of knots we can differentiate as yet; we just have two categories – knots which are tri-colourable, and those which aren't. Within those categories, we have no way to prove two knots are different. Well, the notion of an invariant helped us before – maybe we just need a more versatile one; maybe a function that has more outputs than just Yes or No (which is what tri-colourability gave us) – maybe even one that is unique to each and every knot.

More Invariants

In an effort to come up with more ways to differentiate knots, mathematicians have defined more and more invariants, some of which I will describe below:

- The *unknotting number* is the minimum number of crossings you have to exchange to change a knot into the unknot. By exchanging a crossing, we mean that at that particular crossing, we change the overcrossing into an undercrossing and vice-versa.

For example, the unknotting number of the trefoil knot is one:

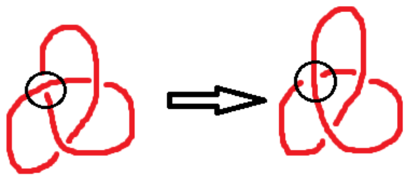


Figure 16

The picture on the right is actually the unknot.

- A knot is *alternating* if there exists at least one projection where the string alternates between crossing over and under.

For example, the trefoil and the figure 8 knot are both alternating knots because if you trace the regular projections of this knot given at the beginning of this article, you will find that the knots alternate over-crossings and under-crossings throughout the entire knot.

- The *crossing number* of a knot is the minimum number of crossings in any projection of a knot.

The crossing number of the unknot is 0. The crossing number of the trefoil is 3. Keep in mind that what this means is that although there are clearly projections of the trefoil knot that can be drawn with more than 3 crossings, the minimum number of crossings that have to be in any projection of the trefoil is 3. Therefore, there is no projection of the trefoil knot that has less than 3 crossings.

- The stick number of a knot is the minimum number of 'sticks' (straight lines) required to make the knot.

The *stick number* of the unknot is 3, and the stick number of the trefoil is 6 as shown below.

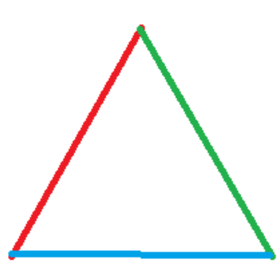


Figure 17

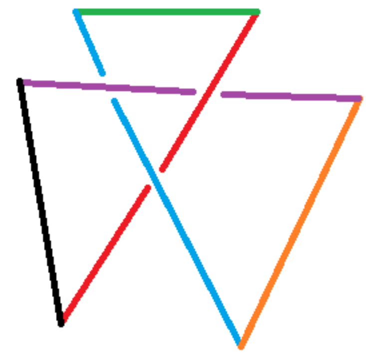


Figure 18

These are but a few of the invariants created in order to differentiate between knots.

The search is still on …

While these invariants are extremely useful in proving knots were different, they are not enough. None of these invariants is unique to each knot – for example, the unknotting number of at least 6 to 7 knots is 1. In order to find something unique to each knot, mathematicians tried to devise something called a knot polynomial – which is an expression in variables that describes the knot. In 1928, the ***Alexander polynomial*** was introduced, the first polynomial of its kind. Later, in 1969, Conway devised the ***Conway polynomial***, which is much simpler to compute than the Alexander Polynomial. In 1984, a few months after the invention of something called the ***Jones polynomial***, a group of six people collectively made the ***HOMFLY polynomial***, which is a superset of all other polynomials made so far. While the HOMFLY polynomial broke ground in this field, the amazing yet frustrating truth is that it is still not enough. Two knots have been discovered having the same HOMFLY polynomial, yet it has been proved using other methods that they are different knots. So, as of now, there is still no definitive answer as to how to prove that two knots are different.

As mentioned earlier, knot theory is a relatively new subject. There is a lot more research that needs to be done. But what we do know is that it is a very interesting subject – and who knows? Maybe you could discover a way to actually prove two knots are different.

References

1. Colin C. Adams, The Knot Book: *An Elementary Introduction to the Mathematical Theory of Knots*
2. Charles Livingston, *Knot Theory* (Mathematical Association of America Textbooks)
3. Richard H. Crowell and Ralph H. Fox, *Introduction to Knot Theory*. Chapter 1 (Dover Books on Mathematics)

Acknowledgements

Some of the diagrams in the article have been taken from the following websites:

1. http://www.wikiwand.com/en/Unknot
2. https://commons.wikimedia.org/wiki/File:Blue_Trefoil_Knot.png
3. https://collegemathteaching.wordpress.com/tag/knot-theory/
4. http://xingyuyang.blogspot.in/2016/02/reidemeister-moves.html
5. https://en.wikipedia.org/wiki/Reidemeister_move#/media/File:Reidemeister_move_2.png
6. https://en.wikipedia.org/wiki/Reidemeister_move#/media/File:Reidemeister_move_3.png

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FILLING A CIRCLE

Rahul tells Devang:

'Imagine you have a circular table and a large number of circular coins. We take turns to place these coins on the table and the player who is unable to find a space to place a coin loses.'

Is there a sure way to win this game?

Does your win depend on whether you are the first player?

Solution on page......63.......

Knot Theory Addendum

At Right Angles met up with author Ramya to discuss her article on Knot Theory. Over coffee at Starbucks, Ramya adeptly made sense of a tangled bunch of wool which I had carried with me to try and see if Knot Theory could help me untangle the web.

Here was the Trefoil in a familiar form:

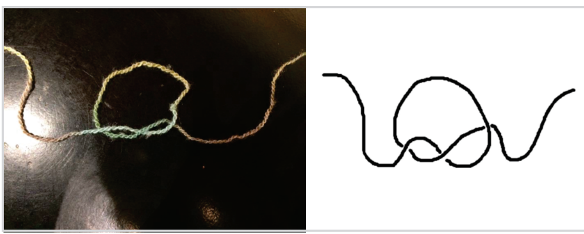


Figure 1. Trefoil (with Ramya's sketch alongside)

And now, with a little loosening of the wool :

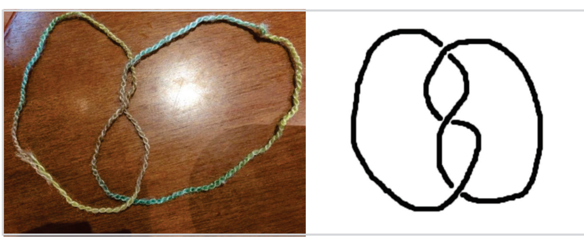


Figure 2. Trefoil

Some tweaking and voila, here is another view of the same knot:

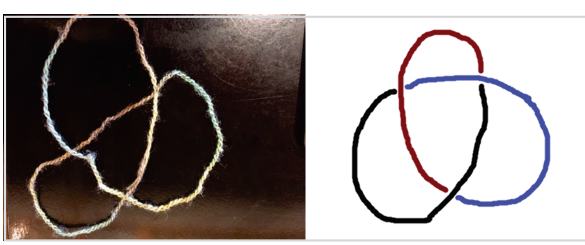


Figure 3. Trefoil

See the 3 crossings in each of the figures.

Now for a double knot:

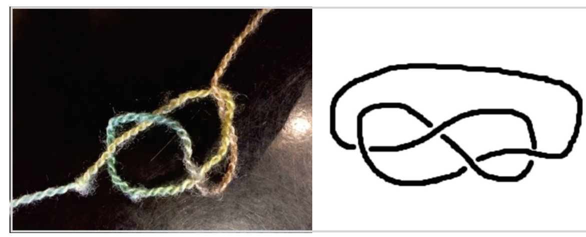


Figure 4. Cinquefoil

Look at the star shape with the 5 crossings:

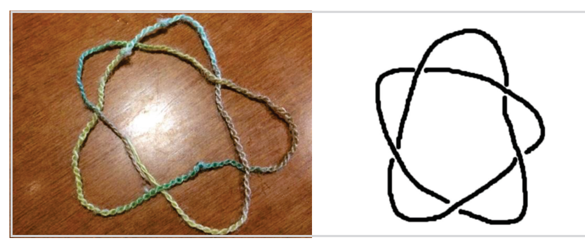
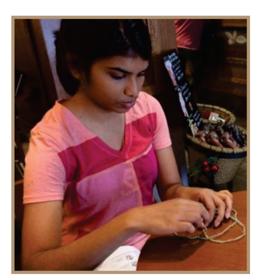


Figure 5. Cinquefoil

Starbucks may not appreciate their product being ignored, but I came away, not wanting to untangle my wool – in fact, I wanted to get it knottier!



P. S. RAMYA suggests the following simple reading on Knot Theory and DNA available at http://www.tiem.utk.edu/~gross/bioed/webmodules/DNAknot.html

Features

Method in Mathness

CATALOGUING 1-UNIFORM TILINGS

HANEET GANDHI

In my earlier two articles on Tessellations – *Covering the Plane with Repeated Patterns* Parts I and II – which appeared in *At Right Angles* (March 2014 and July 2014), I tried to provide a glimpse into the topic of recognizing regular polygons that can tessellate. In an accompanying article, *Enumeration of Semi-regular Tessellations* (AtRiA, March 2014), the authors arithmetically enumerated the combinations of regular polygons whose interior angles could fit together to make a complete angle (i.e., 360°); 17 such combinations were listed. However, not all of these tessellate. The present article may be viewed as an extension of the earlier ones; I have tried to identify and shortlist the combinations that extend to create semi-regular tiling patterns. (*Note from the editor: See the glossary at the end of the article for explanations of terms with which you may not be familiar. Some potentially unfamiliar terms have been highlighted for you.*)

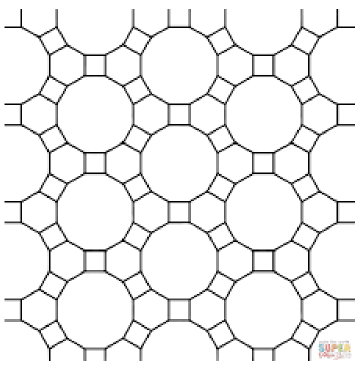
Code	Number of faces						Code	Number of faces					
	n_1	n_2	n_3	n_4	n_5	n_6		n_1	n_2	n_3	n_4	n_5	n_6
A	3	7	42				**K**	6	6	6			
B	3	8	24				**L**	3	3	4	12		
C	3	9	18				**M**	3	3	6	6		
D	3	10	15				**N**	3	4	4	6		
E	3	12	12				**P**	4	4	4			
F	4	5	20				**Q**	3	3	3	4	4	
G	4	6	12				**R**	3	3	3	3	6	
H	4	8	8				**S**	3	3	3	3	3	3
J	5	5	10										

Table 1: The 17 combinations

Keywords: *Tessellation, tiling, enumeration, regular polygon, edge-to-edge tessellation, regular tessellation, semi-regular tessellation, demi-regular tessellation, Archimedean tessellation*

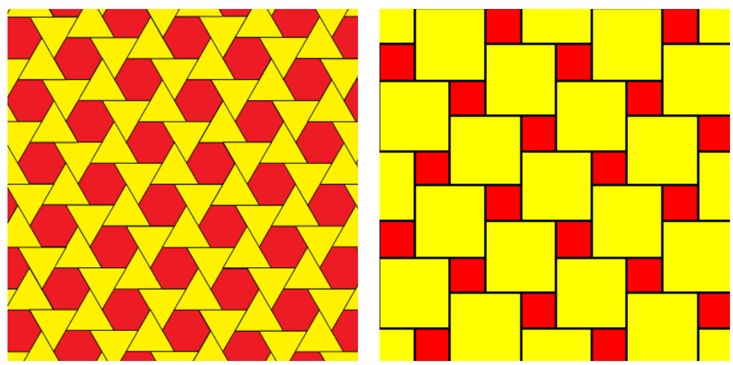
Enumeration of tiling patterns has been sporadic. The credit for categorizing tessellations goes to Johannes Kepler. In Book II of *Harmonices Mundi* (1619), Kepler enumerated tessellations which have the property that the way in which the polygons are arranged around each vertex is the same for all vertices; tessellations with this property are called **1-*uniform vertex-homogeneous tilings***. Subsequently, Krotenheerdt, Chavey and Galebach succeeded in cataloguing tessellations in which there is more than one vertex-homogeneity in the pattern. In other words, they systematised tessellations that have *k* kinds of vertices in the pattern. Tessellations with this property are called ***k-uniform*** **(*k* ≥ 2)** ***vertex tilings***. (See references [1], [2] and [3].)

An ***edge-to-edge tessellating pattern*** is one in which two polygons which touch each other do so along complete edges. An immediate consequence of this requirement is that the edge lengths of all polygons in the pattern are the same. Figure 1a shows an example of such a pattern, and Figure 1b shows an example of a tessellating pattern which is *not* edge-to-edge. In this article, we consider only tessellating patterns which are edge-to-edge.



Example of edge-to-edge tessellation

Figure 1a

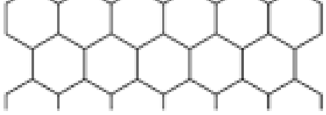


Examples of non-edge-to-edge tessellation of regular polygons

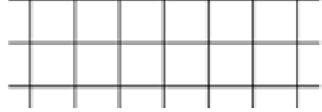
Figure 1b

Edge-to-edge tessellating patterns which also maintain vertex-homogeneity have been classified as ***regular, semi-regular*** and ***demi-regular.*** In regular tessellations and semi-regular tessellations, there is a single uniform vertex configuration all through the pattern (such tilings are also called 1-***uniform Archimedean tilings***), whereas in a demi-regular tessellation there are two or more vertex configurations. In other words, tessellations in which a combination of two or more polygons repeat to cover a plane are termed as semi-regular, and patterns in which two or more vertex configurations co-exist are termed as demi-regular. ***(Note: Some mathematicians define a demi-regular tessellation as a combination of semi-regular tessellations.)*** Though there does not exist much consensus on the number of demi-regular tessellations, we are confident that there are only three regular tessellations and only eight semi-regular tessellations. In this article, we study the eight configurations that give rise to semi-regular tessellations. The ideas presented in this article can be extended for identifying, constructing and cataloguing demi-regular tessellations.

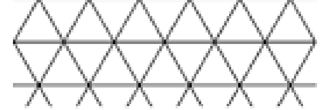
The first category of 1-uniform vertex tessellation is that of regular tessellations, in which the component polygons all have the same number of edges. Combinations K, P and S (Table 1) make regular tessellations (see Figure 2).



6.6.6
Figure 2a



4.4.4.4
Figure 2b



3.3.3.3.3.3
Figure 2c

To catalogue semi-regular tessellations, let us work with the remaining 14 combinations in Table 1. Through the enumeration process we had found all combinations of polygons that meet to form 360° at a vertex. It so happens that this process conveys little about the actual space fitting of the polygons. For example, the lexicon 3.3.3.4.4 only tells us that three equilateral triangles and two squares join together to complete the vertex. It does not, however, convey any information about the spatial arrangement of these triangles and squares. Would the three triangles come together as 3.3.3.4.4, or would they alternate as 3.4.3.3.4? Will both the arrangements tessellate? Will these make 1-uniform vertex tessellations?

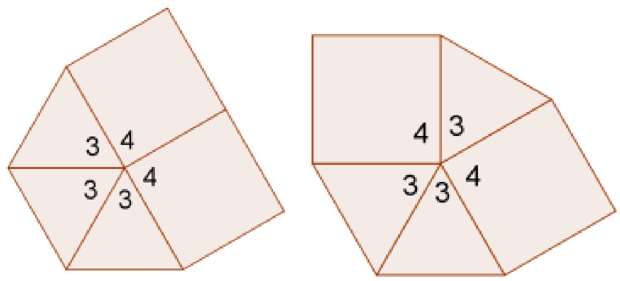


Figure 3

Three Regular Polygons at a Vertex

Let us start with a combination of three polygons of which one is a regular triangle: 3.A.B. In this lexicon, 3 refers to a regular triangle, A and B are regular polygons with A sides and B sides, respectively. The edge lengths of all the polygons are the same. To maintain a uniform vertex-homogeneity, the triangle must have the same spatial configuration of polygons at all its three vertices; i.e., they must be 'surrounded' by polygons in the same manner.

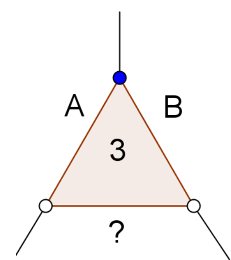


Figure 4

In the above representation, polygons A and B meet to complete the blue vertex. However, the white vertices will get completed only when the polygon at '?' is either A or B. The only way to do this is by setting A = B; so the pattern becomes 3.A.A. This eliminates combinations 3.7.42, 3.8.24, 3.9.18 and 3.10.15. Thus there exists only one combination, E (3.12.12), that extends to make 1-uniform vertex tessellation.

The same argument holds for all odd sided polygons incident to two other regular polygons. This eliminates F (4.5.20, read it as 5.20.4) and J (5.5.10).

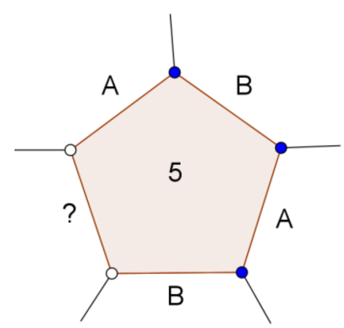


Figure 5

With a four-sided regular polygon, incidental to two other regular polygons, the configuration would be 4.A.B. Figures 6a and 6b explain that there can be only two ways in which 4.A.B would extend while maintaining single vertex-homogeneity in the plane – either A is equal to B or A and B are placed alternately. This qualifies H (4.8.8) and G (4.6.12).

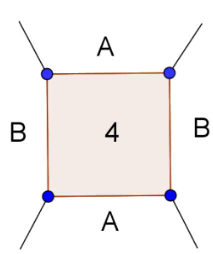


Figure 6a

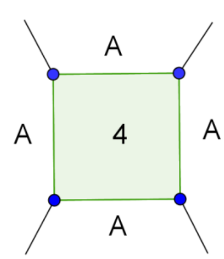
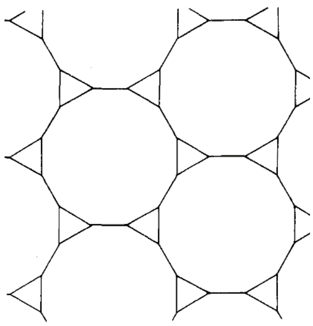
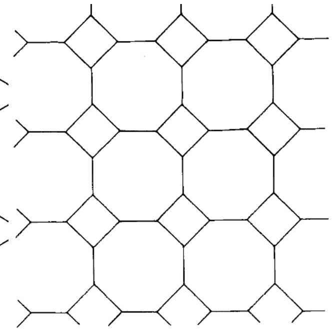


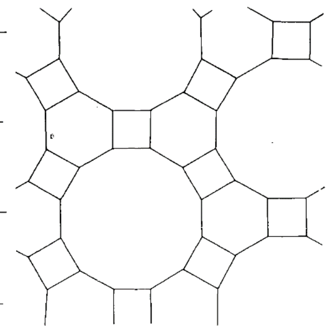
Figure 6b



3.12.12
Figure 7a



4.8.8
Figure 7b



4.6.12
Figure 7c

Summarizing, there can be only three 1-uniform vertex Archimedean tessellations of three regular polygons meeting at a vertex, and these are 3.12.12, 4.8.8 and 4.6.12 (see Figure 7).

Four Polygons

Let the tessellation have two triangles and two other polygons: 3.3.A.B. These polygons can be arranged as 3.3.A.B or 3.A.3.B. Let us begin by placing polygons A and B at the top blue vertex (Figure 8a). The polygons read in cyclic order would be 3.3.A.B. To have the same configuration at the red vertex, the polygons must be arranged in counter-clockwise order. At the white vertex, however, there will be three triangles making the sum total of angles 180°, leaving no possibility of fitting any other polygon. Thus, 3.3.A.B will not make a semi-regular tessellation. This shows the impossibility of both 3.3.4.12 (L) and 3.3.6.6 (M).

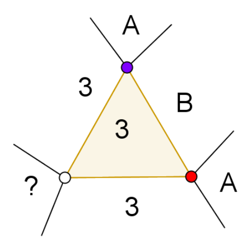


Figure 8a

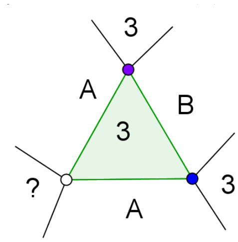
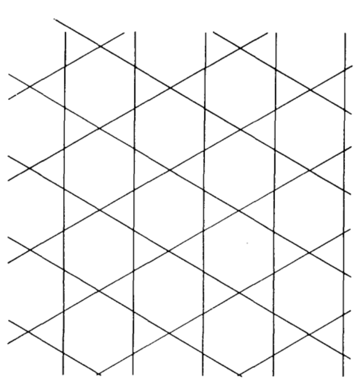


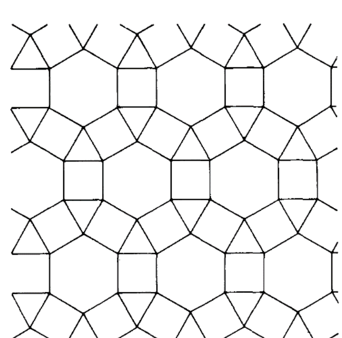
Figure 8b

The arrangement 3.A.3.B is depicted in Figure 8b; to maintain the same configuration at the white vertex, polygons A and B would have to be the same. Thus, two triangles, a square and a dodecagon (L) will not tessellate, but the combination M when rearranged as 3.6.3.6 will lead to a 1-uniform vertex tiling Figure 9a.

Similar reasoning leads to the conclusion that for Code N (3.4.4.6), there cannot be a semi-regular tessellation. However, if the polygons are rearranged as 3.4.6.4, then as shown in Figure 9b we are able to form a semi-regular tessellation.



3.6.3.6
Figure 9a



3.4.6.4
Figure 9b

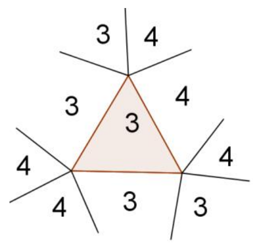


Figure 10a

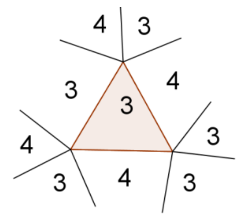
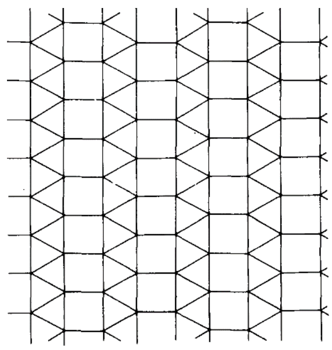
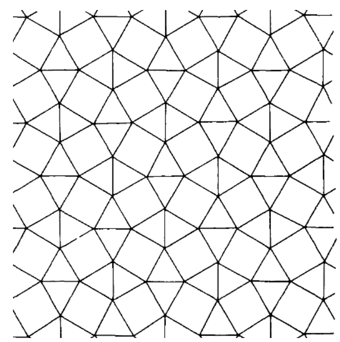


Figure 10b



3.3.3.4.4
Figure 11a

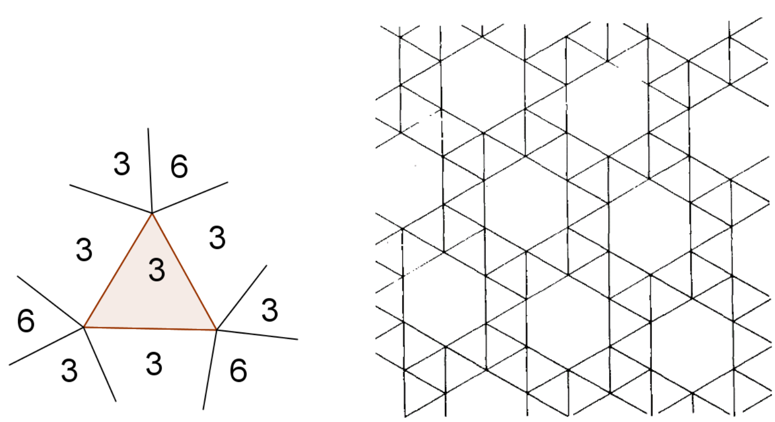


3.4.3.4.3
Figure 11b

Five Polygons
There are two configurations Q and R of five polygons. We will consider these separately.

The configuration Q makes two arrangements: 3.3.3.4.4 and 3.3.4.3.4. The arrangements in Figure 10a and Figure 10b depict schematically how these arrangements partition the plane, maintaining single homogeneity of vertices, to create semi-regular tiling patterns.(Figures 11a and 11b respectively)

And finally, Figure 12 depicts the tessellation of R (3.3.3.3.6):



3.3.3.3.6
Figure 12

Thus, only 8 of the 14 combinations of regular polygons partition the plane while conforming to 1-uniform vertex-homogeneity: 3.12.12, 4.8.8, 4.6.12, 3.6.3.6, 3.4.6.4, 3.3.3.4.4, 3.3.4.3.4 and 3.3.3.3.6. These then are our eight 1-uniform Archimedean or semi-regular tessellations.

Though there is a consensus on the possible number of regular and semi-regular tessellations, there is no precise way of concluding the same for demi-regular tessellations. To explore demi-regular tessellations you may choose to consider tile-homogeneity or vertex-homogeneity or edge-homogeneity as the criterion for listing. Krötenheerdt, 1969 (as stated in Grünbaum and Shephard, 1986) established 124 uniform-vertex tessellations on the basis of vertex-homogeneity. Chavey (1984) considered edge-homogeneity to list more than 165 demi-regular tessellations.

In this article we have taken an explorative approach to catalogue semi-regular tessellations. Middle grade teachers may let students hunt for tessellations combinatorially as well as geometrically and explore their many properties. This work can be extended to identify and catalogue demi-regular tessellations. Including such hands-on activities as part of middle grade mathematics teaching provides opportunities to experience the interplay of shapes, space, position and symmetry.

Further Reading

For more about demi-regular classification, readers can refer to the following:

1. Chavey, D.P. (1984). *Periodic Tilings and Tilings by Regular Polygons*. Unpublished doctoral dissertation submitted at University of Wisconsin.
2. Galebach, B. L. (2002). *N-uniform Tilings* at http://probabilitysports.com/tilings.html.
3. Grünbaum, B. and Shephard, G.C. (1986). *Tiling and Pattern*. W.H. Freeman and Company. New York.
4. www.maa.org/sites/default/files/pdf/upload_library/22/Allendoerfer/1978/0025570x.di021102.02p0230f.pdf
5. www.math.nus.edu.sg/aslaksen/papers/Demiregular.pdf

Acknowledgements:

Images of tessellations have been taken from open-sources:

1. Images of all tessellations are from: Weisstein, Eric W. "Tessellation." From MathWorld--A Wolfram Web Resource. http://mathworld.wolfram.com/Tessellation.html
2. The images of non-edge-to-edge tessellations are from: https://en.wikipedia.org/wiki/Euclidean_tilings_by_convex_regular_polygons#Tilings_that_are_not_edge-to-edge
3. The repeating units were made on Geogebra.

Glossary of terms

- 1-uniform vertex-homogeneous tiling: a tessellation in which the way in which the polygons are arranged around each vertex is the same for all vertices
- *k*-uniform ($k \geq 2$) vertex tessellation: a tessellation in which there are *k* different ways in which the polygons are arranged around the vertices
- Edge-to-edge tessellating pattern: a tessellation which has the feature that the polygons which touch each other do so along complete edges
- Regular tessellation: a tessellation in which the component polygons all have the same number of edges
- Semi-regular tessellation: a tessellation which has two or more kinds of component polygons, but the way in which the polygons are arranged around the vertices is the same for all vertices
- Demi-regular tessellation: a tessellation which has two or more kinds of component polygons, but the way in which the polygons are arranged around the vertices is not the same for all vertices

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FAGNANO'S PROBLEM
A Geometric Solution

UJJWAL RANE

When we see a problem on maximization or minimization, we generally think of calculus or linear programming. But in reality, problems are never bound to a specific tool and we are always free to try something different, like high school geometry and even a bit of physics!

Here is one such minimization problem: *Inspired by the American ministry of defense headquarters, the Pentagon, the Department of Mathematics has built* ***The Triangle*** *– a triangular building with three wings named A, B, C after Aryabhata, Brahmagupta and Chandrasekhar, respectively. The gates of the wings opening into the central courtyard are connected to one another with a triangular walkway. To ensure close interaction among the mathematicians, it is decided to minimize the length of this walkway, by optimally positioning the gates of the wings.*

Can you find the length (perimeter) of the shortest walkway?

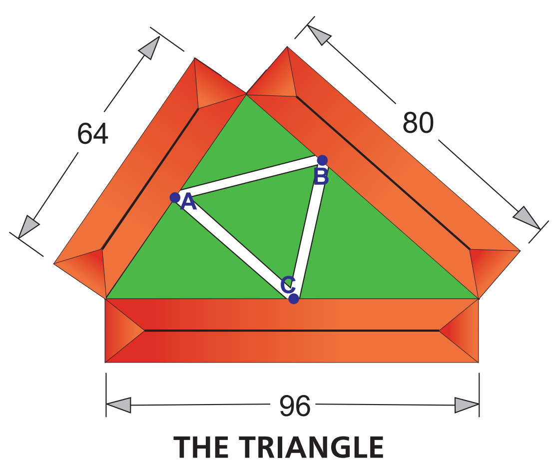

Figure 1

Keywords: optimization, triangle, ellipse, confocal, similar triangle, cosine rule, reflection, Fermat's Principle

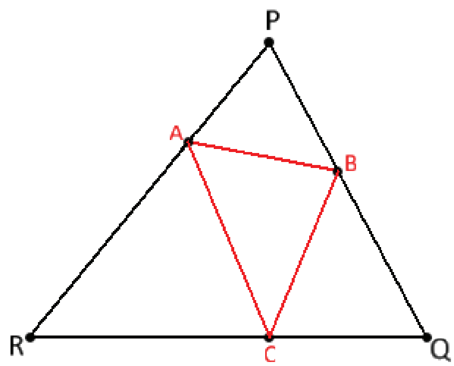

Figure 2

Historical reference: The abstract form of this problem was first solved by Giovanni Fagnano in 1775, stated as: *Given an acute-angled triangle (PQR), find the inscribed triangle (ABC) of minimum perimeter.*

[Comment: The restriction that triangle PQR should be acute-angled may not seem clear. What happens if instead the triangle is right-angled or obtuse-angled? It turns out that if PQR is either right-angled or obtuse-angled, the optimization problem is rather trivial. The reason for this is explained in an addendum to this article.]

Fagnano used calculus to solve it, but let's try some geometry and perhaps a dash of physics!

To start with, imagine that we have partially solved the problem and already found the best positions of gates A and B and now we want to minimize the length A-C-B.

Consider any path A-C-B as a string with its ends A and B fixed with pins. Does that remind you of something? The string and pins method of plotting an ellipse, of course!

So for a given length of A-C-B, point C lies on an ellipse with A and B as its foci, shown by dotted blue lines in the figure.

Such an ellipse, in general, will intersect line RQ in two points (say C_1 & C_1')

Now, if we shorten the 'string', the ellipse shrinks inwards, and we get a smaller 'confocal' ellipse which brings the two points of intersection (C_2 & C'_2) closer. If we continue to shorten the string, ultimately the two points of intersection will merge into a single point (C), which will make the ellipse tangential to line RQ at (C). And then we can use an interesting property of a tangent to an ellipse.

Property: The tangent to an ellipse is equally inclined to the lines (F_1T & F_2T) joining the point of tangency (T) to the two foci (F_1 & F_2).

Think of the two foci as two gates already found and (T) as the gate of the third wing represented by the tangent.

Conclusion: The sides of the optimized inscribed triangle ABC will be equally inclined to the sides of the outer triangle PQR.

Interestingly, this result can be reached via physics too, using what is called as Fermat's Principle.

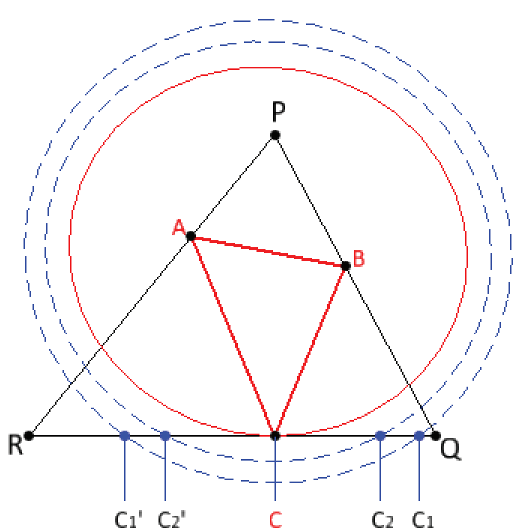

Figure 3

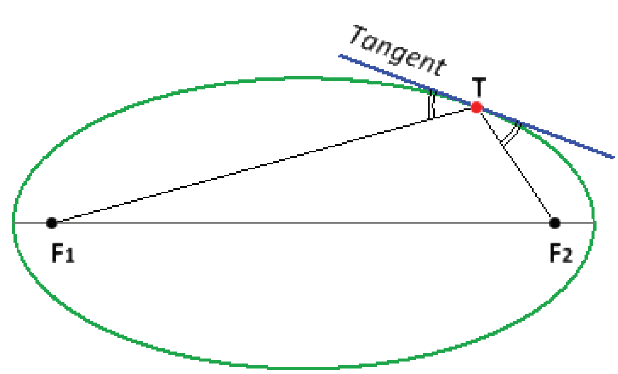

Figure 4

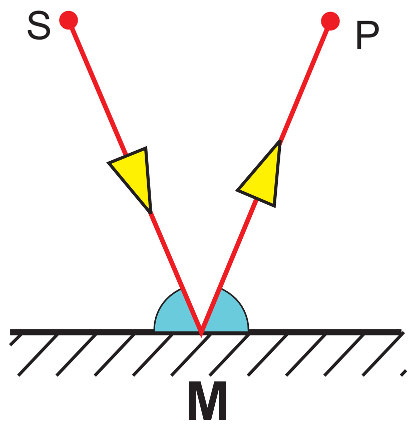

Figure 5

Fermat's Principle: Light always takes the quickest path.

We can apply it to a ray of light coming from a source (S), getting reflected from a plane mirror at (M) and reaching point (P) as shown. From Fermat's principle, S-M-P must be the shortest path from (S) to (P) via the mirror.

We know from basic optics, that the incident ray (SM) and the reflected ray (MP) are equally inclined to the normal and hence also to the mirror – a conclusion reached earlier using geometry!

So we conclude that **all** sides of the optimized inscribed triangle (ABC) must be equally inclined to the sides of the outer triangle (PQR).

In ΔARC, ΔPAB and ΔBCQ (Figures 6 and 7):

$\angle R + \gamma + \alpha = 180^\circ$, $\quad \angle P + \alpha + \beta = 180^\circ$, $\angle Q + \beta + \gamma = 180^\circ$.

Adding the three equations, we get

$(\angle R + \angle P + \angle Q) + 2(\alpha + \beta + \gamma) = 3 \times 180.$

This yields $\alpha + \beta + \gamma = 180^\circ$, and $\angle R = \beta$ $\angle P = \gamma \quad \angle Q = \alpha$.

Thus ΔARC, ΔPAB and ΔBCQ are similar to ΔPQR

Let,

$x, y, z =$ the scale factors of ΔARC, ΔPAB and ΔBCQ relative to ΔPQR respectively.

$p, q, r =$ sides RQ, PR and PQ respectively; then,

$p = xq + zr, \quad q = xp + yr, \quad r = yq + zp.$

Solving simultaneously for x, we get

$x = \dfrac{q^2 + p^2 - r^2}{2pq}$ which is equal to $\cos R$.

Thus,

$RC = RP \cos R.$

This can happen if ΔRPC is a right angled triangle. In other words, (C) is the foot of the altitude from (P).

By symmetry, points (A) and (B) too would lie at the feet of the altitudes from (Q) and (R) respectively.

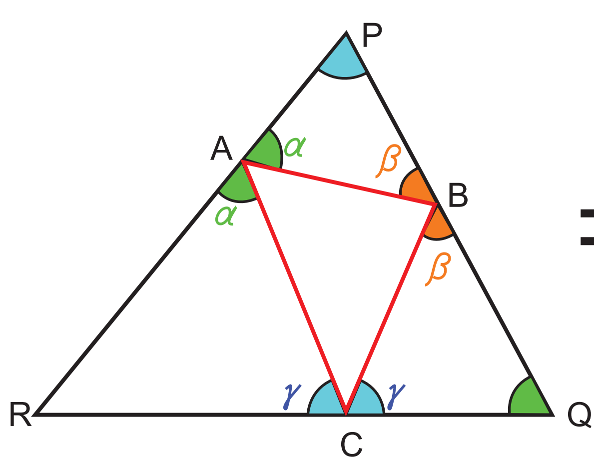

Figure 6

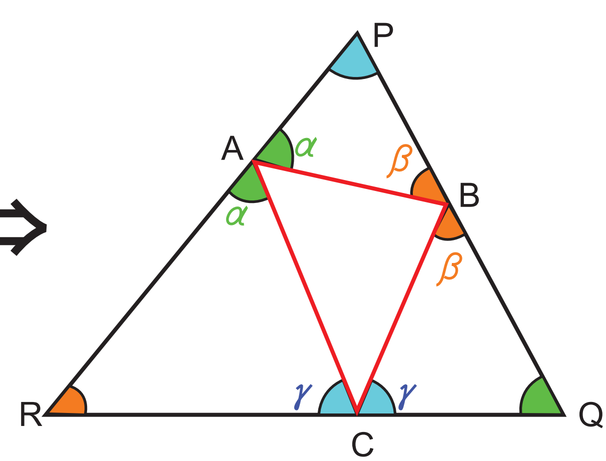

Figure 7

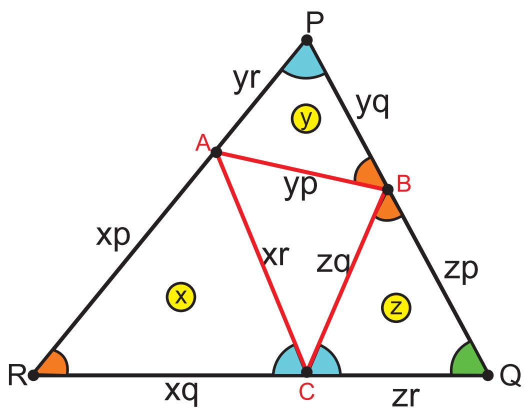

Figure 8

Thus the desired triangle is the **Orthic Triangle** – a triangle with the feet of the three altitudes as its vertices.

Now for the perimeter itself:

Perimeter $= p\cos P + q\cos Q + r\cos R$.

Substituting the given lengths of wings:
$p = 96, q = 64, r = 80,$

$$p \cdot \cos P = 96 \times \frac{64^2 + 80^2 - 96^2}{2(64)(80)} = 12,$$

$$q \cdot \cos Q = 64 \times \frac{80^2 + 96^2 - 64^2}{2(80)(96)} = 48,$$

$$r \cdot \cos R = 80 \times \frac{64^2 + 96^2 - 80^2}{2(64)(96)} = 45.$$

So the minimal perimeter is $45 + 48 + 12 = \mathbf{105}$ units.

Additional note from the author: I have created two videos of the solution and placed them online. Here are their short URLs:

http://tinyurl.com/FagnanoAnalytical

http://tinyurl.com/FagnanoPhysical

Please view them in full screen. Both videos have closed captions for their entire length.

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FAGNANO'S PROBLEM
Addendum

SHAILESH SHIRALI

The problem treated in the accompanying article is this: Given an arbitrary acute-angled triangle PQR, inscribe within it a triangle ABC, with A on side RP, B on side PQ, and C on side QR, having the smallest possible perimeter. The author establishes, using geometrical arguments, that in the optimal configuration, the following triangle similarities must hold (see Figure 1):

$$\triangle ARC \sim \triangle QBC \sim \triangle ABP \sim \triangle QRP,$$

and then shows, using trigonometry, that these conditions force A, B, C to be the feet of the altitudes of the triangle.

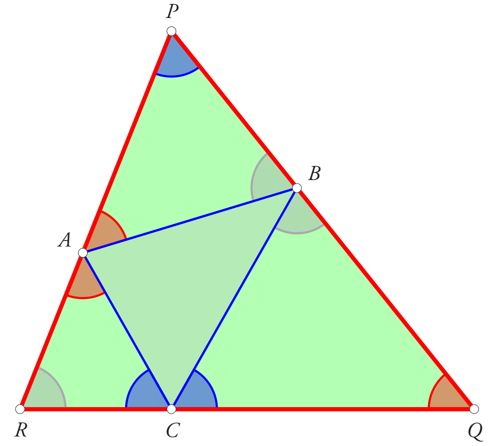
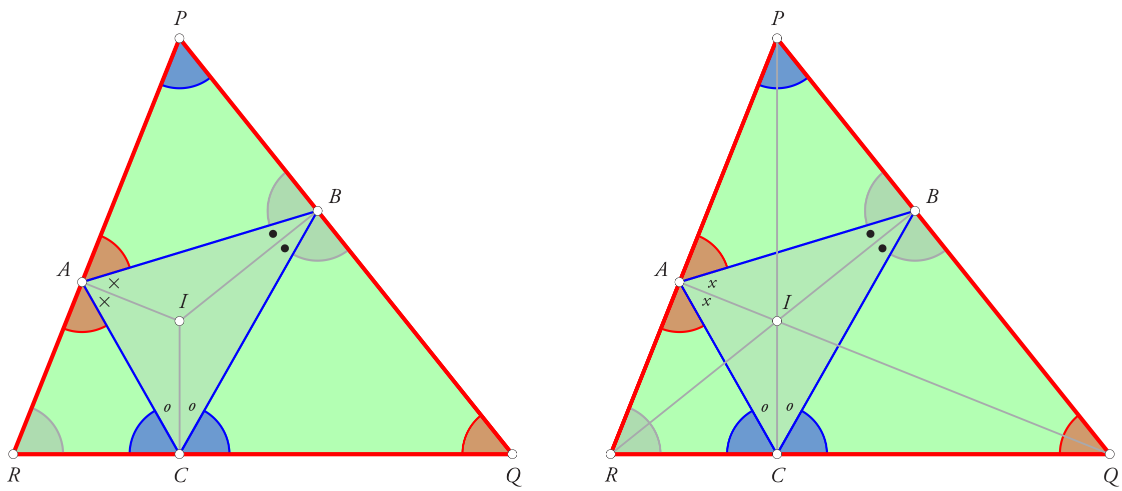


Figure 1. Fagnano's Problem

Here we provide a geometrical proof of this proposition. We also justify the need to impose the condition that triangle PQR should be acute-angled.

Keywords: triangle, acute, obtuse, perpendicular, angle bisector, incentre, excentre, collinear



(a) (b)

Figure 2

Proof of proposition

Construct the internal bisectors of the angles of $\triangle ABC$. The three lines thus constructed meet at the incentre I of $\triangle ABC$; see Figure 2 (a).

It is easy to check, using elementary angle computations, that the sides of $\triangle PQR$ are respectively perpendicular to the three angle bisectors; that is, side QR is perpendicular to the angle bisector CI of $\measuredangle ACB$, and so on. But this implies that the sides of $\triangle PQR$ are respectively the external bisectors of the angles of $\triangle ABC$ (i.e., side QR is the external angle bisector of $\measuredangle ACB$, and so on). This in turn implies that P, Q, R are the ex-centres of $\triangle ABC$. And this in turn implies that P, Q, R lie on the (internal) angle bisectors of $\measuredangle ACB$, $\measuredangle CAB$, $\measuredangle ABC$ respectively. That is, points P, I, C are collinear, as are points Q, I, A and points R, I, B; see Figure 2 (b).

It follows that PC, QA and RB are the altitudes of $\triangle PQR$. This is just what we had set out to prove. $\square$

Why should the triangle be acute angled?

We now justify the need to impose the condition that $\triangle PQR$ should be acute angled. We accomplish this by considering what happens if $\triangle PQR$ is right-angled or obtuse-angled.

Figures 3 (a), 3 (b) and 3 (c) show triangles in each of which the angle at vertex R is successively larger than in Figures 1 and 2; it is getting 'closer' to a right angle, and in the limit, Figure 3 (c), the triangle becomes right-angled at vertex R.

Observe carefully what happens: as $\measuredangle R$ increases, vertices A and C get steadily closer to each other, and in the limit, when the triangle becomes right-angled at vertex R, the two vertices coincide with R. When this happens, $\triangle ABC$ collapses into segment RB. The configuration will now be as depicted in Figure 3 (c). We infer from this that if $\triangle PQR$ is right-angled, then the inscribed triangle with least perimeter is a line segment. (Note that in the limiting situation, segment BR is traced out *twice*, which means that the perimeter of $\triangle ABC$ is twice the length of segment BR.)

It is possible to show directly that if $\triangle PQR$ is right-angled at R and $\triangle DEF$ is inscribed in $\triangle PQR$, then its perimeter cannot be less than twice the length of altitude RB. Let DEF be any inscribed triangle, as in Figure 3 (d). We now perform the following geometrical operations on this figure: we reflect the

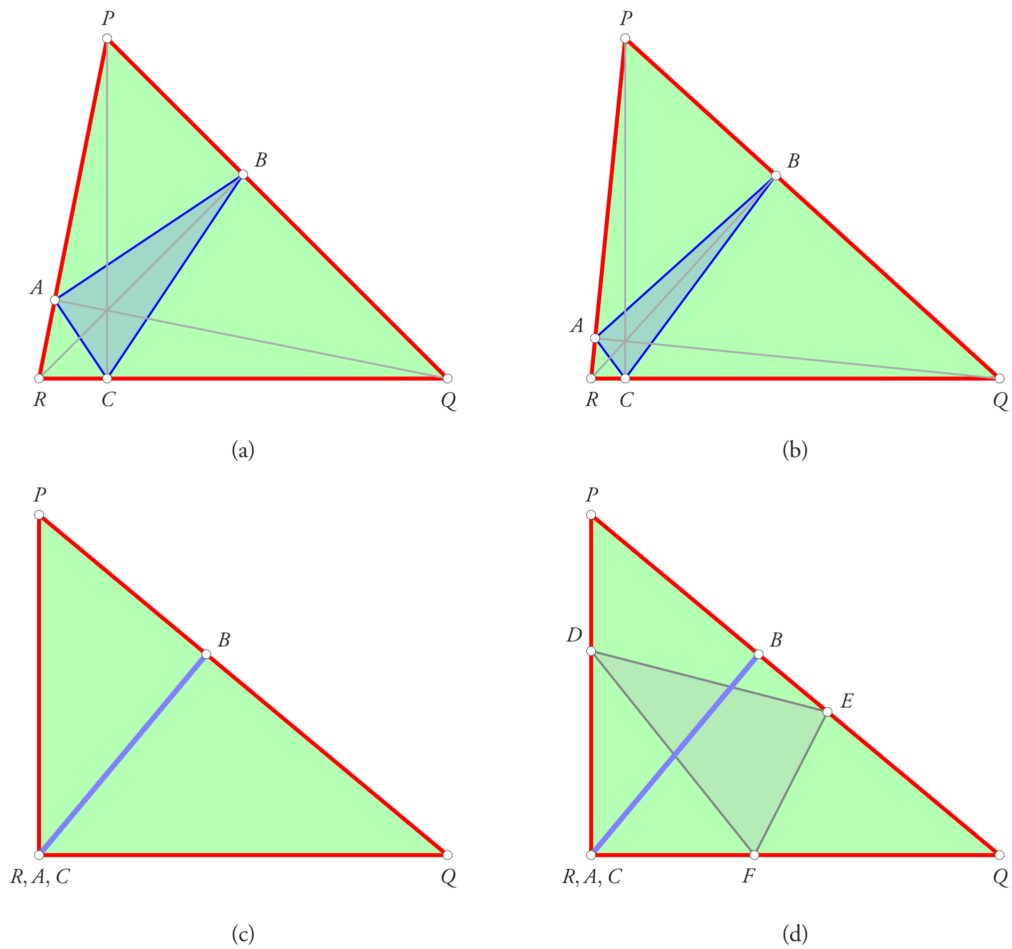


Figure 3

configuration in line QR and again in line PR. The effect is shown in Figure 4; the two mappings take E to points E_1 and E_2 respectively.

We now note the following:

- $E_1F = EF$ and $E_2D = ED$, by the very nature of the reflection operation; hence the perimeter of $\triangle DEF$ is equal to the length of the path E_1FDE_2.
- $\measuredangle ERE_1 = 2\measuredangle ERQ$ and $\measuredangle ERE_2 = 2\measuredangle ERP$, so $\measuredangle E_1RE_2 = 2\measuredangle PRQ = 180^\circ$. That is, points E_1, R, E_2 lie in a straight line.
- The length of path E_1FDE_2 is greater than or equal to the length of segment E_1E_2, i.e., greater than or equal to $2\times$ the length of segment RE. (This follows from several usages of the result that any two sides of triangle are together greater that the third side.) Hence: perimeter of $\triangle DEF \geq 2\times$ the length of segment RE.
- The length of segment RE is greater than or equal to the length of segment RB (because RB is perpendicular to PQ).
- Hence: perimeter of $\triangle DEF \geq 2\times$ the length of segment RB.

The stated claim therefore follows: the optimal inscribed triangle ('optimal' in the sense of having the least possible perimeter) is the degenerate triangle consisting of the segment RB traced twice over. □

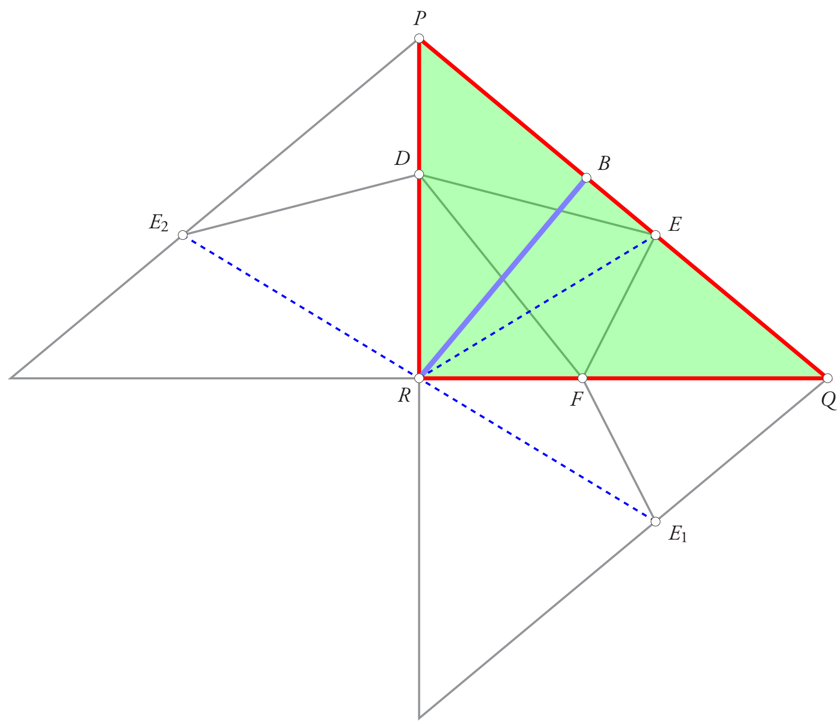


Figure 4

If we continue to increase the size of $\measuredangle PRQ$, then we get a triangle which is obtuse-angled at vertex R. What is the optimal inscribed triangle in this case? It turns out that we cannot do better than opting for the degenerate triangle which consists of the segment RB traced twice over (here, B is the foot of the perpendicular from vertex R to side PQ). We leave the full justification of the statement to you. (Hint: The reflection idea used above will work here as well.) □

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ClassRoom

THE GOLDEN RATIO Unexpectedly

Part 1

MARCUS BIZONY

A **Kepler triangle** is a right-angled triangle whose sides are in Geometric Progression, which requires that its sides are in the ratio $1 : \sqrt{\varphi} : \varphi$ where $\varphi = (1 + \sqrt{5})/2$ is the Golden Ratio.

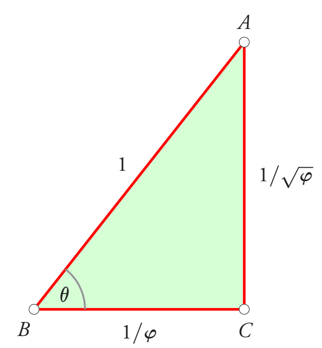


Figure 1

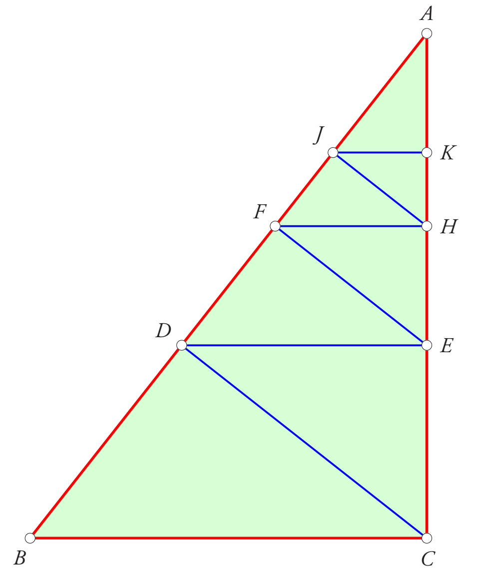
Clearly, if the length of the hypotenuse is 1 (see Figure 1), the sides will have lengths

$$\frac{1}{\varphi}, \quad \frac{1}{\sqrt{\varphi}}, \quad 1.$$

If θ is the larger acute angle of the triangle, then $\tan\theta = \sqrt{\varphi}$. This angle is the acute solution to the equation $\tan^2\theta\,\cos\theta = 1$. (For, since $\varphi^2 = \varphi + 1$, we get $\tan^4\theta = \tan^2\theta + 1$, i.e., $\tan^4\theta = \sec^2\theta$, which yields $\tan^2\theta = \sec\theta$ since θ is acute. Hence $\tan^2\theta\,\cos\theta = 1$.)

For clarity, we note that we are using here the symbol φ to represent the value $1.618\ldots$, which means that $\varphi^2 = \varphi + 1$. The other value which could equally well be called the Golden Ratio is the reciprocal of this number, which is also $\varphi - 1$, and which is here given the symbol G.

Keywords: Kepler triangle, golden ratio, incircle, geometric progression



$AB = 1$

$BC = G$

$AC = G^{1/2}$

$CD = G^{3/2}$ $\quad AD = G$

$DE = G^2$ $\quad AF = G^2$

$EF = G^{5/2}$ $\quad AJ = G^3$

$FH = G^3$

$HJ = G^{7/2}$

$JK = G^4$

Figure 2

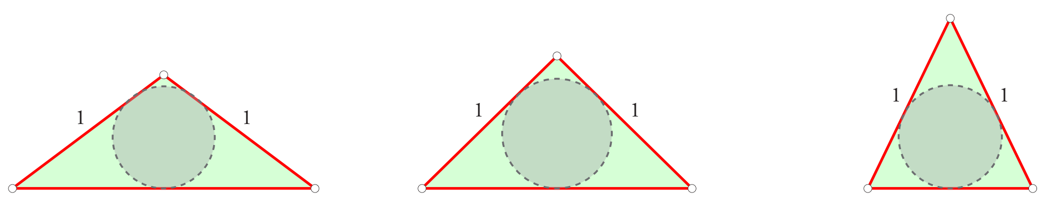


Figure 3

Dropping perpendiculars progressively from the right-angle vertex to the hypotenuse and then back to a leg (see Figure 2) produces lengths which are powers of $G^{1/2}$; continuing the process generates similar triangles, so any length of the form $G^{n/2}$ can be achieved in this manner.

Now consider an isosceles triangle whose equal sides have length 1. Clearly there are lots of possible 'shapes' for it, and presumably therefore different areas for its incircle (see Figure 3). It seems reasonable to ask which of these isosceles triangles has the largest incircle.

An intuitive response might be that the required isosceles triangle is going to be equilateral; certainly if that turned out to be the case, it would fit one's sense of what is 'right'. And, indeed, it is when the isosceles triangle is equilateral that *the largest proportion of its area* is included in its incircle. But if the incircle itself should be as large as possible, we need to make the isosceles triangle not equilateral but in the form of a double Kepler triangle, produced by placing two Kepler triangles alongside each other, with their longer legs coinciding (see Figure 4).

To see why, we need first to understand something about the incentre of a triangle, which is at the same distance r from each of its sides (see Figure 5). The figure shows a triangle and its

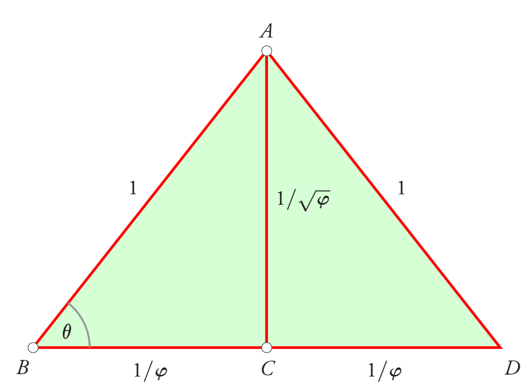


Figure 4

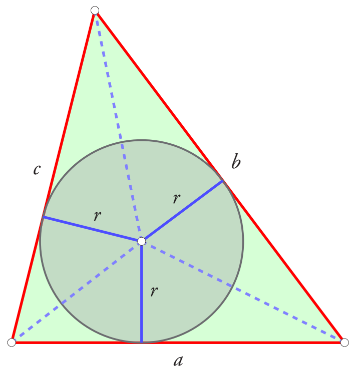


Figure 5

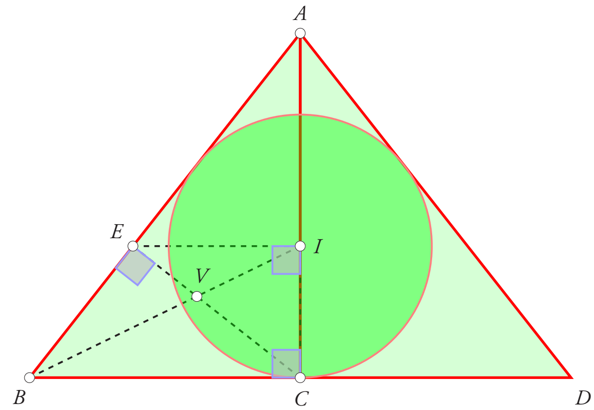


Figure 6

incircle, and also lines connecting the vertices of the triangle to the incentre; they demonstrate how the area of the triangle can be seen as the sum of the areas of three smaller triangles. Of course the radii are perpendicular to the sides and therefore serve as heights on those bases for the smaller triangles.

Thus the area of the whole triangle is $ra/2 + rb/2 + rc/2 = rs$ where s is the semi-perimeter of the triangle, and we deduce that for any triangle the length r of its in-radius is given by the formula

$$r = \frac{\text{area of triangle}}{\text{semi-perimeter of triangle}}.$$

In the case of an isosceles triangle whose equal sides have length 1 and with base angle x, we have:

$$\text{area of triangle} = \cos x \cdot \sin x = \frac{1}{2}\sin 2x,$$

$$\text{perimeter of triangle} = 2 + 2\cos x.$$

Hence the radius of the incircle is given by

$$r = \frac{\sin 2x}{2 + 2\cos x}.$$

Setting the derivative of r equal to zero gives

$$(2 + 2\cos x) 2\cos 2x - \sin 2x(-2\sin x) = 0,$$

$$\therefore\ 4(1 + \cos x)\left(2\cos^2 x - 1\right) + 4\sin^2 x\cos x = 0,$$

$$\therefore\ (1 + \cos x)\left(2\cos^2 x - 1\right) + \cos x\left(1 - \cos^2 x\right) = 0,$$

$$\therefore\ 2\cos^2 x - 1 + \cos x(1 - \cos x) = 0,$$

since $1 + \cos x \neq 0$. The equation in the last line yields

$$\cos^2 x + \cos x - 1 = 0,$$

so that $\cos x = G$. The verification that this indeed yields a maximum value of r is left to the reader.

This particular isosceles triangle ($\triangle ABD$, with $AB : AD : BD = 1 : 1 : 2/\varphi$) has the additional interesting property that if AC is the altitude to its base, then the perpendicular to AC through the incentre I of the triangle meets side AB at the foot E of the perpendicular CE to that side from the midpoint C of the base BD; see Figure 6.

Moreover, I is a Golden Point of AC (i.e., divides it in the Golden Ratio). Further, the point V where BI meets CE is a Golden Point of both CE and BI. Readers might like to tackle these proofs for themselves.

MARCUS BIZONY is now retired, having been Deputy Head (Academic) at Bishops in Cape Town, and before that Head of Mathematics; he taught at Bishops for over thirty years. He was Associate Editor of *Learning and Teaching Mathematics*, a journal published by the Association for Mathematics Educators of South Africa. He was also convenor of the panel that sets the South African Olympiad questions for juniors (Grades 8 and 9). He may be contacted at marcusbizony@gmail.com.

WHEN NUMBERS COMMUNICATE MORE THAN WORDS!!

PERFECT SQUARE AND PERFECT CUBE DATES

Dates often get registered in our brains thanks to the life experiences we encounter: birth, graduation, marriage and death: each incident leaves an imprint on the mind, thus giving the day its true value.

An incident in my life reinforced the importance of numbers and how dates manifest in our thinking process. On a bright Sunday morning, lost in my number world, doodling made me realise that it was not just another day. For the day was 4/9/16 and that to me became a beautiful concoction of mathematical concepts: A perfect square day, $2^2/3^2/4^2$.

The magnitude of my excitement was endless and I felt an urge to communicate my thoughts to my professional Mathematics WhatsApp group. The messages that followed were mesmerising.

The accepted format of writing dates worldwide is DD/MM/YYYY (and its shortened form, DD/MM/YY). My mentor Vaidya Sir wanted to know which would be the nearest 'perfect square day' and how many such days there are in the 21st century. Pages were turned, a maze of calculations done, and the result was mind-blowing:

There are 5 'perfect square dates' between 1 and 31 inclusive: 1^2, 2^2, 3^2, 4^2, 5^2.

There are 3 'perfect square month numbers' between 1 and 12 inclusive: 1^2, 2^2, 3^2.

The nearest upcoming 'perfect square year' in YYYY form is 2025 (or 45^2).

The future may not be known, but to live long enough to witness a mathematically significant day is surely as good as attaining Mathematical Nirvana! Thus, I came up with permutable dates that could achieve the status of 'perfect square day'. These included 5 dates (1, 4, 9, 16, and 25) in each month for three months (January, April and September, i.e., 1, 4 and 9) in the year 2025. As I did my calculations I realised that in each coming century there seemed to be at most one 'perfect square year'. I found that till the 41st century, there are 15 such 'perfect square years' and that such years didn't exist in some centuries.

Vaidya Sir pointed out an interesting aspect of the problem. He asked me to check whether 'square-less' centuries were a random occurrence.

The beauty of numbers is that simple questions can help uncover great facts! The calculations made me conclude that the following (future) centuries did not contain a single 'perfect square year': 35, 41, 45, 48, 52, 55, 58, 61, 63, 66, 69, 71, 74, 76, 78, 80, 83, 85, 87, 89, 91, 93, 95, 97, 99.

This made me indeed feel like the Nostradamus of Mathematics – the king who could predict the future! But in the court of this king, the court jester was not going to be left behind! If the discussion on 'perfect square' was fair and square, then wouldn't the 'perfect cube' feel offended? This led to the next round of deliberations with Vaidya Sir, to calculate the dates in which all three components of the day (dd/mm/yyyy) are 'perfect cubes'. More calculations, more explorations and more magic followed. The very first 'perfect cube day' (dd/mm/yyyy) to come is 1/1/2197. To find the 'perfect cube days' in the year 2197, note that:

The 'Perfect cube dates' between 1 and 31 inclusive are 1^3, 2^3, 3^3;

The 'Perfect cube months' between 1 and 12 inclusive are 1^3, 2^3;

Hence the dates that form the next 'perfect cube days' are: 1/1/2197; 8/1/2197; 27/1/2197; 1/8/2197; 8/8/2197; 27/8/2197.

It is often said that the mystery of the future can be unearthed by the events of the past. So this journey of 'perfect cube days' commenced in the year 1/1/8 followed by 1/1/27 and 1/1/64 (and maybe in the year 1 AD – making it 1/1/1).

If the almighty gives me a chance to be born again, I would choose one and only one date, viz. 'the perfect square and perfect cube day' – 1/1/4096 (4096= 64^2 and 16^3).

I am highly obliged to Vaidya Sir who inspired me to apply various Mathematical concepts to dates making them memorable for a lifetime. He pointed out that the year 4096 is not only 'a perfect square and a perfect cube year' but also a fourth power, sixth power and twelfth power year!

Future aside, let me leave you with a thought provoking fact. If we consider just the dd/mm/yy format, did you realise that the date 25/09/16 is also the first day which is a 'Pythagorean triplet day' ($5^2/3^2/4^2$)?

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DADS Rule!

SNEHA TITUS &
SWATI SIRCAR

We start with a 10×10 grid numbered sequentially and colour the multiples of 11. As you can see, they occur diagonally and, up to 100, the digits repeat.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Figure 1

As we look at multiples of 11 which are greater than 100, the pattern of repeating digits changes. And to find the more general pattern, look at any other diagonal parallel to the colored one in the 10×10 grid. If we take the diagonal starting with 3 say, we notice the numbers 3, 14, 25, 36, 47, 58, 69, 80. It takes only a moment to notice that there is a pattern in the difference between the units digit and the tens digit. In this diagonal, except for the last difference (which is -8), there is a constant difference of 3. If we take the diagonal beginning with 2, we get the differences as 2 and -9 respectively. We can observe that the two integers we get

Keywords: factor, multiple, tables, pattern, differentiated teaching,investigation

11	22	33	44	55	66	77	88	99	110
121	132	143	154	165	176	187	198	209	220
231	242	253	264	275	286	297	308	319	330
341	352	363	374	385	396	407	418	429	440
451	462	473	484	495	506	517	528	539	550
561	572	583	594	605	616	627	638	649	660
671	682	693	704	715	726	737	748	759	770
781	792	803	814	825	836	847	858	869	880
891	902	913	924	935	946	957	968	979	990

Figure 2

as the digit differences for any diagonal are 11 apart from each other. Now notice the difference between the units and tens digits for the diagonal of multiples of 11; we see that this difference is zero for all multiples of 11 less than 100.

This is a good time to take a look at the 3-digit numbers. We get the multiples of 11 as 110, 121, 132, 143 . . . . Interestingly, the sum of the units digit and the hundreds digit less the tens digit is 0 till we hit 209. There the difference is 11. It seems like it is time to zoom in on the multiples of 11 only. To do this, we advise the use of any spreadsheet (we used Excel) to generate these rows of multiples of 11. If necessary a teacher can always take a printout of these in class. There the students can look into the patterns and color numbers with $(U + H) - T = 11$.

Look at Fig. 2. Here we find an interesting triangle of 209, 308, 407 . . . 902 which yields a difference of alternate digit sums of 11 while every other multiple of 11 gives a zero as the difference of the alternate digit sums.

What about 4 digit multiples of 11? Look at Fig. 3. We observe $(U + H) - (T + Th) = 0$ or 11 or -11. At this point, we decided to name this difference DADS (Difference of Alternate Digit Sums). It is interesting to notice how the triangles with DADS 11 start off as fairly large triangles but then shrink to just one number 7909. Similarly the numbers with DADS -11, appear as triangles with the same orientation initially, but then change orientation and seem to wrap across the ends of the grid.

What if we change the number of columns in the hope of finding some better pattern? We tried using 9 columns, i.e., the 1st row become 11 . . . 99, 2nd row 110 . . . 198, etc. Immediately, we were rewarded by a clearer pattern of triangles. See Fig. 4. The DADS 11 triangles shrink and make room for the DADS -11 ones which increase till they cover almost entire rows.

At this point, we would like to move from narrative mode to posing a few questions which follow the Low Floor High Ceiling Pattern.

3971	3982	3993	4004	4015	4026	4037	4048	4059	4070
4081	4092	4103	4114	4125	4136	4147	4158	4169	4180
4191	4202	4213	4224	4235	4246	4257	4268	4279	4290
4301	4312	4323	4334	4345	4356	4367	4378	4389	4400
4411	4422	4433	4444	4455	4466	4477	4488	4499	4510
4521	4532	4543	4554	4565	4576	4587	4598	4609	4620
4631	4642	4653	4664	4675	4686	4697	4708	4719	4730
4741	4752	4763	4774	4785	4796	4807	4818	4829	4840
4851	4862	4873	4884	4895	4906	4917	4928	4939	4950
4961	4972	4983	4994	5005	5016	5027	5038	5049	5060
5071	5082	5093	5104	5115	5126	5137	5148	5159	5170
5181	5192	5203	5214	5225	5236	5247	5258	5269	5280
5291	5302	5313	5324	5335	5346	5357	5368	5379	5390
5401	5412	5423	5434	5445	5456	5467	5478	5489	5500
5511	5522	5533	5544	5555	5566	5577	5588	5599	5610
5621	5632	5643	5654	5665	5676	5687	5698	5709	5720
5731	5742	5753	5764	5775	5786	5797	5808	5819	5830
5841	5852	5863	5874	5885	5896	5907	5918	5929	5940

Figure 3

A brief recap: an activity is chosen which starts by assigning simple age-appropriate tasks which can be attempted by all the students in the classroom. The complexity of the tasks builds up as the activity proceeds so that students are pushed to their limits as they attempt their work. There is enough work for all, but as the level gets higher, fewer students are able to complete the tasks. The point, however, is that all students are engaged and all of them are able to accomplish at least a part of the whole task.

3080	3091	3102	3113	3124	3135	3146	3157	3168
3179	3190	3201	3212	3223	3234	3245	3256	3267
3278	3289	3300	3311	3322	3333	3344	3355	3366
3377	3388	3399	3410	3421	3432	3443	3454	3465
3476	3487	3498	3509	3520	3531	3542	3553	3564
3575	3586	3597	3608	3619	3630	3641	3652	3663
3674	3685	3696	3707	3718	3729	3740	3751	3762
3773	3784	3795	3806	3817	3828	3839	3850	3861
3872	3883	3894	3905	3916	3927	3938	3949	3960
3971	3982	3993	4004	4015	4026	4037	4048	4059
4070	4081	4092	4103	4114	4125	4136	4147	4158
4169	4180	4191	4202	4213	4224	4235	4246	4257
4268	4279	4290	4301	4312	4323	4334	4345	4356
4367	4378	4389	4400	4411	4422	4433	4444	4455
4466	4477	4488	4499	4510	4521	4532	4543	4554
4565	4576	4587	4598	4609	4620	4631	4642	4653
4664	4675	4686	4697	4708	4719	4730	4741	4752
4763	4774	4785	4796	4807	4818	4829	4840	4851
4862	4873	4884	4895	4906	4917	4928	4939	4950

Figure 4

Summing up our findings so far:

1. All 2 digit multiples of 11 have digits repeated in the tens and units place.
2. For 3 digit multiples of 11, the sum of the units digit and the hundreds digit less the tens digit is either 0 or +11.
3. The DADS (Difference of Alternate Digit Sums) is defined as the difference of the sum of the digits in alternate places).
4. For 4 digit multiples of 11, the DADS was 0, +11 or −11.
5. Numbers which gave a particular DADS value appeared in a triangle, clearly visible in a grid with 9 columns.

Questions for Investigation

1. Are there numbers with DADS equal to 22?
2. Which is the smallest number with DADS equal to 22?
3. Are there numbers with DADS equal to -22? Which is the smallest number with DADS equal to -22?
4. What will be the smallest number with DADS equal to $11n$? How many digits will this number have?
5. Find a general formula for the number of digits of the smallest number with DADS equal to $11n$ for a given value of n.
6. For all multiples of 11, will the DADS be a multiple of 11?

Teacher's Notes:

1. Students may construct numbers of the form

   1010101010101010101010101010101010101010101

   to obtain a number with DADS 22. From there it will be a matter of time before they start shortening the number to 20202020202020202020 and seeing if they can get smaller numbers.
2. This is a good chance for students to proceed systematically in an investigation. Following the reasoning in step 1, they obtain the significantly shorter number 909 which has a DADS of 18 (it is not a multiple of 11). To get a DADS of 22, the number will have to be 40909.
3. This is a very interesting variation- following the reasoning in the steps above, we see that 409090 is a number with a DADS of -22. Is this the smallest number? Clearly, if there are non-zero digits in the places alternating with the units place, then the digits in the other places will have to be larger so that the difference remains as -22. Students may notice that the smallest number with a negative DADS will have an even number of digits and the smallest number with a positive DADS will have an odd number of digits. A table recording the smallest number with a particular DADS value will help them make this observation.
4. Going forward, we can ask what will be the number of digits of the smallest number with DADS $11n$ for any natural number. The reasoning is exactly the same as before. E.g. the smallest number with 55 as DADS should have six 9s in every alternate place starting with the units digit and then a 1 in the leading digit ($55 \div 9 = 6$ with remainder 1). It will, therefore be the 13 digit number 1090909090909. So the general formula for the smallest number with a DADS of $11n$ will be as follows: if q and r are natural numbers such that $11n \div 9 = q$ with remainder r (i.e., $r < 9$), then, the number will be

   $$r \times 10^{2q} + 9 \times \left(10^{2(q-1)} + 10^{2(q-2)} + \ldots + 1\right).$$

   It will be a number with $2q + 1$ digits. We call this the DADS Rule!
5. **Proof that if N is a multiple of 11, then DADS is also a multiple of 11, and vice versa:**

   Let us take any number N with $(2n + 1)$ digits as $a_0 + 10a_1 + 100a_2 + \ldots + 10^{2n} \times a_{2n}$

   The alternate digit sums are $a_0 + a_2 + \ldots + a_{2n}$ and $a_1 + a_3 + \ldots + a_{2n-1}$

   And therefore DADS for N is $(a_0 + a_2 + \ldots + a_{2n}) - (a_1 + a_3 + \ldots + a_{2-1})$

   Let us consider N $-$ DADS which is

   $$a_0 + 10a_1 + 100a_2 + \ldots + 10^{2n} \times a_{2n} - [(a_0 + a_2 + \ldots + a_{2n}) - (a_1 + a_3 + \ldots + a_{2n-1})]$$
   $$= a_0 + 10a_1 + 100a_2 + \ldots + 10^{2n} \times a_{2n} - (a_0 + a_2 + \ldots + a_{2n}) + (a_1 + a_3 + \ldots + a_{2n-1})$$

$= 11a_1 + 99a_2 + 1001a_3 + 9999a_4 + \ldots + (10^{2n-1} + 1)a_{2n-1} + (10^{2n} - 1)a_{2n}$

$= \sum_{k=1}^{n} [(10^{2k-1} + 1)a_{2k-1} + (10^{2k} - 1)a_{2k}]$

Now $10^{2k-1} + 1 = (10 + 1)(10^{2k-2} - 10^{2k-3} + \ldots. + 1) = 11b$ for some natural number b, i.e., $10^{2k-1} + 1$ is divisible by 11. [This step of successive decomposition may be easier for students to understand if we use an example say

$$\begin{aligned}
10^5 + 1 &= 10 \cdot 10^4 + 1 = 11 \cdot 10^4 - 1 \cdot 10^4 + 1 \\
&= 11 \cdot 10^4 - 10 \cdot 10^3 + 1 = 11 \cdot 10^4 - 11 \cdot 10^3 + 1 \cdot 10^3 + 1 \\
&= 11 \cdot 10^4 - 11 \cdot 10^3 + 10 \cdot 10^2 + 1 = 11 \cdot 10^4 - 11 \cdot 10^3 + 11 \cdot 10^2 - 10^2 + 1 \\
&= 11 \cdot 10^4 - 11 \cdot 10^3 + 11 \cdot 10^2 - 10 \cdot 10 + 1 \\
&= 11 \cdot 10^4 - 11 \cdot 10^3 + 11 \cdot 10^2 - 11 \cdot 10 + 1 \cdot 10 + 1 \\
&= 11 \cdot 10^4 - 11 \cdot 10^3 + 11 \cdot 10^2 - 11 \cdot 10 + 11 - 1 + 1 = 11(10^4 - 10^3 + 10^2 - 10 + 1)
\end{aligned}$$

which is a multiple of 11.

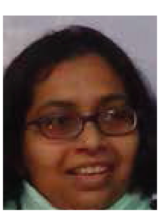
From this step, students may find it easier to generalise. They could also investigate if $10^n + 1$ is a multiple of 11 for all n or only odd n.]

Similarly $10^{2k} - 1 = (10^2)^k - 1 = 100^k - 1 = (100 - 1)(100^{k-1} + \ldots + 1) = 99c$ for some natural number c, $10^{2k} - 1$ is divisible by 99 and hence by 11.

Since N − DADS is divisible by 11, either both N and DADS are divisible by 11 or neither one is; so if DADS is a multiple of 11 so is the original N, and if DADS is not, neither is N.

Conclusion: Mathematical investigations are perfect for Low Floor High Ceiling activities. Here, we have described how a simple pattern can be recognized, investigated, played with and generalized. If your students have enjoyed DADS Rule, do let them try the same strategies with other number patterns; we hope they rule!

And don't forget to share your students' findings with At Right Angles.



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Drawing A SPIRAL OF SQUARE ROOTS

ClassRoom

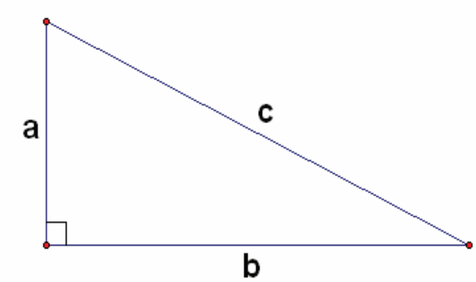
KHUSHBOO AWASTHI

"Mathematics possesses a beauty cold and austere, yet sublimely pure, and capable of a stern perfection such as only the greatest art can show." - Bertrand Russell

Pythagoras theorem has been a perennially interesting topic among mathematicians – both young and old. In this investigation, we will create a spiral of sequential square roots using Pythagoras theorem and will explore several related exciting patterns and relationships. Most of these investigation questions can be taken up by students in middle school and high school alike, who have been introduced to the theorem.

Pythagoras Theorem

In any right angle triangle, the square of the hypotenuse, 'c' is equal to the sum of squares of the base and the opposite side 'a' and 'b'. (See Figure 1)



$$a^2 + b^2 = c^2$$

Figure 1

Steps to create a square root spiral

1. Take an A-4 sized paper and draw a line segment AB of unit length in the middle of the paper.
2. At point B, construct a perpendicular line segment of unit length (same as AB), named BC. The hypotenuse AC will hence be of length $\sqrt{\mathbf{2}}$. (See Figure 2)

   Here, $AC^2 = AB^2 + BC^2 = 2$

   So, $AC = \sqrt{\mathbf{2}}$

Keywords: Investigation, Pythagoras, square roots, spiral, angle

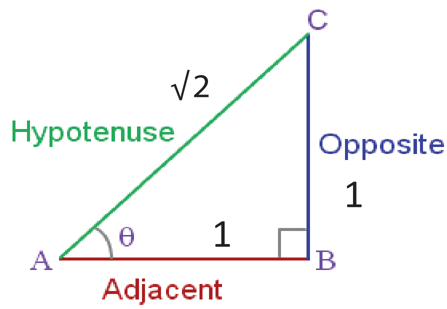


Figure 2

Construct a line at point C perpendicular to segment AC.

Construct line segment CD of unit length (same as AB) on this line.

Then, draw AD to form the hypotenuse with AC as base and CD as the opposite side of the new right-angled triangle. So, $\mathbf{AD} = \sqrt{\mathbf{3}}$. (See Figure 3)

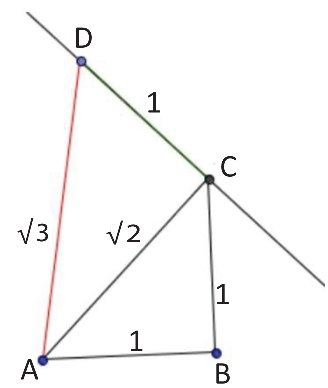


Figure 3

3. Similarly, construct a line segment DE of unit length (same as AB or CD) from point D perpendicular to AD. Join AE to form the hypotenuse of the right-angled triangle with AD as base.

Repeat this process to get more right-angled triangles. The only point to remember is that all the perpendicular opposite sides are of unit length (same as AB).

How many triangles have you drawn? 5, 6,…, 11, 12, 13,…, 16…, n, $n + 1$,…?

Investigation questions

1. As you keep drawing, do you notice any shape emerging?
2. At which iteration (n), does this spiral cross the Y-axis?
3. At which iteration (n), does this spiral cross the X-axis?
4. The fan of the spiral seems to be growing longer. Do you notice the same? Is there a way to prove it?
5. How does the angle of triangle at vertex 'A', θ_n, vary with the iterations?
6. What would be the highest possible value of θ_n?
7. What will happen to the spiral if, at each iteration, we vary the length of the opposite side and make it equal to the base? How will the length of the hypotenuse vary now? And how would the angle of the triangle change?

Teachers' Note:

In the following section, the explanations for the above-mentioned investigation questions have been given. Teachers can use it for their reference. Please ensure that students get ample time to explore these questions on their own.

A. As you keep drawing, do you notice any shape emerging?

Yes. You get a shape like a **spiral** as shown in Figure 4.

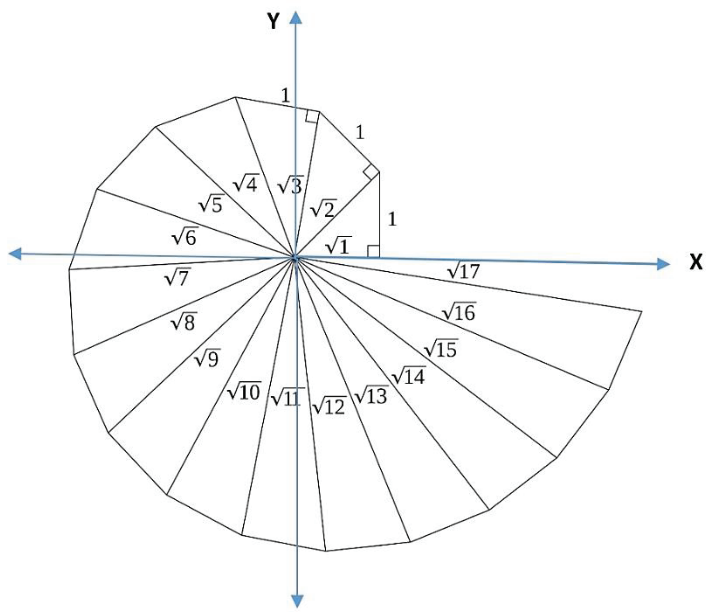


Figure 4

B. At which iteration (n), does this spiral cross the Y-axis?

As seen in Figure 4, the spiral crosses the positive side of the Y-axis at $\boldsymbol{n} = \mathbf{3}$. To see why, note that the sum of the central angles of the spiral is less than 90° at $n = 2$ but greater than 90° at $n = 3$. See Table 2.

When do you think it will cross the negative side of the Y-axis?

Find this out for yourself!

C. At which iteration (n), does this spiral cross the X-axis?

As seen in Figure 4, the spiral crosses the negative side of the X-axis at $\mathbf{n} = \mathbf{6}$. Again, we can confirm this by calculation, the sum of the central angles of the spiral will be less than 180° at $n = 5$ but greater than 180° at $n = 6$.

When do you think it will cross the positive side of the X-axis?

Let us keep drawing. See Figure 5.

D. The fan of the spiral seems to be growing longer. Do you notice the same? Is there a way to prove it?

Remember, when we started, the base and the opposite side of the right angle triangle were of unit length. This made the hypotenuse AC of length $\sqrt{\mathbf{2}}$. (See Figure 2)

Let us record our observations in a tabular format. Enter the length of the base, the opposite side and the hypotenuse. The table is shown below. (See Table 1)

Iteration (n)	**No. of Triangle (n)**	**Base (b)**	**Opposite side (a)**	**Hypotenuse (c)**
1	1	1	1	$\sqrt{2}$
2	2	$\sqrt{2}$	1	$\sqrt{3}$
3	3	$\sqrt{3}$	1	…..
4	4	…...	1	…..
5	5	…...	1	…..
N	n	…...	1	…..

Table 1. Length of base, opposite side and hypotenuse

Looking at the table, it is clear that the opposite side remains of unit length. Only the base and the hypotenuse change. For the n^{th} iteration, what will be the length of the base and the hypotenuse? Can you derive that?

From the tabulated data in **Table 1**, it is clear that for the n^{th}iteration,

Base $= \sqrt{n}$

Hypotenuse $= \sqrt{n+1}$

Thus, for (n+1)$^{\text{th}}$ iteration,

Base $= \sqrt{n+1}$ and

Hypotenuse $= \sqrt{n+2}$

Clearly if $n > 0$,

$\sqrt{n+2} > \sqrt{n+1}$.

It follows that the fan of the spiral (hypotenuse) gets longer with each iteration.

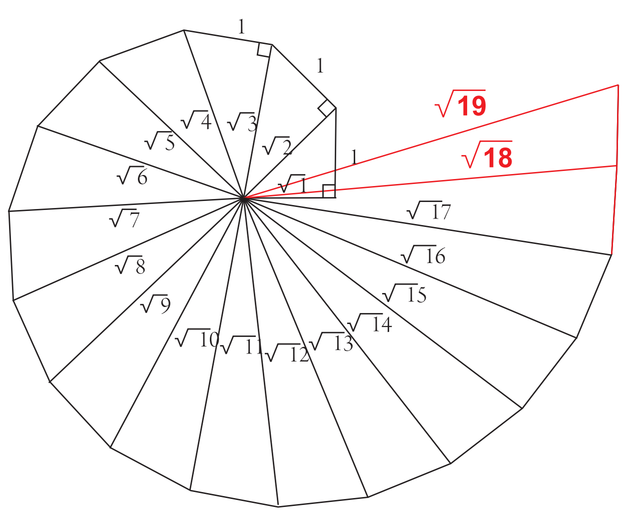


Figure 5

Since we are working with triangles, how can one ignore focusing on the angles?

We have established that with each successive iteration the hypotenuse increases. Can we say the same about the angle, θ_n (the angle of the triangle at the vertex A)? Let us investigate.

E. How does the angle $\boldsymbol{\theta_n}$ of triangle at vertex A, vary with the iterations?

One possible way is to measure the angle $\boldsymbol{\theta_n}$ using a protractor and tabulate the findings as shown in **Table 2.** It would show that the angle steadily decreases.

No. of Triangle (n)	Angle, $\boldsymbol{\theta_n}$
1	45^0
2	……
…..	……

Table 2. Angle of the triangle at each iteration, n

Is there another way to find how the angles vary?

In a right-angled triangle, as the length of base becomes steadily larger, the angle formed by the base and the hypotenuse, $\boldsymbol{\theta_n}$**,** keeps gets steadily smaller. See Figure 6.

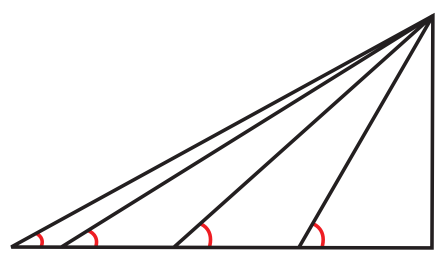


Figure 6

F. What is the largest possible value of $\boldsymbol{\theta_n}$?

Clearly, if the angle decreases with each iteration, the greatest angle is at $n = 1$.

From **Table 2**, at $n = 1$, $\boldsymbol{\theta_1 = 45^0}$**.**

This implies that $\boldsymbol{45^0 \geq \theta_n > 0^0}$**.**

For students in high school, who have studied **trigonometric ratios**, it will be interesting to observe how Sin $\boldsymbol{\theta_n}$ **and** Cos $\boldsymbol{\theta_n}$ vary with each iteration.

We know,

$$\text{Sin } \boldsymbol{\theta_n} = \frac{\text{(Length of opposite side)}}{\text{(Length of hypotenuse)}}$$

From **Table 1,** we know that with each iteration, the length of the opposite side remains equal to 1, but length of the hypotenuse keeps on increasing. Thus, the denominator keeps becoming greater than the numerator. That implies that the value of Sin $\boldsymbol{\theta_n}$ will keep on getting smaller with increase in n.

As shown in **Table 3**,

At $n = 1$, Sin $\boldsymbol{\theta_n} = 1/\sqrt{2}$

i.e. $\boldsymbol{\theta_n = 45^0}$

No. of Triangle (n)	Opposite Side	Base	Hypotenuse	Sin θ_n	Cos θ_n
1	1	1	$\sqrt{2}$	$1/\sqrt{2}$	$1/\sqrt{2}$
2	1	$\sqrt{2}$	$\sqrt{3}$	$1/\sqrt{3}$	$\sqrt{2}/\sqrt{3}$
3	1	$\sqrt{3}$	…..	…..	……
4	…...	……	…..	…..	……
5	…...	……	…..	…..	……

Table 3. Length of opposite side, base, hypotenuse and trigonometric ratios at each iteration

G. What will happen to the spiral if, at each iteration, we vary the length of the opposite side and make it equal to the base?

If at each iteration the length of opposite side is made equal to that of the base, it will become an isosceles right-angled triangle and hence, the angle of the triangle will be always equal to 45^0.

What happens to the length of the hypotenuse, now that the length of the base and the opposite side remains the same? Will the hypotenuse grow longer as in the previous case?

Let's draw a table similar to Table 1, capturing how the hypotenuse varies with the iteration. See to Table 4. It shows that the hypotenuse continues to become longer with every iteration.

Iteration (n)	No. of Triangle (n)	Base (b)	Opposite side (a)	Hypotenuse (c)	Angle, θ_n
1	1	1	1	$\sqrt{2}$	45^0
2	2	$\sqrt{2}$	$\sqrt{2}$	$\sqrt{4}$	45^0
3	3	$\sqrt{4}$	$\sqrt{4}$	$\sqrt{8}$	45^0
4	4	$\sqrt{8}$	$\sqrt{8}$	$\sqrt{16}$	45^0
5	5	$\sqrt{16}$	$\sqrt{16}$	$\sqrt{32}$	45^0
6	6	$\sqrt{32}$	$\sqrt{32}$	….	45^0
n	n	$\sqrt{2}\hat{}(n-1)$	$\sqrt{2}\hat{}(n-1)$	$\sqrt{2}\hat{}(n)$	45^0

Table 4. Length of base, opposite side and hypotenuse

In this case, it is also to be noted that the hypotenuse of the triangle at even iterations will overlap with the axes. (Refer *Figure7*)

This can also be established from Table 4 that shows that the angle of the triangle, $\boldsymbol{\theta_n}$, is always equal to 45^0. Hence the sum of vertex angles of triangles of any two consecutive iterations (n and $n+1$) will be 90^0.

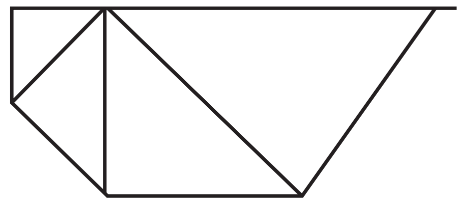


Figure 7

Since at $n = 1$, $\boldsymbol{\theta_n}$=45^0, the hypotenuse of the triangle at even iterations will overlap either with the X-axis or with the Y-axis.

Another interesting observation is that at $n = 8$, sum of angles of triangles at the vertex is equal to 360^0. Thus, the spiral will close.

Will the spiral in the previous case close too? Can you prove it?

It will also be interesting to note the patterns in the length of the hypotenuse and in the angle of the triangles if more variations of this investigation are tried by varying the length of the opposite sides in different manners. For instance,

i. Length of opposite side to be made double the length of the opposite side from previous iteration.

ii. Length of opposite side to be made double the length of the base / adjacent side.

iii. Length of opposite side to be kept equal to the number of iterations, i.e., 1, 2, 3, 4 and so on.

Keep investigating..!

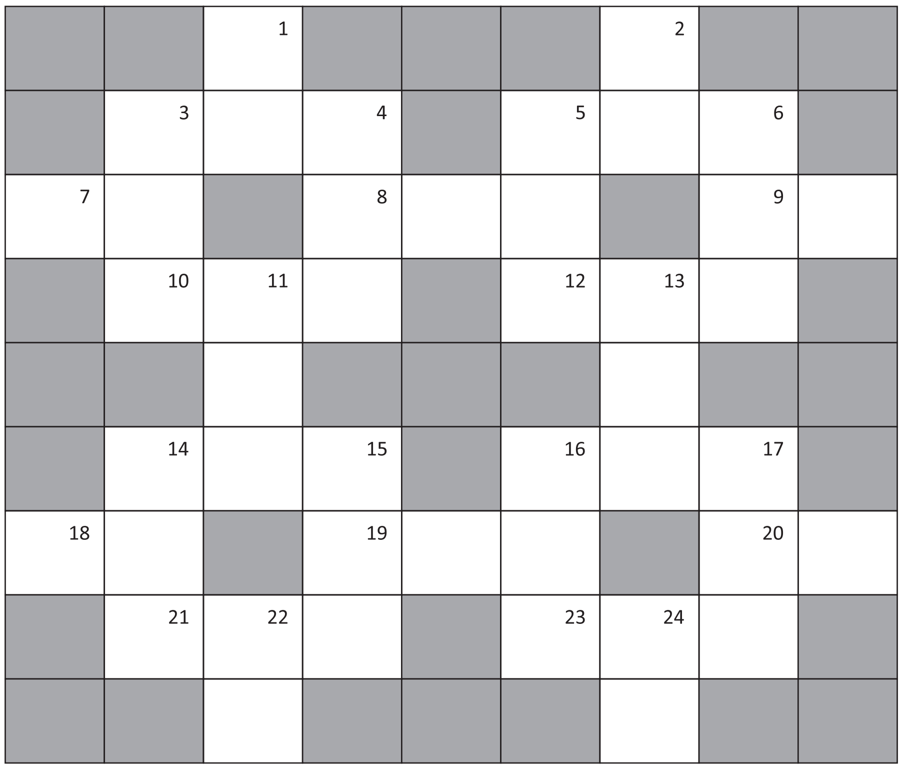
KHUSHBOO is the co-founder and Director at MANTRA Social Services (aka Mantra4Change)- a Bengaluru-based NGO that works to improve the quality of education for the under-served. A management graduate from Tata Institute of Social Sciences (TISS), she has over 8 years of work experience across diverse domains of technology, healthcare, education, product management, business development and stakeholder management. With her newly discovered interest in school education, Khushboo spends her time exploring ways how meaningful learning opportunities can be created in classrooms.

NUMBER CROSSWORD

Solution on Page 75

Solution on Page 75

D.D. Karopady & Sneha Titus



CLUES ACROSS

3: Both a perfect square and a perfect cube
5: 19 A – 14 A
7: A prime between 30 and 40
8: Square of a number minus the number
9: One score
10: Six factorial
12: Multiple of 7
14: Number of days in 2020
16: New currency note divided by 4 plus 6
18: Number base for Babylonian counting system
19: Model number of a Boeing plane
20: Unlucky number
21: Repeated digits
23: Last 3 digit prime

CLUES DOWN

1: Square root of 3 A reversed
2: A perfect number
3: 223 less than a millennium
4: Half past nine
5: Has more than two prime factors
6: Digits in geometric progression
11: A perfect cube
13: Old currency note in a new avatar
14: 14A – 20 A times 5
15: Permutation of 17D
16: Digits in arithmetic progression
17: Sum of first two digits gives the third
22: Square root of a gross
24: 20 A times 7

A simpler way to Bisect an Angle

C⊗MαC

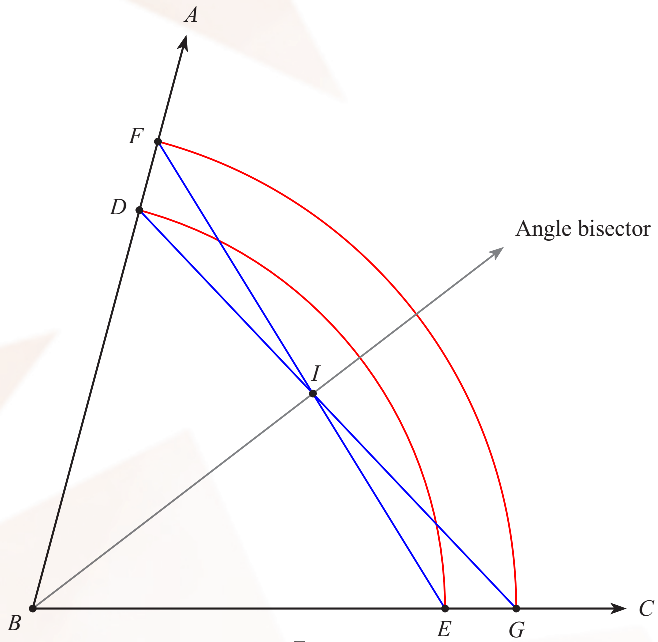


Figure 1

Angle bisection using ruler and compass is part of the standard geometry syllabus at the upper primary level. There is a standard procedure for doing the job, and it is so simple that one would be hard put to think of an alternative to it that is just as simple, if not simpler. But here is such a procedure, announced in a Twitter post [1].

It can be depicted using practically no words. In Figure 1, the angle to be bisected is $\measuredangle ABC$. Draw two arcs DE and FG as shown, centred at B. Next, draw the segments DG and FE; let them intersect at I. Draw the ray BI. This is the required angle bisector.

References

1. Solve my maths (Twitter), *A simpler way to bisect an angle using a compass*; https://twitter. com/solvemymaths/status/776842316938113024?s=03

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ClassRoom

3 4 5...
STRIKES AGAIN!

𝒞⊗ℳα𝒞

A few days back, I came across the following problem on Dan Meyer's blog **dy/dan** (entry dated October 27, 2016: "How I'm Learning to Step into Math Problems"; I have restated the problem in my own words): *In the figure shown* (Figure 1), *the circle touches the base BC of the square and passes through the two upper vertices, A and D. Find the ratio of the radius of the circle to the side of the square.*

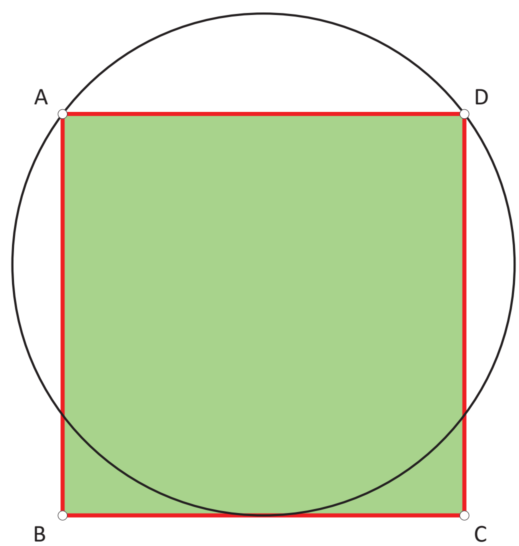

Figure 1

A related problem: *Find a way to construct such a figure*. (That is: given the square, show how to construct the circle; or: given the circle, show how to construct the square.)

Keywords: Pythagoras, problem solving, construction, circle, square

Solution. Denote the radius of the circle by r, and the side of the square by $2a$ (the '2' is only to avoid unnecessary fractions). See Figure 2.

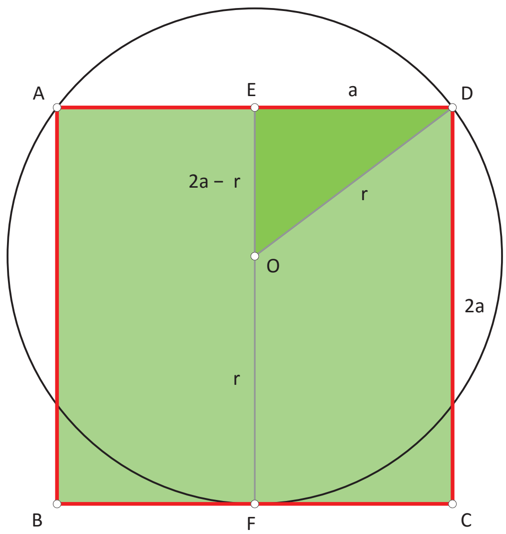

Figure 2

Let O be the centre of the circle, and let EF be the midline of the square, connecting midpoints E and F of AD and BC. (Note that O lies on EF. Why should this be so? In other words: *Given only the information that the circle touches the base BC and passes through the upper two vertices A and D, can we conclude that the midline EF of the square will be an axis of symmetry of the figure?* We leave this question for the reader.) Then we have $OF = r$, $OE = 2a-r$, $DE = a$, $OD = r$. Hence from ΔODE we get, using Pythagoras theorem:

$$a^2 + (2a - r)^2 = r^2, \quad \therefore\ 4ar = 5a^2, \quad \therefore\ \frac{a}{r} = \frac{4}{5}.$$

It follows that

$$2a - r : a : r = 3 : 4 : 5,$$

i.e., ΔODE is a 3–4–5 triangle!

Once we have deduced this, the construction procedure becomes clear. All we need to use is the fact that $EO : OF = 3 : 5$ and $FC : r = 4 : 5$.

Remark. A natural extension to the question studied above is obtained by replacing the word ***square*** throughout by ***regular hexagon***. The configuration is depicted in Figure 3. A similar question can now be asked: *Compute the ratio of the radius of the circle to the side of the hexagon.* We leave this question too for the reader to solve. Likewise for more extensions of this kind.

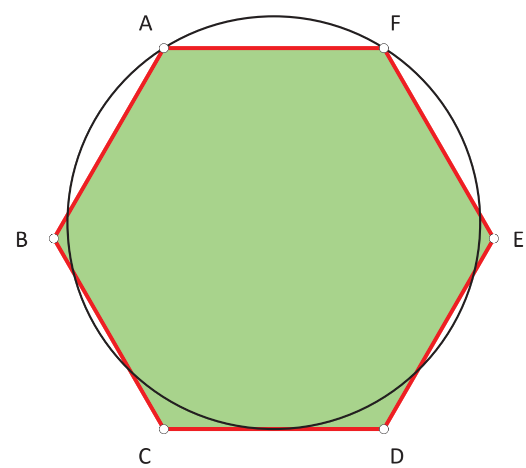

Figure 3

Pedagogical note

These are excellent GeoGebra exercises for students helping them to develop and practise skills of visualisation, logical sequencing, making connections and recalling theory.

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ClassRoom

CONGRUENCY AND CONSTRUCTIBILITY IN TRIANGLES

A. RAMACHANDRAN

Every criterion of congruency in triangles is linked to a method of constructing a unique triangle from given data, using ruler, compass and protractor. Typically we first draw a 'base' and then draw rays or arcs from its ends fitting the given data. The point of intersection of these is the third vertex of the required triangle.

For instance, take the 'SSS' rule. If we are given the three side lengths in a triangle we can construct a unique triangle with those side lengths. Of course, they have to comply with the basic existence requirements for a triangle – that the sum of any two sides exceeds the third side, and that the difference of any two sides is less than the third side.

Let us see what happens if the first condition is not fulfilled – i.e., taking a, b, c to be the sides, let us assume that $b + c < a$. Draw a base of length a. Taking its two ends to be the centres, draw circles of radii b and c, respectively. Note that the circles do not intersect at all, so that no third vertex can be located (Figure. 1). If $b + c = a$, the circles just touch each other and the vertices of the triangle are collinear. We get a flat triangle, with angles of 180°, 0° and 0° respectively. Triangles of this kind are called "degenerate" by mathematicians.

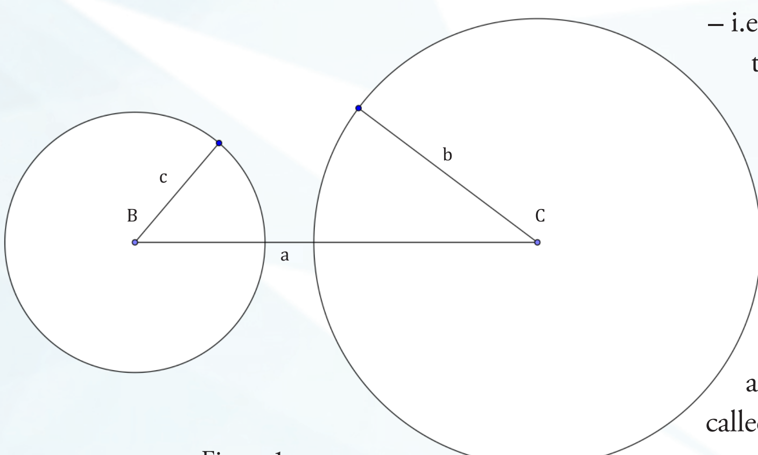

Figure 1

Keywords: Constructibility, congruency, triangle, inequality

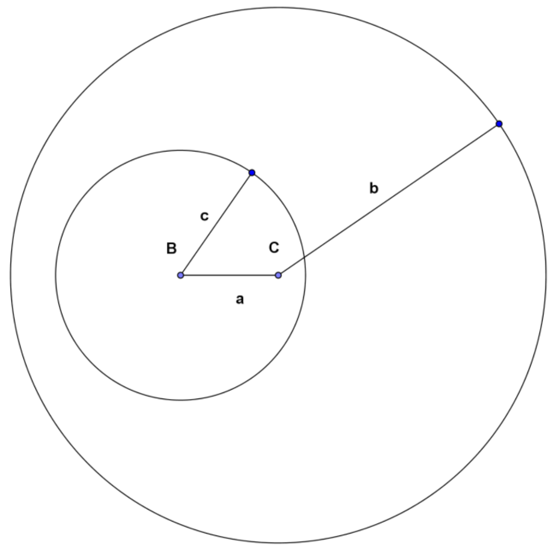

Figure 2

What happens if the second condition is not met? Again, let us take a, b, c to be the sides, and assume that $b - c > a$. As before, draw a base of length a, and taking its ends as centres draw circles of radii b and c. Now one of the circles is fully inside the other and so no third vertex can be located (Figure. 2). If $b - c = a$, the circles touch each other internally, yielding a third vertex that is collinear with the other two. This possibility too yields a degenerate triangle.

In the case of the 'SAS' rule, there is no restriction on the data except that the given angle must be less than 180°. In the case of the 'ASA' rule too, the only restriction is that the given angles add up to less than 180°. Of course it must be specified to which angle the given side is opposite.

The RHS rule is similar to the SAS rule in that two sides and an angle are given. However the given angle is not the angle included between the given sides, and it is specified to be a right angle. Also, the hypotenuse needs to be longer than the other given side. Other than these, no restrictions need to be placed on the data.

Can there be other such rules of congruency/constructibility where the restriction on the angle being 'included' is relaxed, but other conditions are imposed?

Given the following data, it is possible to construct a unique triangle.

In ΔABC, $\measuredangle A = x > 90°$, $AB = c\ cm$, $BC = a\ cm$, $a > c$.

We first draw a line segment AB of length c. We then draw a ray from A making angle x with AB. Keeping the point of a compass opened to radius a on B, we draw an arc to cut the above ray at C (Figure. 3).

Note that the arc would also cut ray CA extended backwards to yield another triangle, but then $\measuredangle$ A would not be obtuse.

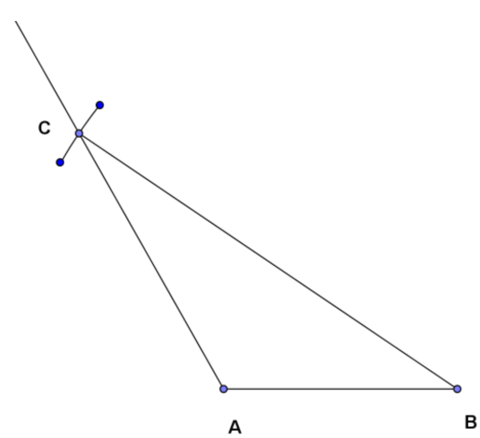

Figure 3

Given the following data, it is possible to construct a unique triangle.

In ΔPQR, $\measuredangle P = x < 90°$, $PQ = r\ cm$, $QR = p\ cm$, $p \geq r$.

We first draw a line segment PQ of length r. We then draw a ray from P making an angle x with PQ. Keeping the point of a compass opened to radius p on Q, we draw an arc to cut the above ray at R (Figure. 4).

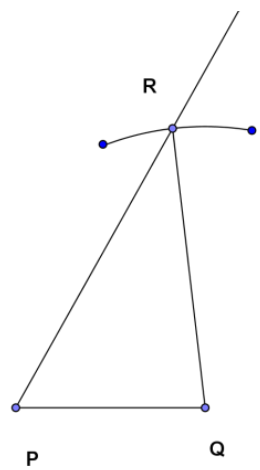

Figure 4

Two cases arise:

- $p = r$: In this case, the arc from Q cuts the ray from P at point R, yielding an isosceles triangle;
- $p > r$: In this case, the arc from Q cuts the ray from P at a point R 'further downstream'. It would also cut the ray extended backwards to yield another triangle, but then $\measuredangle$P would not be acute. Note that the resulting triangle need not be an acute angled triangle. The angle formed at Q could be acute, right or obtuse, depending on the data given.

Though the case $p < r$ is not under consideration here, we could explore what happens in such a case. There are three possibilities. If $p > r \sin x$, the arc cuts the ray at two places, which means there are two different triangles fitting the given data. If $p < r \sin x$, the arc does not cut the ray at all. If $p = r \sin x$, the arc is tangent to the ray, with a single point of contact.

Note that in both the above situations, we are given two side lengths of a triangle and the magnitude of an angle not included between those sides, but additional constraints have been imposed. We are able to construct unique triangles in both cases.

The above deliberations suggest two congruency situations supplementary to the commonly encountered ones. The first one could be called the "OLA" rule (O – obtuse angle, L – longest side, A – adjacent side). The second one could be called the "AAELO" rule (A – acute angle, A – adjacent side, ELO – equal or longer opposite side).

Alternatively, the two rules suggested above and the RHS rule could be absorbed into a single generalisation: an "AALO" rule [A – angle (which could be acute, right or obtuse), A – adjacent side, LO – longer opposite side (i.e., longer than the given adjacent side)].

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INEQUALITIES in Algebra and Geometry Part 1

SHAILESH SHIRALI

This article is the first in a series dealing with inequalities. We shall show that in the world of algebra as well as the worlds of geometry and trigonometry, there are numerous inequalities of interest which can be proved in ways that are easy as well as instructive.

At first encounter, inequalities tend to unsettle students. Possibly this is because they have gotten used to studying equalities and exact relations. Now all of a sudden they are faced with approximations and inexact relations. Moreover, there is no longer a unique, correct answer to a question! This fact alone can seriously unsettle many students.

Periodically, in our study of inequalities, we will come across implications that are quite counterintuitive; indeed, quite paradoxical. These may be regarded as a bonus.

Preliminaries: general facts about inequalities

(1) For any two real numbers a, b, precisely one of the following statements is true:

$$a < b; \qquad a = b; \qquad b < a.$$

This is expressed by saying: *The set $\mathbb{R}$ of real numbers is **totally ordered**.*

Remark. No such statement can be made about the set $\mathbb{C}$ of complex numbers. For example, the numbers 1 and i are non-comparable; no order relation can be placed between them. Note: This is a matter which greatly puzzles students. They ask: *Why cannot we declare that $1 < i$ or that $i < 1$? What would go wrong if we choose to do this?* This is a nontrivial question. We will take it up for study later.

(2) If x, y are real numbers, the following statements are true:

$$x > 0 \text{ and } y > 0 \implies x + y > 0,$$
$$x < 0 \text{ and } y < 0 \implies x + y < 0,$$

and:

$$x > 0 \text{ and } y > 0 \implies xy > 0,$$
$$x > 0 \text{ and } y < 0 \implies xy < 0,$$
$$x < 0 \text{ and } y < 0 \implies xy > 0.$$

Keywords: inequalities, positive, negative, generalisation

(3) If x, y, k are real numbers, the following statements are true:

$$x < y \quad \text{and} \quad k > 0 \quad \Longrightarrow \quad kx < ky,$$
$$x < y \quad \text{and} \quad k < 0 \quad \Longrightarrow \quad kx > ky.$$

We similarly have the rules which govern the taking of reciprocals:

$$0 < x < y \quad \Longrightarrow \quad 0 < \frac{1}{y} < \frac{1}{x},$$
$$x < y < 0 \quad \Longrightarrow \quad \frac{1}{y} < \frac{1}{x} < 0.$$

For example we have: $0 < 3 < 5$ and $0 < 1/5 < 1/3$; and $-5 < -3 < 0$ and $-1/3 < -1/5 < 0$.

(4) Lastly, we note that the inequality relation has the property of ***transitivity***. This means that if a, b, c are real numbers such that $a < b$ and $b < c$, then we also have $a < c$.

This may seem obvious and innocuous, but in fact the property of transitivity makes its presence felt quite often. Here is how it happens. Say we want to prove the inequality $a < b$ for some two quantities a, b; but the nature of the expressions involved makes it difficult to proceed. In such cases an often-used strategy is to look for a third quantity c such that the two inequalities

$$a < c \quad \text{and} \quad c < b$$

are both true and both easily proved. Or we may look for two other quantities c, d such that the separate inequalities

$$a < c, \quad c < d, \quad d < b$$

are all true and all readily proved. Once these steps have been accomplished, the desired inequality follows immediately. The challenge in this case is to find good choices for the quantities c and d.

Remark. Observe that no rules have been set down governing *subtraction*. Students sometimes imagine that the following is true: if $a < b$ and $c < d$, then $a - c < b - d$ (or some variant of this). But it is very easy to find counterexamples to this claim (it would be an interesting task to set the students: find your own counterexamples). Here is one such: $7 < 8$ and $3 < 5$, but $7 - 3$ is not less than $8 - 5$. Similarly, no rules have been set down governing *division*.

On the other hand, the following *is* true and often made use of: If $a < b$ and $c < d$, then $a - d < b - c$. See if you can justify it for yourself, drawing on the basic facts listed above.

Theorems and Problems

The most fundamental fact about inequalities is Theorem 1 (below) which may be said to underpin the entire theory of inequalities! We will see this fact at work repeatedly in the next few pages. Two other results of importance are Theorems 2 and 3.

Theorem 1 (Fundamental fact about inequalities)**.** ***The square of any real number is non-negative.*** *That is, if x is any real number, then $x^2 \geq 0$; equality holds in this relation precisely when $x = 0$.*

Theorem 2. *Let a be a positive real number. Then:*

- *If $a > 1$, then $a < a^2 < a^3 < a^4 < \cdots$.*
- *If $a < 1$, then $a > a^2 > a^3 > a^4 > \cdots$.*

Theorem 3. *Let a, b be positive real numbers with $a < b$. Then:*

$$a^2 < b^2, \quad a^3 < b^3, \quad a^4 < b^4, \quad a^5 < b^5, \quad \ldots,$$

and

$$\frac{1}{a} > \frac{1}{b}, \quad \frac{1}{a^2} > \frac{1}{b^2}, \quad \frac{1}{a^3} > \frac{1}{b^3}, \quad \frac{1}{a^4} > \frac{1}{b^4}, \quad \ldots.$$

We will leave it to you to prove Theorems 2 and 3, by drawing upon facts stated earlier.

In connection with Theorem 2, the following additional statement can be made.

Theorem 4. *Let a be a positive real number. Then:*

- *If $a > 1$, then the sequence $1, a, a^2, a^3, a^4, \ldots$ diverges to infinity.*
- *If $a < 1$, then the sequence $1, a, a^2, a^3, a^4, \ldots$ converges to 0.*

For now, we will not have occasion to use this result.

Isoperimetric property of the square

As an example of how problems in this topic are handled, we study an extremely well-known problem with a geometrical flavour. Specifically, we prove the following:

Theorem 5. *Among all rectangles sharing the same perimeter, the square has the largest area. Among all rectangles sharing the same area, the square has the least perimeter.*

Proof. Consider a rectangle with sides a and b. Let p be its perimeter, and k its area. Then we have:

$$p = 2(a+b), \quad k = ab.$$

Hence $\dfrac{p^2}{4} = a^2 + 2ab + b^2$, which yields:

$$\frac{p^2}{4} - 4k = a^2 - 2ab + b^2,$$

$$\text{i.e., } \frac{p^2}{4} - 4k = (a-b)^2.$$

The desired results can be deduced from the last line. Thus we have:

$$4k = \frac{p^2}{4} - (a-b)^2 \leq \frac{p^2}{4},$$

and equality holds precisely when $a = b$; hence if p is fixed, then k assumes its largest value when $a = b$, i.e., when the figure is a square. Also:

$$\frac{p^2}{4} = 4k + (a-b)^2 \geq 4k,$$

and equality holds precisely when $a = b$; hence if k is fixed, then p assumes its least value when $a = b$, i.e., when the figure is a square. $\square$

Observe the effective use made of the fact that the square of any real number is non-negative.

Problems

We now pose a few problems for the reader to work on and if possible solve fully. They can all be solved using the principles listed above. Later in this article we shall present our solutions for your study.

(1) Which is larger: $2^{1/2}$ or 1.5?

(2) Is the square root of 2 closer to 1.4 or to 1.5? (Base your answer to this question on elementary mathematical arguments, and *not* on your existing knowledge of the value of $\sqrt{2}$.)

(3) Is the square root of 3 closer to 1.7 or to 1.75?

(4) Which is larger: $2^{1/2}$ or $3^{1/3}$ (also written as $\sqrt{2}$ and $\sqrt[3]{3}$)?

(5) Which is the largest among the following quantities:

$$\sqrt{1}+\sqrt{19}, \quad \sqrt{2}+\sqrt{18}, \quad \sqrt{3}+\sqrt{17}, \quad \ldots, \quad \sqrt{9}+\sqrt{11}, \quad \sqrt{10}+\sqrt{10}\,?$$

Solutions

(1) Which is larger, $2^{1/2}$ or 1.5?

We use the following: if $a, b > 0$, then $a > b \iff a^2 > b^2$. The squares of the two given numbers are 2 and 2.25 respectively, and 2.25 is obviously the larger quantity. Hence $1.5 > \sqrt{2}$.

(2) Is the square root of 2 closer to 1.4 or to 1.5?

We do away with the decimals and ask: which is larger, a or b, where

$$a = \left|10\sqrt{2} - 14\right|, \quad b = \left|10\sqrt{2} - 15\right| \;?$$

We have:

$$a^2 = 396 - 280\sqrt{2}, \quad b^2 = 425 - 300\sqrt{2}.$$

Hence:

$$b^2 - a^2 = 29 - 20\sqrt{2}.$$

We need to check whether $b^2 - a^2$ is positive or negative. Since

$$29^2 = 841, \quad \left(20\sqrt{2}\right)^2 = 800,$$

and $841 > 800$, it follows that $b^2 > a^2$, and hence that $b > a$. Therefore the square root of 2 is closer to 1.4 than to 1.5.

(3) Is the square root of 3 closer to 1.7 or to 1.75?

We do away with the decimals and ask: which is larger, a or b, where

$$a = \left|20\sqrt{3} - 34\right|, \quad b = \left|20\sqrt{3} - 35\right| \;?$$

We have:

$$a^2 = 2356 - 1360\sqrt{3}, \quad b^2 = 2425 - 1400\sqrt{3}.$$

Hence:

$$b^2 - a^2 = 69 - 40\sqrt{3}.$$

We need to check whether $b^2 - a^2$ is positive or negative. Since

$$69^2 = 4761, \quad \left(40\sqrt{3}\right)^2 = 4800,$$

and $4800 > 4761$, it follows that $b^2 < a^2$, and hence that $b < a$. Therefore the square root of 3 is closer to 1.75 than to 1.7.

(4) Which is larger, $2^{1/2}$ or $3^{1/3}$?

Let $a = 2^{1/2}$ and $b = 3^{1/3}$. It is obviously very easy to compare two irrational quantities when they have the same exponent. For example, we can readily see that $2^{1/2} < 3^{1/2}$ or that $10^{1/3} < 11^{1/3}$. Here the difficulty is caused by the fact that the two exponents (1/2, 1/3 respectively) are different. So we do the obvious thing: transform the quantities so that the exponents are the same. Since the LCM of 2 and 3 is 6, we raise both a and b to the sixth power. We have:

$$a^6 = \left(2^{1/2}\right)^6 = 2^3 = 8,$$

$$b^6 = \left(3^{1/3}\right)^6 = 3^2 = 9.$$

Since $8 < 9$, it follows that $8^{1/6} < 9^{1/6}$, i.e., $2^{1/2} < 3^{1/3}$.

(5) Which is the largest among the quantities $\sqrt{1}+\sqrt{19}$, $\sqrt{2}+\sqrt{18}$, $\sqrt{3}+\sqrt{17}$, ..., $\sqrt{9}+\sqrt{11}$, $\sqrt{10}+\sqrt{10}$? We give two solutions which are both worthy of close study.

Method 1. The terms are of the form $\sqrt{x}+\sqrt{20-x}$ for $x = 1, 2, 3, \ldots, 10$. The square of this quantity is

$$x + 2\sqrt{x}\sqrt{20-x} + (20-x) = 20 + 2\sqrt{x(20-x)}.$$

It should be evident from this expression that it suffices to find the value of x in the set $\{1, 2, 3, \ldots, 9, 10\}$ for which $x(20-x)$ is the largest. This may be done by multiplying out all the quantities; we find that $x(20-x)$ takes its largest value when $x = 10$. A rigorous proof for this is the following:

$$x(20-x) = 20x - x^2 = 100 - \left(x^2 - 20x + 100\right) = 100 - (x-10)^2,$$

and from the final expression we deduce that

$$x(20-x) \leq 100,$$

with equality precisely when $x = 10$. Hence for $x \in \{1, 2, 3, \ldots, 10\}$,

$$\sqrt{x} + \sqrt{20-x}$$

takes its largest value when $x = 10$. Therefore the largest among the given quantities is $\sqrt{10}+\sqrt{10} = 2\sqrt{10}$.

Method 2. We shall make clever use of the following identity, which follows from the identity $a^2 - b^2 = (a-b)(a+b)$: for $a \neq b$,

$$a + b = \frac{a^2 - b^2}{a - b}.$$

Comparing $\sqrt{1}+\sqrt{19}$ with $\sqrt{2}+\sqrt{18}$ is the same as comparing $\sqrt{19}-\sqrt{18}$ with $\sqrt{2}-\sqrt{1}$. Now we have, by the identity just quoted:

$$\sqrt{19} - \sqrt{18} = \frac{1}{\sqrt{19}+\sqrt{18}},$$

$$\sqrt{2} - \sqrt{1} = \frac{1}{\sqrt{2}+\sqrt{1}}.$$

Note carefully how use has been made of the fact that $\{1, 2\}$ and $\{18, 19\}$ are pairs of consecutive numbers. Since $\sqrt{19} + \sqrt{18}$ is clearly larger than $\sqrt{2} + \sqrt{1}$, it follows that $\sqrt{19} - \sqrt{18} < \sqrt{2} - \sqrt{1}$, and hence that $\sqrt{1} + \sqrt{19} < \sqrt{2} + \sqrt{18}$.

Next we compare the quantities $\sqrt{2} + \sqrt{18}$ and $\sqrt{3} + \sqrt{17}$. Following exactly the same kind of reasoning, we deduce that $\sqrt{18} - \sqrt{17} < \sqrt{3} - \sqrt{2}$, and therefore that $\sqrt{2} + \sqrt{18} < \sqrt{3} + \sqrt{17}$.

In just the same way, we deduce that:

$$\sqrt{3} + \sqrt{17} < \sqrt{4} + \sqrt{16},$$

$$\sqrt{4} + \sqrt{16} < \sqrt{5} + \sqrt{15},$$

$$\sqrt{5} + \sqrt{15} < \sqrt{6} + \sqrt{14},$$

and so on; and finally:

$$\sqrt{9} + \sqrt{11} < \sqrt{10} + \sqrt{10}.$$

Therefore, the largest of the quantities $\sqrt{1} + \sqrt{19}$, $\sqrt{2} + \sqrt{18}$, $\sqrt{3} + \sqrt{17}$, …, $\sqrt{9} + \sqrt{11}$, $\sqrt{10} + \sqrt{10}$ is $\sqrt{10} + \sqrt{10} = 2\sqrt{10}$.

Problems for you to solve

We close by offering a small list of problems for you to tackle. (No calculators to be used, please!)

(1) Which is larger:

(a) $3^{1/3}$ or $4^{1/4}$?

(b) $4^{1/4}$ or $5^{1/5}$?

(2) Which is larger:

(a) $2^{1/3}$ or $3^{1/4}$?

(b) $3^{1/4}$ or $4^{1/5}$?

(3) Which is larger: $1.1 + \dfrac{1}{1.1}$ or $1.01 + \dfrac{1}{1.01}$?

(4) If a, b are nonnegative real numbers with constant sum s, what are the least and greatest values taken by $a^2 + b^2$? Express the answers in terms of s.

(5) Let a, b, c be real numbers. Show that:

$$a^2 + b^2 + c^2 \geq ab + bc + ca.$$

Under what circumstances does the equality sign hold? In other words, when is it true that $a^2 + b^2 + c^2 = ab + bc + ca$?

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HOW to PROVE it

SHAILESH SHIRALI

In this episode of "How To Prove It", we discuss some more applications of Ptolemy's theorem.

In the last episode of this series, we stated and proved Ptolemy's theorem. We also showcased a few elegant applications of the theorem. In this episode, we study more such applications. We also state and prove a related theorem which is associated with the names of the ancient Indian mathematicians Brahmagupta (7th century), Mahavira (9th century) and Paramesvara (15th century).

Ptolemy's theorem states the following (see Figure 1):

Theorem 1 (Ptolemy of Alexandria).
If ABCD is a cyclic quadrilateral, then we have the equality $AB \cdot CD + BC \cdot AD = AC \cdot BD$.

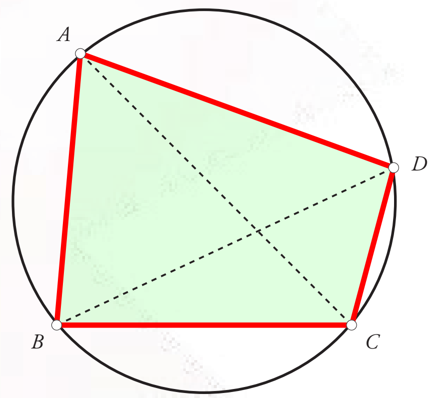


Figure 1. Ptolemy's theorem

Some applications of Ptolemy's theorem

We showcase below a few more applications of the theorem (we had discussed a few such in the previous article). The first is a proof of the most venerable theorem of all.

Keywords: Ptolemy, similar triangle, power of a point

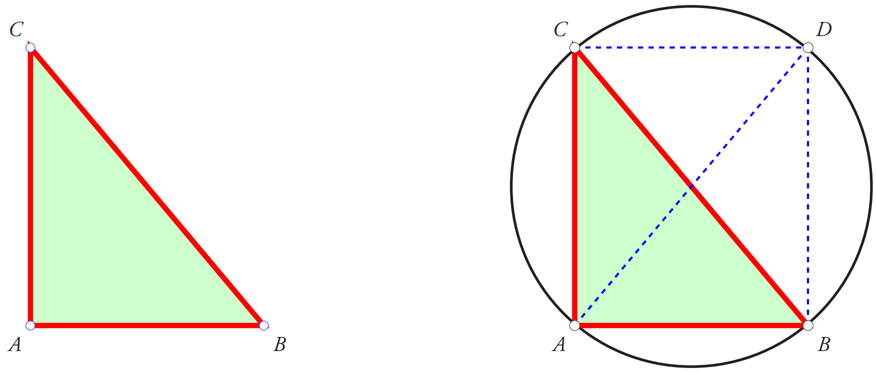


Figure 2. Application of Ptolemy's theorem--I: Proof of Pythagoras theorem

Theorem 2 (Pythagoras).
Let ABC be a right-angled triangle, with the right angle at vertex A. Then $b^2 + c^2 = a^2$.

Proof. Locate point D such that $ABDC$ is a rectangle (note the cyclic order of the vertices: $ABDC$, not $ABCD$; see Figure 2). Since $ABDC$ is cyclic (a rectangle is necessarily cyclic), Ptolemy's theorem applies to it.

Ptolemy's theorem tells us that $AB \cdot CD + AC \cdot BD = BC \cdot AD$. Since $AB = CD$, $AC = BD$ and $AD = BC$ (all these follow because $ABDC$ is a rectangle), we get: $AB^2 + AC^2 = BC^2$, i.e., $b^2 + c^2 = a^2$. $\square$

Cosine rule. A slight alteration of the above yields a proof of the cosine rule.

Theorem 3 (Cosine rule).
Let ABC be an arbitrary triangle. Then we have the following relation: $a^2 = b^2 + c^2 - 2bc\cos A$.

Proof. Locate point D such that $ABDC$ is an isosceles trapezium, with $AB \parallel CD$, $AC = BD$ and $\angle CAB = \angle DBA$ (note again the cyclic order of the vertices: $ABDC$, not $ABCD$; see Figure 3). Since $ABDC$ is an isosceles trapezium, it is cyclic, so Ptolemy's theorem applies to it.

Since $AD = BC$ and $AC = BD$, we get: $BC^2 = AC^2 + AB \cdot CD$, or:

$$a^2 = b^2 + c \cdot CD.$$

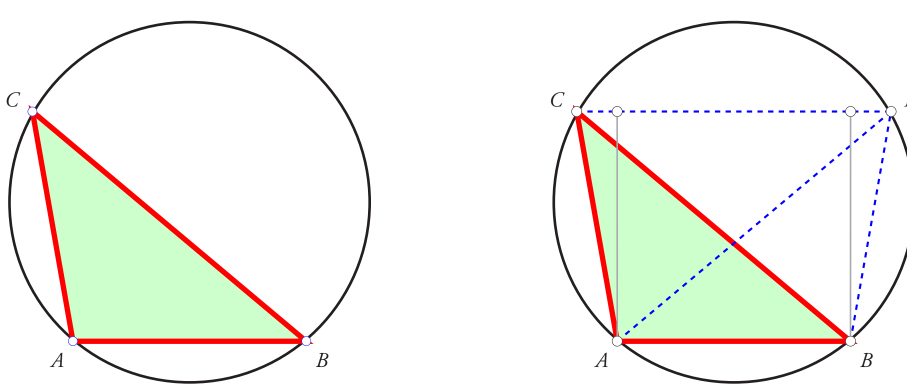
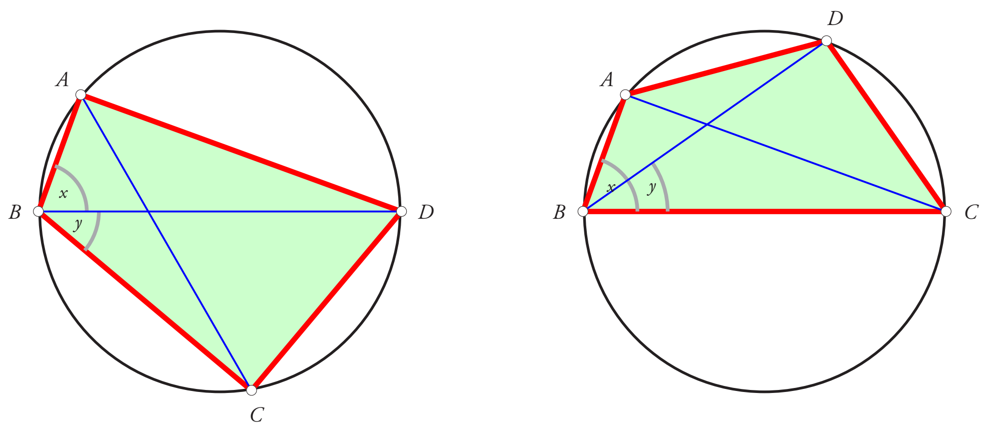


Figure 3. Application of Ptolemy's theorem--II: Proof of the cosine rule



$BD = 1$, $\angle ABD = x$, $\angle DBC = y$ $\qquad$ $BC = 1$, $\angle ABC = x$, $\angle DBC = y$

Figure 4. Derivation of the addition and difference rules for the sine function

By dropping perpendiculars from A and B to line CD (they have been shown using a very light colour to avoid creating a visual clutter), it follows readily that

$$CD - AB = 2\,AC \cos \angle ACD,$$

and since $\angle ACD$ and $\angle CAB$ are supplementary, it follows that $CD = c - 2b \cos A$. Therefore $a^2 = b^2 + c^2 - 2bc \cos A$, as required. □

Our third application features the proofs of two important trigonometric identities: the addition and difference rules for the sine and cosine functions. Historically, Ptolemy used his theorem to draw up tables of values of the trigonometric functions by traversing this very route. It is thus of interest to mathematicians as well as historians. Here is how it is done.

The addition and difference rules for the sine function.

The single result from geometry-trigonometry that we shall use repeatedly is this: in a circle with radius R, if a chord with length d subtends an angle x at the circumference, then we have: $d = 2R \sin x$.

Let acute angles x, y be given, $0 \leq y \leq x \leq \pi/2$. Draw a circle with diameter $2R = 1$ unit; see Figures 4 (a) and (b).

In (a), BD is a diameter of the circle, and angles x, y are non-overlapping. Using the geometric-trigonometric result mentioned above, we get: $AB = \cos x$, $BC = \cos y$, $CD = \sin y$, $AD = \sin x$, $AC = \sin(x+y)$, $BD = 1$. Hence by Ptolemy's theorem:

$$\sin(x+y) = \sin x \,\cos y + \cos x \,\sin y,$$

and we have obtained the addition formula.

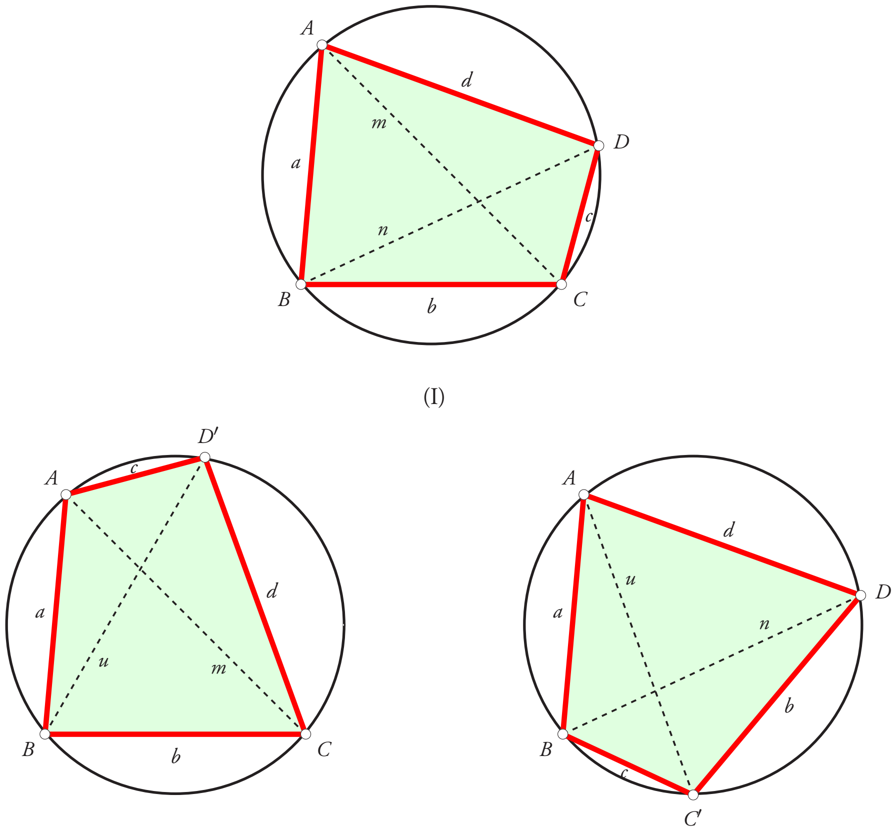
In (b), BC is a diameter of the circle, and angles x, y are overlapping. This time we get: $AB = \cos x$, $BC = 1$, $CD = \sin y$, $AD = \sin(x-y)$, $AC = \sin x$, $BD = \cos y$. Hence by Ptolemy's theorem:

$$\sin x \,\cos y = \cos x \,\sin y + \sin(x-y),$$

and hence:

$$\sin(x-y) = \sin x \,\cos y - \cos x \,\sin y.$$

We have obtained the difference formula.



(I)

(I) → (II): Swap c and d (I) → (III): Swap b and c

Figure 5. Formulas for the lengths of the diagonals

The Brahmagupta-Mahavira Identities

The formulas presented here give the lengths m, n of the diagonals of the cyclic quadrilateral, expressed in terms of the sides a, b, c, d, the sides being in the cyclic order a, b, c, d; see Figure 5 (I). The proofs of the formulas are amazingly simple; they are the work of Paramesvara (15th century), though the formulas themselves were discovered by Brahmagupta (7th century) and again by Mahavira (9th century).

We perform two independent operations on (I), giving Figures (II) and (III). To get (II), we swap sides c and d (this is equivalent to reflecting D in the perpendicular bisector of AC). The sides of the resulting quadrilateral (which is still cyclic) are in the order a, b, d, c, and the diagonals are m, u (diagonal m stays unchanged).

To get (III), we swap sides b and c (this is equivalent to reflecting C in the perpendicular bisector of BD). The sides of the resulting quadrilateral (which is still cyclic) are in the order a, c, b, d, and the diagonals are n, u (diagonal n stays unchanged). The crucial point here is that diagonal AC' has the same length as diagonal BD'. Hence the choice of the symbol u to denote their common length.

Using Ptolemy's theorem, we obtain the following three relations:

$$mn = ac + bd,$$

$$mu = ad + bc,$$

$$nu = ab + cd.$$

The last two relations yield:

$$\frac{m}{n} = \frac{ad+bc}{ab+cd}.$$

Combining this with the first relation, we get the Brahmagupta-Mahavira identities:

$$m^2 = \frac{(ac+bd)(ad+bc)}{ab+cd}, \qquad n^2 = \frac{(ac+bd)(ab+cd)}{ad+bc}.$$

We have obtained formulas which yield the lengths of the diagonals. The relation

$$\frac{m}{n} = \frac{ad+bc}{ab+cd}$$

is sometimes called the *ratio form of Ptolemy's theorem*.

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3. Ptolemy's Theorem, https://en.wikipedia.org/wiki/Ptolemy's_theorem

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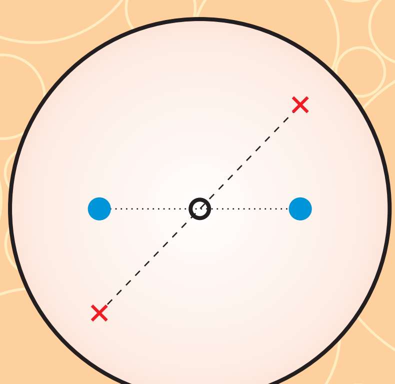
SOLUTION TO FILLING A CIRCLE

Problem on page 13

Rahul: 'The winning strategy is based on the perfect symmetry of a circle. I can ensure my win if I play first and if I do, then I place the first coin exactly at the centre of the table. Subsequently, I simply imitate your actions by mirroring them in the centre of the table. If you have space for a coin, on the table, then so will I. So you will be the first to be unable to find space for a coin.'



The winning strategy. Clearly, Rahul is a winner!

TechSpace

An example of constructive defining:

From a GOLDEN RECTANGLE to GOLDEN QUADRILATERALS and Beyond

Part 1

MICHAEL DE VILLIERS

There appears to be a persistent belief in mathematical textbooks and mathematics teaching that good practice (mostly; see footnote[1]) involves first providing students with a concise definition of a concept before examples of the concept and its properties are further explored (mostly deductively, but sometimes experimentally as well). Typically, a definition is first provided as follows:

- *Parallelogram*: A parallelogram is a quadrilateral with half turn symmetry. (Please see endnotes for some comments on this definition.)
- The number $e = \lim_{n\to\infty}\left(1+\dfrac{1}{n}\right)^n = 2.71828\ldots$
- *Function*: A function f from a set A to a set B is a relation from A to B that satisfies the following conditions:
  (1) for each element a in A, there is an element b in B such that $\langle a, b\rangle$ is in the relation;
  (2) if $\langle a, b\rangle$ and $\langle a, c\rangle$ are in the relation, then $b = c$.

[1] It is not being claimed here that all textbooks and teaching practices follow the approach outlined here as there are some school textbooks such as Serra (2008) that seriously attempt to actively involve students in defining and classifying triangles and quadrilaterals themselves. Also in most introductory calculus courses nowadays, for example, some graphical and numerical approaches are used before introducing a formal limit definition of differentiation as a tangent to the curve of a function or for determining its instantaneous rate of change at a particular point.

Keywords: constructive defining; golden rectangle; golden rhombus; golden parallelogram

Following such given definitions, students are usually next provided with examples and non-examples of the defined concept to 'elucidate' the definition. The problem with this overwhelmingly popular approach is that it creates the misconception that mathematics always starts with definitions, and hides from students that a particular concept can often be defined in many different equivalent ways. Moreover, students are given no idea where the definition came from and on what grounds this particular definition was chosen. By providing students with a ready-made definition, they are also denied the opportunity to engage in the process of mathematical defining themselves, and hence it unfortunately portrays to them an image of mathematics as an 'absolutist' science (Ernest, 1991).

In general, there are essentially two different ways of defining mathematical concepts, namely, *descriptive* (a posteriori) and *constructive* (a priori) defining. Descriptive definitions systematize already existing knowledge, whereas constructive definitions produce new knowledge (Freudenthal, 1973).

The purpose of this article is to heuristically illustrate the process of constructive defining in relation to a recent exploration by myself of the concept of a 'golden rectangle' and its extension to a 'golden rhombus', 'golden parallelogram', 'golden trapezium', 'golden kite', etc. Though these examples are mathematically elementary, it is hoped that their discussion will illuminate the deeper process of constructive defining.

Constructively Defining a 'Golden Rhombus'

> "... [The] *algorithmically constructive and creative definition ... models new objects out of familiar ones.*"
>
> – Hans Freudenthal (1973: 458).

Constructive (a priori) defining takes place when a given definition of a concept is changed through the exclusion, generalization, specialization, replacement or addition of properties to the definition, so that a new concept is constructed in the process.

Since there is an interesting *side-angle* duality between a rectangle (all angles equal) and a rhombus (all sides equal) (see De Villiers, 2009:55), I was recently considering how to define the concept of a 'golden rhombus'. Starting from the well-known definition of a golden rectangle as a rectangle which has its adjacent *sides* in the ratio of the golden ratio $\phi = 1.618\ldots$, I first considered the following analogous option in terms of the *angles* of the rhombus (Please see endnotes for the definition of the golden ratio):

A golden rhombus is a rhombus with adjacent *angles* in the ratio of ϕ.

Assuming the acute angle of the rhombus as x, this definition implies that:

$$\frac{180^\circ - x}{x} = \phi, \therefore x = \frac{180^\circ}{1+\phi} \approx 68.75^\circ.$$

An accurate construction of a 'golden rhombus' fulfilling this angle condition is shown in Figure 1.

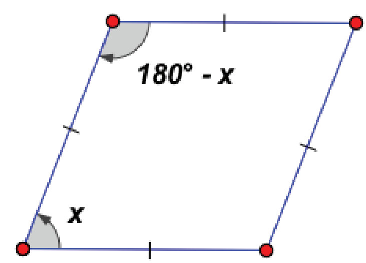


Figure 1. Golden rhombus with angles in ratio phi

Though this particular rhombus looks reasonably visually appealing, I wondered how else one might reasonably obtain or define the concept of a golden rhombus. Since a rectangle is cyclic and a rhombus has an inscribed circle, I hit upon the idea of starting with a golden rectangle *EFGH* (with $\frac{EH}{EF} = \phi$) and its circumcircle, and then constructing the rhombus *ABCD* with sides tangent to the circumcircle at the vertices of the rectangle. (Note that it follows directly from the symmetry of the rectangle *EFGH* that *ABCD* is a rhombus). Much to my surprised delight, I now found through accurate construction and measurement with dynamic geometry software as shown in Figure 2 that though the angles were no longer in the ratio phi, the diagonals for this rhombus now were!

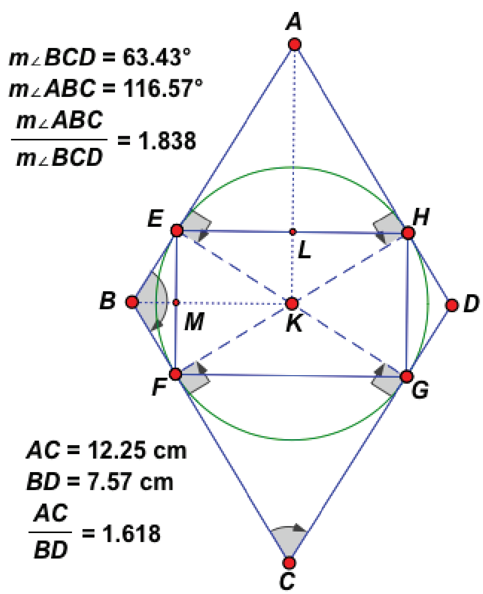


Figure 2. Golden rhombus with diagonals in ratio phi

It is not difficult to explain why (prove that) the diagonals of rhombus $ABCD$ are in the ratio ϕ. Clearly triangles ABK and KEM are similar, from which follows that $\frac{AK}{BK} = \frac{KM}{EM}$. But $KM = LE$; so $\frac{AK}{BK} = \frac{LE}{EM}$. But these lengths (AK, BK); (LE, EM) are respectively half the lengths of the diagonals of the rhombus and the sides of the rectangle; hence the result follows from the property of the golden rectangle $ABCD$.

The size of the angles of the golden rhombus in Figure 2 can easily be determined using trigonometry, and the task is left to the reader. Another interesting property of both the golden rectangle and golden rhombus in this configuration is that $\tan \angle EKF = \tan \angle BCD = 2$. One way of easily establishing this is by applying the double angle tan formula, but this is also left as an exercise to the reader to verify.

Since definitions in mathematics are to some extent arbitrary, and there is no psychological reason to prefer the one to the other from a visual, aesthetic point of view (footnote[2]), we could therefore choose either one of the aforementioned possibilities as our definition. However, it seems that a better argument can be made for the second definition of a 'golden rhombus', since it shows a nice, direct connection with the golden rectangle. Also note that the second definition can be stated in either of the following equivalent forms: 1) a quadrilateral with sides constructed tangential to the circumcircle, and at the vertices, of a golden rectangle as illustrated in Figure 2; or more simply as 2) a rhombus with diagonals in the ratio of ϕ (footnote[3]).

The case for the second definition is further strengthened by the nice duality illustrated between the golden rectangle and golden rhombus in Figure 3, which shows their respective midpoint quadrilaterals (generally called '*Varignon parallelograms*'). Since the diagonals of the golden rectangle are equal, it follows that its corresponding Varignon parallelogram is a rhombus, but since its diagonals are equal to the sides of the golden rectangle, they are also in the golden ratio, and therefore the rhombus is a golden rhombus. Similarly, it follows that the Varignon parallelogram of the golden rhombus is a golden rectangle.

Constructively Defining a 'Golden Parallelogram'

Since the shape of a parallelogram with sides in the ratio of phi is variable, it seemed natural from the aforementioned to define a 'golden parallelogram' as a parallelogram $ABCD$ with its sides and diagonals in the ratio phi, e.g., $\frac{AD}{AB} = \frac{BD}{AC} = \phi$ as shown in Figure 4. Experimentally dragging a dynamically constructed general parallelogram until its sides and diagonals were approximately in the golden ratio gave a measurement for $\angle ABC$ of approximately 60°.

[2]It is often claimed that there is some inherent aesthetic preference to the golden ratio in art, architecture and nature. However, several recent psychological studies on peoples' preferred choices from a selection of differently shaped rectangles, triangles, etc., do not show any clear preference for the golden ratio over other ratios (e.g., see Grossman et al, 2009; Stieger & Swami, 2015). Such a finding is hardly surprising since it seems very unlikely that one could easily visually distinguish between a rectangle with sides in the golden ratio 1.618, or say with sides in the ratio of 1.6, 1.55 or 1.65, or even from those with sides in the ratio 1.5 or 1.7.

[3]A later search on the internet revealed that on https://en.wikipedia.org/wiki/Golden_rhombus, a golden rhombus is indeed defined in this way in terms of the ratio of its diagonals and not in terms of the ratio of its angles. A further case for the preferred choice of this definition can be also made from the viewpoint that several polyhedra have as their faces, rhombi with their diagonals in the golden ratio.

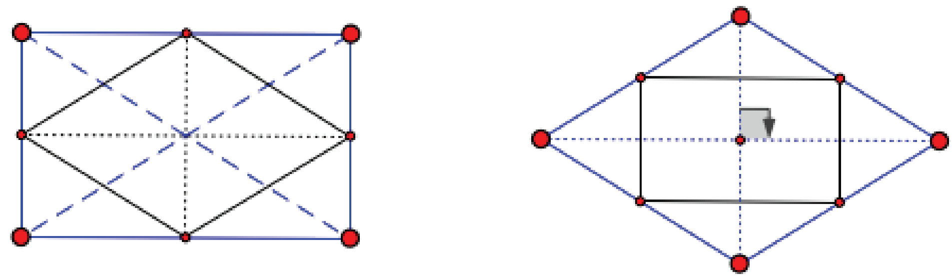


Figure 3. The Varignon parallelograms of a golden rectangle and golden rhombus

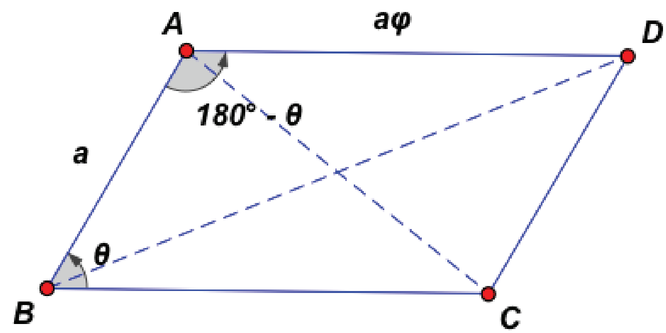


Figure 4. Golden parallelogram with sides and diagonals in the golden ratio

To prove this conjecture was not hard. Assuming $a = 1$ in Figure 4, it follows from the cosine rule that:

$$AC^2 = 1^2 + \phi^2 - 2\phi\cos\theta,$$
$$BD^2 = 1^2 + \phi^2 + 2\phi\cos\theta.$$

But since $\frac{BD}{AC} = \phi$ is given, it follows that:

$$\frac{1^2 + \phi^2 + 2\phi\cos\theta}{1^2 + \phi^2 - 2\phi\cos\theta} = \phi^2.$$

Solving this equation for $\cos\theta$ and substituting the value of ϕ gives:

$$\cos\theta = \frac{\phi^4 - 1}{2(\phi + \phi^3)} = \frac{1}{2},$$

which yields $\theta = 60^\circ$. So my experimentally found conjecture was indeed true. Accordingly, a golden parallelogram defined as a parallelogram with both its sides and diagonals in the golden ratio has 'neat' angles of 60° and 120°, and it also looks more or less visually pleasing. Equivalently, and more conveniently, we could define the golden parallelogram as a parallelogram with an acute angle of 60° and sides in the golden ratio[4] or as a parallelogram with an acute angle of 60° and diagonals in the golden ratio. That the remaining property follows from these convenient, alternative definitions is left to the interested reader to verify.

An appealing property of this golden parallelogram, consistent with that of a golden rectangle, is shown in the first two diagrams in Figure 5, namely, that respectively cutting off a rhombus at one end, or two equilateral triangles at both ends, produces another golden parallelogram. This is because in each case a parallelogram with an acute angle of 60° is obtained, and letting $a = 1$, we see that it has sides in the ratio $\frac{1}{\phi - 1}$, which is well known to equal ϕ.

In addition, constructing the Varignon parallelogram determined by the midpoints of the sides of any parallelogram as shown by the third diagram in Figure 5, it is easy to see that the sides and diagonals of the Varignon parallelogram will be in the same ratio as those of the parent

[4]Somewhat later I found that Walser (2001, p. 45) had similarly defined a golden parallelogram as a parallelogram with an acute angle of 60° and sides in the golden ratio.

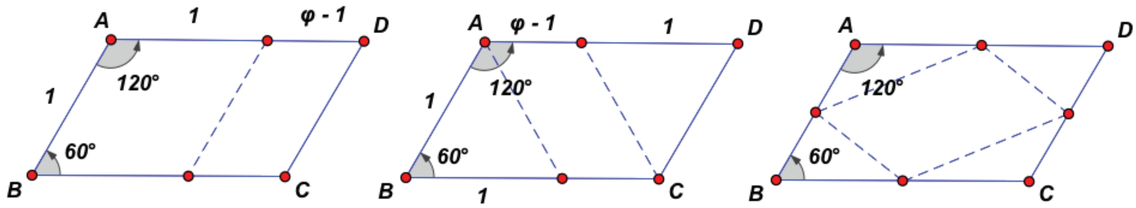


Figure 5. Construction of golden parallelograms by subdivision

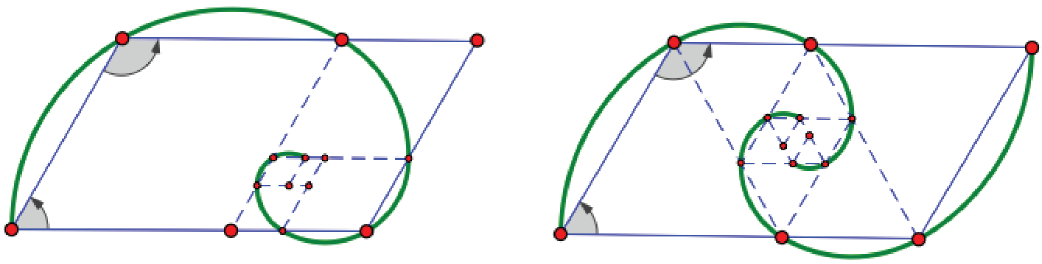


Figure 6. Spirals related to the golden parallelogram

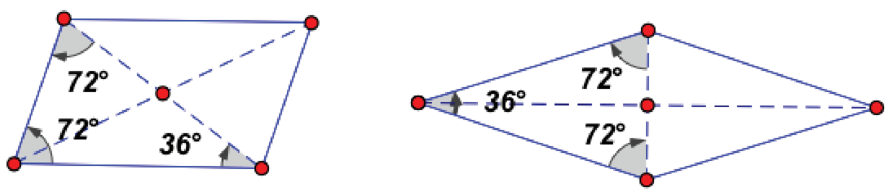


Figure 7. Alternative definitions for golden parallelogram and golden rhombus

parallelogram. Hence, the Varignon parallelogram of a parallelogram will be a golden parallelogram if and only if, the parent parallelogram is a golden parallelogram.

Of further recreational interest is that the subdividing processes of the first two diagrams in Figure 5, can be continued iteratively as shown in Figure 6, just like the golden rectangle, to produce rather pleasant looking spirals.

As was the case with the rhombus, a 'golden parallelogram' can also be constructively defined differently in terms of what is called a 'golden triangle', namely an isosceles triangle with an angle of 36° and two angles of 72° each. (It is left as an exercise to readers to verify that such a triangle has one of its legs to the base in the ratio ϕ). A golden parallelogram can therefore be obtained differently from the aforementioned by a half-turn around the midpoint of one of the legs of the golden triangle to obtain a parallelogram with sides in the ratio ϕ (see footnote[5]) as shown in the first diagram in Figure 7.

Note that using a golden triangle we can also constructively define a golden rhombus in a third way as shown in the second diagram in Figure 7. By simply reflecting a golden triangle around its 'base', we obtain a rhombus with its side to the shorter diagonal in the golden ratio. Though this 'golden rhombus' may appear too flattened out to be visually pleasing, it is of some mathematical interest as it appears in regular pentagons, regular decagons, and in combination with a regular pentagon, can create a tiling of the plane. So this is a case where visual aesthetics of a concept have to be weighed up against its mathematical relevance.

In Part-II of this article, we will explore some possible definitions for golden isosceles trapezia, golden kites, as well as a golden hexagon.

[5]Loeb & Varney (1992, pp. 53-54) define a golden parallelogram as a parallelogram with an acute angle of 72° and its sides in the golden ratio. They then proceed using the cosine rule to determine the diagonals of such a parallelogram to prove that the short diagonal is equal to the longer side of the parallelogram and hence divides it into two golden triangles.

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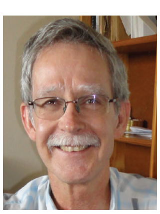
Endnotes

1. This is not the common textbook definition. (The usual definition is: A parallelogram is a four-sided figure for which both pairs of opposite sides are parallel to each other.) I want to emphasize that concepts can be defined differently and often more powerfully in terms of symmetry. As argued in De Villiers (2011), it is more convenient defining quadrilaterals in terms of symmetry than the standard textbook definitions. Reference: De Villiers, M. (2011). Simply Symmetric. *Mathematics Teaching*, May 2011, p34–36.

2. The Golden Ratio can be defined in different ways. The simplest one is: it is that positive number x for which $x = 1 + 1/x$; equivalently, that positive number x for which $x^2 = x + 1$. The definition implies that $x = (\sqrt{5} + 1)/2$, whose value is approximately 1.618034. A rectangle whose length : width ratio is $x : 1$ is known as a golden rectangle. It has the feature that when we remove the largest possible square from it (a 1 by 1 square), the rectangle that remains is again a golden rectangle.

3. The term Golden Rectangle has by now a standard meaning. However, terms like Golden Rhombus, Golden Parallelogram, Golden Trapezium and Golden Kite have been defined in slightly different ways by different authors.



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Problems for the MIDDLE SCHOOL

Problem Editors: SNEHA TITUS & ATHMARAMAN R

We hope that you enjoyed the reworked Middle Problems and found the Handy Reference Sheet useful. This time we will continue to focus on parity with some nice problems which use the properties given in the sheet.

Problem VI-1-M.1

Can a single coin of value 25 be swapped for 10 coins of value 1, 3 or 5? If yes, describe how this can be done. If no, describe a possible swap by changing any one (and only one) of the numbers given.

Problem VI-1-M.2

A 17 digit number is chosen and its digits are reversed, forming a new number. These two numbers are added together. Show that their sum contains at least one even digit. Is this true for an 18 digit number? If not, give an exception.

Problem VI-1-M.3

Place 4 ones and 5 zeroes around a circle in any order. Now, between any 2 numbers place a zero if the numbers are different and a one if the numbers are the same, and then erase the old numbers. Repeat the same operation on the new numbers. The game continues till all 9 numbers in the circle are the same. How long will this game continue?

These three problems are from *Mathematical Circles* by Dmitri Fomin, Sergey Genkin and Ilia Itenberg.

Problem VI-1-M.4

This is a divisibility game with a set of 0-9 digit cards.

Player 1 chooses any one of the 10 digit cards. Player 2 then chooses any of the remaining cards and places it to the right of the card already there, so that the two digit- number is divisible by 2. The game proceeds in this manner- They take it in turns to choose and place a card to the right of the cards that are already there.

- After two cards have been placed, the two-digit number must be divisible by 2.
- After three cards have been placed, the three-digit number must be divisible by 3.
- After four cards have been placed, the four-digit number must be divisible by 4.

And so on! They keep taking it in turns until one of them gets stuck.

Are there any good strategies to help you to win? What's the longest number you can make that satisfies the rules of the game? Is it possible to use all ten digits to create a ten-digit number?

Is there more than one solution?

http://nrich.maths.org/796

Solutions
Problem VI-1-M.1
The central observation we use to solve the problem is: the sum of an odd number of odd numbers is necessarily odd, and the sum of an even number of odd numbers is necessarily even. Try this with any collection of odd numbers and you will see why it is true. We apply this observation to our problem in which we have to select 10 coins, each of value 1, 3 or 5. Note that the numbers 1, 3, 5 are all odd, whereas 10 is an even number. So the total value of 10 coins is even, no matter how you select the coins. It follows that no collection of 10 such coins can have a value of 25.

The problem becomes possible if 25 is changed to an even number, for example, 30.

Or the number of coins changes to an odd number, for example, 11. If the 11 coins are expressed as the sum of 6,3 and 2 coins then $6 \times 1 + 3 \times 3 + 2 \times 5 = 25$.

Teacher's Note: An easy entry into problems on parity and one that develops logical thinking and communication skills. Students need to defend their reasoning. Opening up the second part of the question gives students a chance to give varied answers all of which could be correct (provided, of course, that they can justify their choices).

Problem VI-1-M.2
Since 17 is an odd number, when it is reversed, the middle (9^{th}) digit does not change position. Since the 9^{th} digit is doubled, it will be an even number (provided there is no carry over from the 8^{th} digit from the right), so the answer will have at least one even digit in this case. If there is a carry over, then let us assume that every digit in the sum is odd. A carry over from the 8^{th} column to the 9^{th}, implies that there is a carry over from the 10^{th} column to the 11^{th} (the digits being added are the same, but in reverse order). Since the sum has given an odd digit and the carry over is 1, obviously the digits in the 11^{th} column are of the same parity. Hence the digits in the 7^{th} column are also of the same parity and there has been a carry over from the 6^{th} to the 7^{th} column. We repeat this argument; there will be a carry over from every even column to every odd column. But it is not possible to have a carry over into the 1^{st} column. So the 1^{st} and the 17^{th} digit should be of opposite parity for the sum in the 1^{st} column to be an odd digit. But then, the numbers in the 17^{th} column will be of opposite parity with a carry over from the 16^{th} column and this will result in an even digit in the 17^{th} column of the sum.

When the number of digits is even, all the digits change position and if the digits which are added are pairs of even and odd numbers then no digit in the sum will be even.

Example: The 18 digit number 123456789012345678. Or, even simpler: 121212121212121212 (with nine repetitions of '12').

Teacher's Note: Again, a chance for reasoning and communication. Plus a chance for students to make judicious use of terms such as *always*, *sometimes* or *never*. For example, it is possible to get an 18 digit number which yields a sum with an even digit. Best of all, students learn to reduce a problem to a simpler version; they should be encouraged to experiment with numbers having a fewer number of digits if they cannot get started on the original version of the problem. Caution: The 3 digit number 637 when reversed and added, gives 1373, a number with no even digit, students can investigate why this happens!

Problem VI-1-M.3
Again, the solution to this problem emerges easily when students take a fewer number of ones and zeros.

A second strategy which helps is to look at the desired outcome- the numbers should be all ones or all zeroes.

If they are all zeroes, then in the previous step, no number should have the same neighbours, or alternate numbers should be the same. And in the step previous to that, the same situation should exist, i.e., no number should have the same neighbours or alternate numbers should be the same. This means that this situation should exist in the original configuration which is not possible, since the number of ones is not equal to the number of zeroes.

If they are all ones, then in the previous step, all the numbers should be the same and that is not possible when both zeroes and ones are used.

Try with 1 one and 1 zero. This works and proceeds to a solution.

With 2 ones and 1 zero, it is impossible to arrange them so that alternate numbers are the same. And this happens whenever the number of zeroes and ones are unequal. So the game goes on forever in this case.

Problem VI-1-M.4
A lot of theory gets internalized when students play games. Divisibility rules can never be learnt by heart, they need to be turned to instinctively and used as quick and effective checks. This problem has terrific opportunities for reasoning; the solution is given on the nrich website, so we will only reveal that if all 10 digits are used, the only possible answer is 3816547290.

Pedagogy: Problem solving strategies that have been imparted with this problem set are:

1. Trying the same problem with a smaller number of cases.
2. Understanding the benefits of trial and error.
3. Learning to generalize and being aware of the pitfalls when this is done.

Problems for the SENIOR SCHOOL

Problem Editors: PRITHWIJIT DE & SHAILESH SHIRALI

Problem VI-1-S.1
Let $ABCD$ be a cyclic quadrilateral in which the diagonals AC and BD are perpendicular. Let X be the point of intersection of the diagonals. Let P be the midpoint of BC. Prove that PX is perpendicular to AD.

Problem VI-1-S.2
Let a, b, c be three distinct non-zero real numbers. If a, b, c are in arithmetic progression and b, c, a are in geometric progression, prove that c, a, b are in harmonic progression and find the ratio $a : b : c$.

Problem VI-1-S.3
Prove that the following quantity is not an integer:

$$\frac{2}{3}+\frac{4}{5}+\frac{6}{7}+\cdots+\frac{2016}{2017}.$$

Problem VI-1-S.4
Prove that the product of six consecutive positive integers cannot be a perfect cube.

Problem VI-1-S.5
A square with side a is inscribed in a circle. Find the side of the square inscribed in one of the segments thus obtained.

Solutions to Problems in Issue-V-3 (November 2016)

Solution to problem V-3-S.1

Let ABCD be a convex quadrilateral. Let P, Q, R, S be the midpoints of AB, BC, CD, DA respectively. What kind of a quadrilateral is PQRS? If PQRS is a square, prove that the diagonals of ABCD are perpendicular to each other.

Solution. By the midpoint theorem, PQ and RS are parallel to AC and $PQ = RS = AC/2$. Hence $PQRS$ is a parallelogram. If $PQRS$ is a square, then $PQ \perp QR$. But PQ is parallel to AC and QR is parallel to BD. Thus $AC \perp BD$.

Solution to problem V-3-S.2

Let ABCD be a convex quadrilateral. Let P, Q, R, S be the midpoints of AB, BC, CD, DA respectively. Let U, V be the midpoints of AC, BD, respectively. Prove that lines PR, QS and UV are concurrent.

Solution. Observe that $PQRS$ is a parallelogram. Let PR and QS intersect at X. Note that X is the midpoint of both PR and QS. In triangle ADB, S and V are midpoints of the sides AD and BD respectively. Thus SV is parallel to AB and $SV = AB/2$. In $\triangle ABC$, U and Q are the midpoints of sides AC and BC respectively. Thus UQ is parallel to AB and $UQ = AB/2$. Therefore in quadrilateral $QUSV$ a pair of opposite sides, SV and UQ, are parallel and equal to one another; hence $QUSV$ is a parallelogram. As the diagonals of the parallelogram bisect each other, X must lie on UV. Thus the lines PR, QS and UV are concurrent.

Solution to problem V-3-S.3

There are 12 lamps, initially all OFF, each of which comes with a switch. When a lamp's switch is pressed, its state is reversed, i.e., if it is ON, it will go OFF, and vice-versa. One is allowed to press exactly 5 different switches in each round. What is the minimum number of rounds needed so that all the lamps will be turned ON? (Hong Kong Preliminary Selection Contest 2015)

Solution. Suppose all lamps are turned ON after n rounds. Then we have pressed the switches $5n$ times in all. Note that each lamp must change state an odd number of times. As there are 12 lamps, the total number of times the lamps have changed state is an even number; hence n is even. Clearly $n \neq 2$, since at most $5 \times 2 = 10$ lamps can be turned on in 2 rounds. On the other hand, we can turn on all lamps in 4 rounds as follows:

Round 1: Press switches 1, 2, 3, 4, 5.
Round 2: Press switches 6, 7, 8, 9, 10.
Round 3: Press switches 7, 8, 9, 10, 11.
Round 4: Press switches 7, 8, 9, 10, 12

It follows that the answer is 4.

Solution to problem V-3-S.4

The greatest altitude in a scalene triangle has length 5 units, and the length of another altitude is 2 units. Determine the length of the third altitude, given that it is integer valued.

Solution. Let the triangle be named ABC; let $BC = a$, $CA = b$ and $AB = c$. Since the triangle is scalene, we may assume that $a < b < c$. If h_a, h_b, h_c denote the lengths of the altitudes on sides BC, CA, AB respectively, then $h_a > h_b > h_c$ and $ah_a = bh_b = ch_c = 2\Delta$ where Δ is the area of the triangle. By the condition of the problem, $h_a = 5$. As $a + b > c$ it follows that

$$\frac{1}{h_a} + \frac{1}{h_b} > \frac{1}{h_c}.$$

If $h_b = 2$, then as h_c is an integer the only possibility is $h_c = 1$. But then we get $7/10 > 1$ which is absurd. Thus $h_c = 2$. From the fact that $h_b > h_c = 2$ we get $2 < h_b < 10/3$. Hence $h_b = 3$.

Solution to problem V-3-S.5

If the three-digit number $\overline{ABC}$ is divisible by 27, prove that the three-digit numbers $\overline{BCA}$ and $\overline{CAB}$ are also divisible by 27.

Solution. Since 9 divides $\overline{ABC}$, it divides $A+B+C$. Let $A+B+C=9m$ for some positive integer m. Now

$$\begin{aligned}\overline{BCA}-\overline{ABC} &= (100B+10C+A)\\ &\quad -(100A+10B+C)\\ &= 90B+9C-99A\\ &= 9(A+B+C)-108A+81B\\ &= 27(-4A+3B+3m),\end{aligned}$$

and

$$\begin{aligned}\overline{CAB}-\overline{ABC} &= (100C+10A+B)\\ &\quad -(100A+10B+C)\\ &= 99C-90A-9B\\ &= -81A+108C-9(A+B+C)\\ &= -81A+108C-81m\\ &= 27(-3A+4C-3m).\end{aligned}$$

Hence $\overline{BCA}$ and $\overline{CAB}$ are divisible by 27.

SOLUTIONS
NUMBER CROSSWORD

Page 46

		[1] 7				[2] 2		
	[3] 7	2	[4] 9		[5] 3	8	[6] 1	
[7] 3	7		[8] 3	8	0		[9] 2	0
	[10] 7	[11] 2	0		[12] 8	[13] 5	4	
		1				0		
	[14] 3	6	[15] 6		[16] 5	0	[17] 6	
[18] 6	0		[19] 7	4	7		[20] 1	3
	[21] 1	[22] 1	1		[23] 9	[24] 9	7	
		2				1		

Are you interested in creating the number crossword for the next issue? Send a mail to AtRiA.editor@apu.edu.in and we can help you puzzle your fellow readers.

TWO Problem Studies

In this edition of Adventures in Problem Solving, we continue the theme of studying two problems in detail. Problem 1 is adapted from Problem 6 of the Regional Mathematics Olympiad (India) held in October 2016, and Problem 2 is from the Tournament of the Towns which we had featured in the previous issue (see https://en.wikipedia.org/wiki/Tournament\of\the\Towns). The problem studied here appeared in 1997; it was also featured in an Olympiad in South Africa. As usual, we state the problems first, so that you have an opportunity to tackle them before seeing the solutions.

Problem 1

ABC is an equilateral triangle (Figure 1). Points $P_1, P_2, \ldots, P_{10}$ are taken on side BC, in that order, dividing that side into 11 equal parts. Similarly, points $Q_1, Q_2, \ldots, Q_{10}$ are taken on side CA, in that order, dividing that side into 11 equal parts, and points $R_1, R_2, \ldots, R_{10}$ are taken on side AB, in that order, dividing that side into 11 equal parts. Count the number of triangles $P_iQ_jR_k$ such that the centroid of $\triangle P_iQ_jR_k$ coincides with the centroid of $\triangle ABC$.

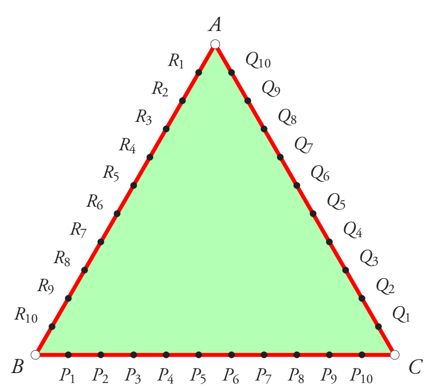


Figure 1

Problem 2

$ABCD$ is a square (Figure 2), and K is any point on side BC. The internal bisector of $\measuredangle KAD$ cuts side CD in point M. Show that $AK = BK + DM$.

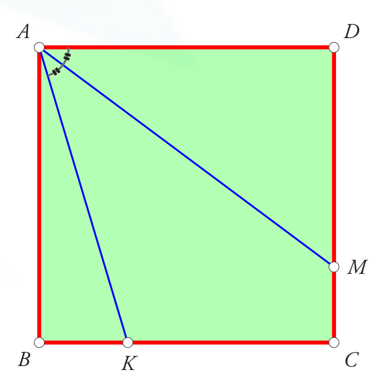


Figure 2

Solution to Problem 1.
This problem is easier than it looks. Also, the answer is what you would intuitively expect!

We consider a more general question. *Suppose that P, Q, R are points on the sides BC, CA, AB respectively of a given $\triangle ABC$. Suppose further that the centroid of $\triangle PQR$ coincides with the centroid of $\triangle ABC$. What, if anything, can be said about the placement of P, Q, R on the respective sides?*

Let us regard A as the origin of the coordinate system. Since P, Q, R lie on sides BC, CA, AB respectively, real numbers u, v, w exist such that (here we are using the name of each point to also denote its coordinates):

$$\begin{aligned} P &= uB + (1-u)C, \\ Q &= vC + (1-v)A, \\ R &= wA + (1-w)B. \end{aligned} \tag{1}$$

Since $A = 0$, this yields:

$$\begin{aligned} P &= uB + (1-u)C, \\ Q &= vC, \\ R &= (1-w)B. \end{aligned} \tag{2}$$

As the centroids of the two triangles coincide, we have

$$\frac{A+B+C}{3} = \frac{P+Q+R}{3}, \tag{3}$$

which yields:

$$B + C = uB + (1-u)C + vC + (1-w)B. \tag{4}$$

This yields, on simplifying:

$$(w-u)B = (v-u)C. \tag{5}$$

Now $(w-u)B$ is a point on ray $\overrightarrow{AB}$, while $(v-u)C$ is a point on ray $\overrightarrow{AC}$. The only point shared by these two rays is A. Hence $w - u = 0$ and $v - u = 0$, which yields $u = v = w$. Therefore: *P, Q, R divide the sides BC, CA, AB in the same ratio.*

Applying this result to the RMO problem, we see that $\triangle P_iQ_jR_k$ has the same centroid as $\triangle ABC$ if and only if $i = j = k$; i.e., if and only if the triangle is of the form $P_iQ_iR_i$. It follows that there are 10 such triangles. □

Remark. The shape of $\triangle ABC$ has absolutely nothing to do with the problem! So the information that the given triangle is equilateral is of no relevance. The same result would be true regardless of what shape the triangle has.

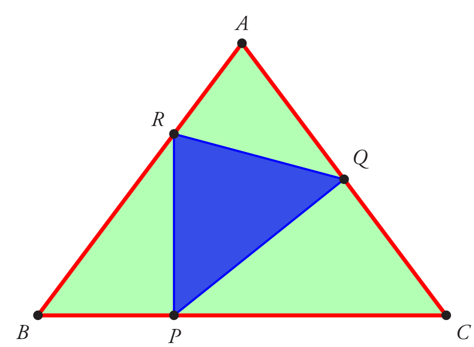


Figure 3

Solution to Problem 2.
We offer three different solutions to this problem! The first one, which uses trigonometry, is perhaps the most straightforward.

Trigonometric solution. Denote $\measuredangle KAM$ by x; then $\measuredangle MAD = x$ as well, and $\measuredangle BAK = 90 - 2x$ (see Figure 4). Let the side of square $ABCD$ have length a units.

We now have:

$$\begin{aligned} AK &= \frac{AB}{\cos(90-2x)} = \frac{a}{\sin 2x}, \\ BK &= a \tan(90-2x) = a \cdot \frac{\cos 2x}{\sin 2x}, \\ DM &= a \tan x. \end{aligned}$$

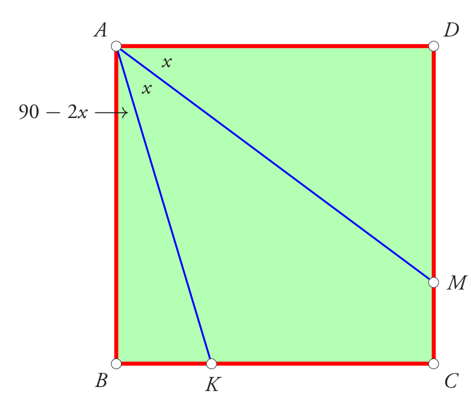


Figure 4

Hence the relation to be proved is the equivalent to the following trigonometric identity:

$$\frac{1}{\sin 2x} = \frac{\cos 2x}{\sin 2x} + \tan x.$$

This in turn is equivalent to the following:

$$\frac{1-\cos 2x}{\sin 2x} = \tan x,$$

$$\text{i.e.,}\quad \frac{2\,\sin^2 x}{2\,\sin x\,\cos x} = \tan x.$$

But this is immediate! Hence the stated relation is true. $\square$

Solution using coordinates. Our second solution makes use of coordinates. Let the side of the square have length 1 unit. Assign coordinates as shown to the right of the figure (see Figure 5).

The slopes of lines AK and AM are respectively:

$$\text{slope of } AK = \frac{0-1}{k-0} = -\frac{1}{k},$$

$$\text{slope of } AM = \frac{m-1}{1-0} = m-1.$$

Since the slope of AD is 0, we get (using the formula for the tangent of the angle between two straight lines):

$$\frac{(m-1)-(-1/k)}{1+(m-1)(-1/k)} = \frac{0-(m-1)}{1+0\cdot(m-1)},$$

which reduces to:

$$\frac{k(m-1)+1}{k-m+1} = -(m-1).$$

This yields an expression for k in terms of m:

$$2k(m-1) = (m-1)^2 - 1,$$

$$\therefore\quad k = \frac{m(2-m)}{2(1-m)}.$$

This is the condition relating k and m, if AM is to be the bisector of $\measuredangle KAD$. Now we must find what condition relates k and m if the condition $AK = BK + DM$ is to hold. We have:

$$AK = \sqrt{k^2+1},$$

$$BK = k,$$

$$DM = 1-m.$$

The condition $AK = BK + DM$ is equivalent to $AK^2 = BK^2 + DM^2 + 2\,BK\cdot DM$, i.e.,

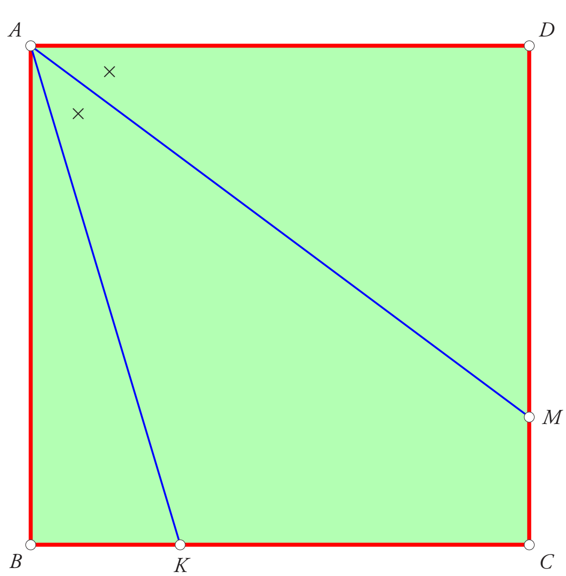
$$k^2 + 1 = k^2 + m^2 - 2m + 1 + 2k(1-m),$$

which simplifies to

$$2k(1-m) = 2m - m^2,$$

$$\text{i.e.,}\quad k = \frac{m(2-m)}{2(1-m)}.$$

Observe that we have obtained exactly the same condition as the one obtained earlier. We conclude that $AK = BK + DM$ if and only if AM bisects $\measuredangle KAD$. $\square$



$A = (0,1),\ B = (0,0)$

$C = (1,0),\ D = (1,1)$

$K = (k,0)$

$M = (1,m)$

$\measuredangle KAM = \measuredangle MAD$

Figure 5

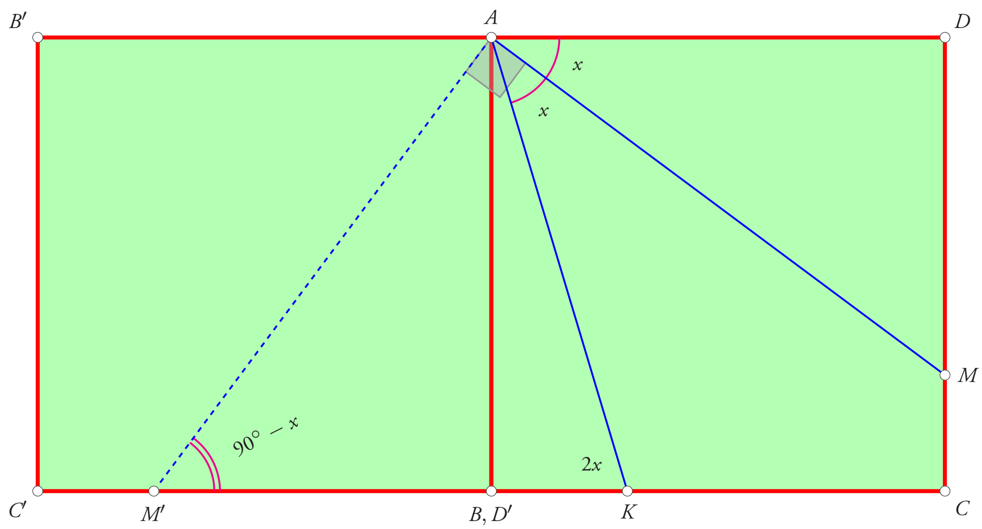


Figure 6

Pure geometry solution. We close by offering a third solution: using pure geometry. As always, such solutions are difficult to find but look extremely easy in hindsight. But if we are able to find such a solution, it constitutes an extremely pleasing discovery; for such solutions are also very elegant. The construction is shown in Figure 6.

We subject the entire configuration (i.e., square $ABCD$ together with segments AK and AM) to a $90°$ rotation centred at A and oriented in the clockwise direction. (Here we are assuming that the vertices of $ABCD$ have been labelled in a counterclockwise direction.) The rotation takes vertices B, C, D, K, M to B', C', D', K', M' respectively. Note that D' coincides with B. Hence $M'B = MD$. Denote $\measuredangle KAM$ by x, as in the trigonometric solution. An easy computation now reveals that $\measuredangle AKB = 2x$, and $\measuredangle M'AK = 90° - x$. This implies that $\measuredangle KM'A = 90° - x$ as well. Since $\measuredangle KM'A = \measuredangle M'AK$, we must have $AK = M'K$. Hence $AK = BK + DM$. $\square$

COMMUNITY MATHEMATICS CENTRE (CoMaC) is an outreach arm of Rishi Valley Education Centre (AP) and Sahyadri School (KFI). It holds workshops in the teaching of mathematics and undertakes preparation of teaching materials for State Governments and NGOs. CoMaC may be contacted at shailesh.shirali@gmail.com.

Review

Review: Liping Ma (1999)

KNOWING AND TEACHING ELEMENTARY MATHEMATICS

Lawrence Erlbaum Associates: New Jersey.
Anniversary Edition, 2010, Routledge: New York.

KAMALA MUKUNDA

In 2010, the Indian Government decided to take part in an international test called PISA (Programme for International Student Assessment). About 5000 fifteen-year-old students from 200 schools in Himachal Pradesh and Tamil Nadu took tests of reading and mathematical and scientific literacy—along with over a million students in 75 other countries! The results were worse than our worst nightmares: Indian students were literally at the bottom of the list. Nearly 90% of the Indian children who took the PISA were at or below what the PISA 2012 document describes as the 'lowest levels' of literacy and numeracy:

"...students can answer questions involving familiar contexts where all relevant information is present and the questions are clearly defined. They are able to identify information and to carry out routine procedures according to direct instructions in explicit situations. They can perform actions that are obvious and follow immediately from the given stimuli."

The Indian press were sharply critical of our government education system, but in its defence, the government said that PISA is linguistically and culturally biased against our government school children. There may be some truth to this[1], and yet, there were several other developing countries that also took part in the test and fared better, which makes it difficult to shrug off the results completely.

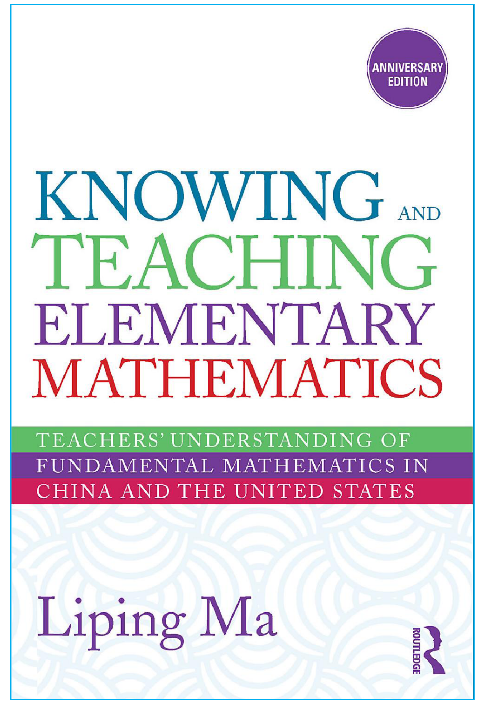
[1]You be the judge! Read the released 2012 mathematics test items and instructions for translation here: http://www.oecd.org/pisa/pisaproducts/pisa2012-2006-rel-items-maths-ENG.pdf

Keywords: PISA, Indian students, elementary school mathematics, teacher training, in-service training

Perhaps the outcome should not be surprising, since we are somewhat aware that we are failing to impart mathematical literacy to our students. Large-scale testing initiatives within India such as ASER and ASSET[2] tell similar stories of the poor learning levels of the majority of our country's children. The reasons range from the lack of quality in textbooks and infrastructure to inadequate teacher education. One thing is clear, though—right from a young age, children are not learning to **think mathematically**. Primary school mathematics, which is the foundation on which all further mathematics learning must rest, is itself not being taught and learned well.

Here is where Liping Ma's book **Knowing and Teaching Elementary Mathematics** comes in. It is a detailed, loving description of primary school teachers' thoughts about basic concepts in mathematics. It is also, by the way, a comparative account of Chinese and North American teachers, in which several Chinese teachers emerge as having a better understanding of the fundamentals. But Ma's aim is not to make a big deal of cross-national differences. Rather, the book is a revelation of how a really good primary school mathematics teacher *thinks* about her students' learning. Good teachers have what she calls a *Profound Understanding of Fundamental Mathematics* (PUFM), and the book provides plenty of examples to illustrate what that means. Ma uses an appealing analogy of the way people know the town they live in. Some newcomers know only their own home; some know a few places and a few fixed routes. But taxi drivers know all the roads extremely well and, "They are very flexible and confident when going from one place to another and know several alternative routes." A good teacher's PUFM is very much like the knowledge a good taxi-driver has of his town!

Ma's book is a distillation of in-depth interviews with primary school teachers in both countries.



She constructs four scenarios, each a topic in mathematics paired with a particular teaching process.

1. Subtraction with regrouping: how to approach the topic?
2. Multi-digit multiplication: how to deal with mistakes?
3. Division by a fraction: how to represent it in a story?
4. Perimeter and area: how to explore the relationship?

Ma closely analyses the teachers' responses, revealing how some ways of thinking have conceptual depth while others are more superficial, not moving beyond the procedural. You may protest—what is there beyond the procedures in primary school mathematics—it's only arithmetic after all! But, as Ma makes clear through the book, one of the most important assumptions that good teachers make is that elementary mathematics is vast, deep, and full of rich conceptual learning.

[2]Annual Status of Education Report (http://www.asercentre.org/) and Assessment of Scholastic Skills through Educational Testing (http://www.ei-india.com/introduction/)

The word *elementary* is not taken to mean *simple*, but rather *at the foundation of all further learning*. These teachers use what Ma calls **arithmetic with a reasoning system**, which takes the four operations (+, -, x and ÷) and all their associated algorithms, from the practical into the theoretical realm, from 'how' to 'why'.

For example, instead of using terms like carrying and borrowing, good teachers prefer to use the terms composing and decomposing bundles of ten units. One of them says:

"The term 'borrowing' does not mean the composing-decomposing process at all. 'Borrowing one unit and turning it into 10' sounds arbitrary. My students may ask me how we can borrow from the tens. If we borrow something, we should return it later on. How and what are we going to return?"

Before going into multi-digit multiplication, these teachers first make sure that students have understood place value and the distributive rule. Only then will the 'staircase' procedure of long multiplication make sense. For example, one teacher says that to multiply 123 x 645:

"I'd review place value and show them that those partial products you can separate them out, just multiply 123 times 5 and then 123 times 40 and then 123 times 600 and then add them all up."

They also said that they would first teach students how to multiply any number by 10, 100 and so on. This would be necessary to understand the way a long multiplication sum is written out.

Good teachers thus also think about the skills necessary before attempting a new lesson. When asked how they would approach subtraction with regrouping, they told Ma that they would first have to teach the children subtraction within 20:

"Given that my students do not have a firm grasp of problems within 20, how could they solve problems like 37–18=? and 52–37=? Whenever they follow the algorithm, they will face problems like 17–8=? and 12–7=? Are we going to rely on counting sticks all the time? All the subtraction procedures in problems with bigger numbers, after all, are transformed into subtraction within 10 and within 20."

Misconceptions are generally avoided by Ma's good teachers. They don't tell their students that a larger number cannot be taken from a smaller number, because this is not true. They don't tell them that the smaller number can borrow from its neighbour, because that implies that the two are independent numbers, rather than two parts of one number.

Ma's good teachers also demonstrated multiple ways of solving the same problem. Alternative ways were often faster than using standard procedures. Remember the taxi driver who can choose flexibly among several alternative routes? Ma writes, "The reason that one problem can be solved in multiple ways is that mathematics does not consist of isolated rules, but connected ideas." Subtraction with decomposing was connected with addition with composing; division was connected with multiplication. According to her, teachers with PUFM have transcended algorithms to reach the essence of an operation.

In contrast, her interviews with teachers who did not have this kind of understanding either showed a lack of confidence, or a misplaced confidence! Here are two such responses to the question of how to divide 1 ¾ by ½.

"For some reason it is in the back of my mind that you invert one of the fractions. Like, you know, either 7/4 becomes 4/7, or ½ becomes 2. I am not sure."

"You could be using pie, a whole pie, one, and then you have three fourths of another pie and you have two people, how will you make sure that this gets divided evenly, so that each person gets an equal share."

*****

Ma's own story is quite unusual, and moving. She was sent away from the city to rural China as part of the Cultural Revolution when she was just thirteen years old. The local school needed a teacher and she was asked to step in. For seven years she taught everything to the children of that village! Later she returned to Shanghai and studied education formally, ending up in the US for a PhD in mathematics education. In the anniversary edition of her book, published in 2010, she says, "Deep in my memory is the image of eyes—the bright eyes of my students in the rural area of South China where, as a teenager, I became a teacher. Whether they belong to Chinese or U.S. children, young students' eyes revealing a desire for learning have set the direction of my work."

The foreword to Ma's book is written by her PhD advisor, educational psychologist Lee Shulman. His praise for the book is glowing, and one of the main reasons is worth going into a little bit. Shulman's own work focussed on what he believed was a **false content-pedagogy divide**. In a 1986 classic paper, he explains how teacher education in the US has shifted emphasis from content to pedagogy. He gives the example of a typical test item in an 1875 teacher licensing examination for elementary school certification: *Divide 88 into two such parts that shall be to each other as 2/3 is to 4/5.* Not a trivial question! Today, teacher education is much more heavily tilted towards *"how teachers manage their classrooms, organize activities, allocate time and turns, structure assignments, ascribe praise and blame, formulate the levels of their questions, plan lessons, and judge general student understanding…"*

Current teacher education programmes in India assume that content knowledge has been covered in whatever basic education the student-teacher has gained. But surely it is extremely important to deepen her knowledge of mathematics *in order to be a better teacher*? Otherwise, as Shulman asks, "...how does he or she employ content expertise to generate new explanations, representations, or clarifications? What are the sources of analogies, metaphors, examples, demonstrations, and rephrasings? How does the novice teacher (or even the seasoned veteran) draw on expertise in the subject matter in the process of teaching?"

In his paper, Shulman quotes two great thinkers on what defines a teacher. One is Bernard Shaw: "He who can, does. He who cannot, teaches." The other is Aristotle: "What distinguishes the man who knows from the ignorant man is an ability to teach."

When great minds disagree so fundamentally, we are forced to think for ourselves! Is content more important, or pedagogy? Ma might actually refuse to answer this question; in fact she doesn't even use the word pedagogy in her book. She seems to see content and pedagogy as inextricable parts of the 'whole' of being a great teacher. Chinese elementary school teachers have studied arithmetic with a reasoning system while at school themselves. When they become teachers, their classrooms do not look pedagogically 'cutting-edge'. Students sit in rows, the textbook is very much in use, and the teacher sets the agenda for class. However, one also sees in these classrooms a clear focus on conceptual understanding, and enthusiastic student participation. In progressive classrooms, where students are working in small groups and using concrete materials, there is no guarantee that conceptual learning is going on. As Ma says, *"The real mathematical thinking going on in a classroom, in fact, depends heavily on the teacher's understanding of mathematics."*

Ma's good teachers keep that learning frame of mind strongly throughout their teaching career. When she asks how they build upon their understanding, they describe four key things:

- **Lesson planning:** *"I always spend more time on preparing a class than on teaching, sometimes three, even four times the latter."*

- **Learning from colleagues:** *"I am the eldest one and have taught for the longest time, yet...my young colleagues...are usually more open-minded than I am in their ways of solving problems."*
- **Learning from students:** *"The little ones have surprised me so many times...I had never thought that [the problem] could be solved in so many different ways."*
- **Learning by doing math:** *"To improve myself, I first of all did in advance all the problems which I asked my students to do."*

These certainly inspire me to re-examine my 20-year-old teaching practice!

The final chapter is devoted to a clear and insightful analysis of teacher education, taking examples from the US and China. We in India can find ourselves in several of her descriptions. A strong message from this section is that if a textbook is excellent, it serves as a script for the teacher, because *"in China, teaching a course is considered to be like acting in a play."* The teacher is not expected to write or rewrite the script, but can and *must* explore creativity in the enactment of it. But of course, we need good playwrights as well as good actors—textbook writing then becomes a supremely important activity.

Within a few years of its publication in 1999, Ma's book became such a runaway bestseller that in 2010 an anniversary edition was brought out, including a few additional sections that frame the educational context of the book in more recent terms. The original was a collector's item; this edition is a must for any educational library. Ma's examples come from the primary school, but mathematics teachers *at all levels* can learn the principles of how we must approach the fundamental and interconnected mathematical understanding inherent in what we teach. It also stands as a counter-example to a common assumption: 'If a book has a great deal of conceptual depth, it must be difficult to read!' The language and clear organisation of chapters allow the reader to move quickly while understanding a great deal.

For several years now, my primary interest has been in educational psychology, but I have also taught mathematics and statistics from primary to high school. This book is, for me, the perfect confluence of these two loves. The strong message I hope any reader will take away from it is that of humility as a teacher of this beautiful subject. As one of Ma's teachers says: "It is easy to be an elementary school teacher, but it is difficult to be a good elementary school teacher." QED.

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1. Shulman, L S (1986). Those Who Understand: Knowledge Growth in Teaching. Educational Researcher, Vol. 15, No. 2, pp. 4-14.

The Closing Bracket . . .

We are living in chaotic times indeed. Forces of totalitarianism and fascism are on the rise in all corners of the world. People are displaying behaviour patterns that we would not have dreamt of a little while back. Truth is turning out to be stranger than fiction, but in a particularly unpleasant manner. Against such a backdrop, one must ask what the significance of education is, and in particular what the significance of mathematics education is. The question acquires a particular bite when one realises that it is the so-called advanced nations that are at the forefront of much of the violence in the world, and when one realises additionally how closely the scientific establishment is involved in that violence. What is one to make of such a situation? How is one to respond to it?

Mathematics at its essence is the exploration, understanding and expression of pattern. To come across a pattern in any form is a delight; to explore the pattern, and then to find the same pattern occurring in a perfectly natural way in a different context is a source of wonder.

There is another aspect of mathematics. In mathematics we also explore implications and consequences of postulates and assumptions. We explore patterns of implication and check how one statement leads to another. In doing so, we often find instances of statements which seem at the outset to have nothing to do with each other to be in fact closely related. It is surely one of the deep joys of mathematics to discover instances of this kind.

But is this all that mathematics is? Can mathematics not enter into 'real living'? I am not referring here to so-called Applied Mathematics. Mathematicians like to quote how instances of pure mathematics enter the fields of industry and application; one example that is always being quoted in such discussions is the RSA algorithm in public key cryptography. But that is not what I am talking about here. I am referring to the entry of precise and accurate thinking into our living, and to the entry of honesty and accuracy and an uncompromising demand for truth in daily life. If this does not happen, it is contradictory to the very essence and spirit of mathematics.

I came across the following quotation the other day: *People must understand that science is inherently neither a potential for good nor for evil. It is a potential to be harnessed by man to do his bidding* (Glenn T. Seaborg). This is of course true – but in a completely trivial sense: it is human beings who are responsible for mischief, not mathematics or science. Taking refuge behind such thinking – this non-recognition of our responsibility – has helped bring about what we see around us. As mathematicians and as mathematics teachers, we need to take a greater interest in the world and in the ways of the human mind (which means our own minds); we need to take a greater responsibility for what we are doing to each other and to nature. This does not merely mean taking a greater interest in politics and gossip! It means taking a deep interest in understanding how our minds operate, in understanding why we are so susceptible to the whims and fancies of political leaders and business leaders and religious leaders, in understanding how groups of people drive their hidden agendas. We need to talk about these things in our classes. We need to talk with our students about the nature of dogma and prejudice, about how unquestioned assumptions lie at the root of conflict. By not doing so, we neglect an important responsibility and thus help maintain a cruel status quo.

Shailesh Shirali

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2. Title the article with an appropriate and catchy phrase that captures the spirit and substance of the article.

3. Avoid a 'theorem-proof' format. Instead, integrate proofs into the article in an informal way.

4. Refrain from displaying long calculations. Strike a balance between providing too many details and making sudden jumps which depend on hidden calculations.

5. Avoid specialized jargon and notation — terms that will be familiar only to specialists. If technical terms are needed, please define them.

6. Where possible, provide a diagram or a photograph that captures the essence of a mathematical idea. Never omit a diagram if it can help clarify a concept.

7. Provide a compact list of references, with short recommendations.

8. Make available a few exercises, and some questions to ponder either in the beginning or at the end of the article.

9. Cite sources and references in their order of occurrence, at the end of the article. Avoid footnotes. If footnotes are needed, number and place them separately.

10. Explain all abbreviations and acronyms the first time they occur in an article. Make a glossary of all such terms and place it at the end of the article.

11. Number all diagrams, photos and figures included in the article. Attach them separately with the e-mail, with clear directions. (Please note, the minimum resolution for photos or scanned images should be 300dpi).

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13. Include a high resolution photograph (author photo) and a brief bio (not more than 50 words) that gives readers an idea of your experience and areas of expertise.

14. Adhere to British spellings – organise, not organize; colour not color, neighbour not neighbor, etc.

15. Submit articles in MS Word format or in LaTeX.

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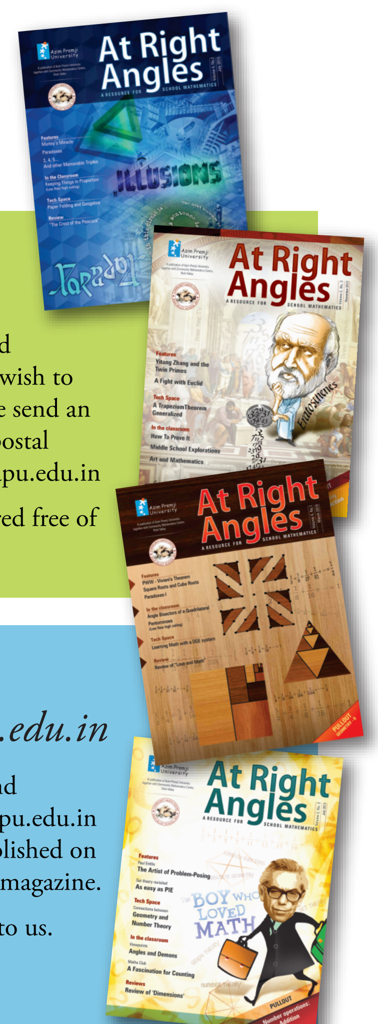

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A Resource for School Mathematics

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TEACHING TIME

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Azim Premji University

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TIME

Introduction

Time is one concept that is used by human beings on a daily basis, and perhaps in an instinctive way by animals and plant life too! The cock knows when to go 'cock-a doodle-doo'. Flowers know when to open their petals. Trees know when to shed their leaves.

Children get exposure to the concept of time organically, well before they come to school.

Yet the teaching and learning of time poses peculiar challenges. Time is an abstract concept. It involves measurement of something that is invisible and intangible. Consequently, challenges arise in understanding the concept of time. Learning the mechanics of reading time can also be a difficult task. It is therefore important to build the scaffolding carefully and match the activities to the children's level of understanding.

Here are some challenges which arise in the mechanics of reading and recording time:

- Reading of time on an analogue clock: The hour hand on the analogue clock does two complete turns in a day. This can be conceptually difficult to understand. Children also need to register the direction of the turn (clockwise). There needs to be an understanding of half turn and quarter turn. The fact that the numbers on the clock are written as 1 to 12 but read as multiples of 5 (to count the minutes) increases the difficulty. It becomes necessary to ignore the second hand which can be a source of distraction to children.
- Conversion of one unit into another: Time conversions do not follow the metric system. An hour is 60 minutes, a minute is 60 seconds. While a week is fixed at 7 days, a month could be 30 or 31 days. The number of days in February varies.
- Multiple ways in which time is described: 12:30 can be 'half past twelve'; 9:55 can be '5 minutes to 10'; 5:45 can be 'quarter to 6'. It takes time for children to understand the equivalences of these statements.
- Conversion from 24 hour clock to 12 hour clock and vice versa: AM and PM have to be taken note of while using a 12 hour clock.

Apart from the above, it takes a long time to develop a sense of time: of a second, a minute, five minutes, ten minutes, one hour. Perhaps many adults lack it too. When somebody says, "I will be back in 5 minutes", we all know that it could extend to 30 minutes!

Keywords: Measure, time, unit, activities, timer, non-standard, standard, clock, analogue, digital

At the outset one needs to take note of the fact that all time activities should be related to the personal experiences of students. They should relate to real life situations.

TEACHING TIME TO 3 to 5 Year olds

Objectives:

- Use and understand vocabulary related to time (daytime, night time, yesterday, today, tomorrow, morning, afternoon, a long time, a short time);
- Use after and before with respect to an event;
- Sequence events in their daily routine;
- Associate events with yesterday, today and tomorrow, and sequence them;
- Become familiar with the passage of time.

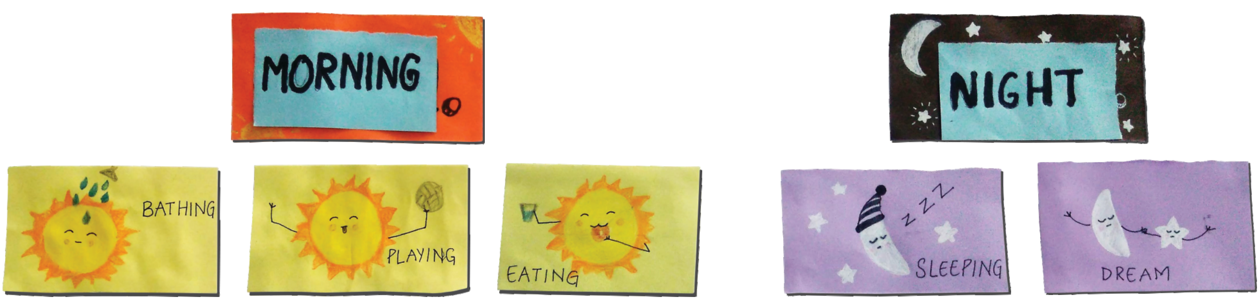
Three-year olds who enter nursery school already associate time with their daily routine. They associate brushing and bathing with morning time, lunch and nap with the afternoon, outdoor play with evening, and darkness with bed time. But one cannot assume that they are familiar with time related vocabulary. Children may use a word like yesterday to refer to an event which happened weeks ago. They may not have a sense of time and prepare to go home before the school day is over.

Development of time related vocabulary will help students sequence events of the day.

Teachers should frequently refer to the time as well as the passage of time in their conversations: “It is 9 o’clock now; assembly time.” “We have games for one hour in the afternoon.”

ROLE PLAY ACTIVITY

Young children love to enact role play! They can be divided into two groups. One group can be asked to act out daytime actions (bathing, going to school, riding a cycle). Another group can be asked to act out night-time actions (going to bed, watching TV, reading a book).

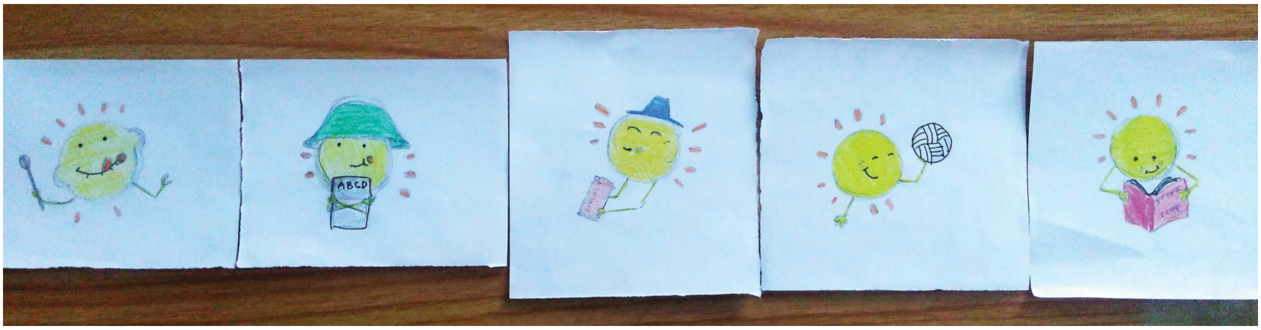


Children could follow up the role-play by making drawings of these activities. The teacher can paste these on a chart paper making two categories (daytime and night-time activities).

In the same way, the children can draw pictures for morning activities (assembly and particular classes) and afternoon activities (lunch and games) which happen in the school.

EVENT TRAIN ACTIVITY

Frequent usage of 'after' and 'before' by the teacher with respect to events helps children to sequence activities. "After assembly we have story time." "Before games we have a music class."



The teacher can help the children make a set of cards which depict the activities which children perform over one day in the school: assembly, storytelling, recess, art, lunch, games, ... They can sequence these cards and fix them on the bulletin board.

Similarly, the teacher can ask the children to illustrate the activities they perform at home and help them to sequence them in the right order.

NEWS TIME ACTIVITY

Teacher can use the time immediately after assembly for sharing information and personal stories. It can be used to familiarise children with the usage of 'yesterday', 'today' and 'tomorrow' through various contexts. "Yesterday we had our music class. Did you like the song you learnt?" "Today is library day. Let us borrow a book on dogs." "Tomorrow is the indoor games day."

Help children to relate stories about their lives. "What game did you play yesterday?" Children may talk about some event which they remember from the previous day. "I bought a toy yesterday." "My dog jumped on me yesterday." "I went to my aunt's house yesterday."

Sequences can also be prepared for yesterday, today and tomorrow, naming the events specific to those days. Children can then display this information on the bulletin board.

ACTION CARDS ACTIVITY

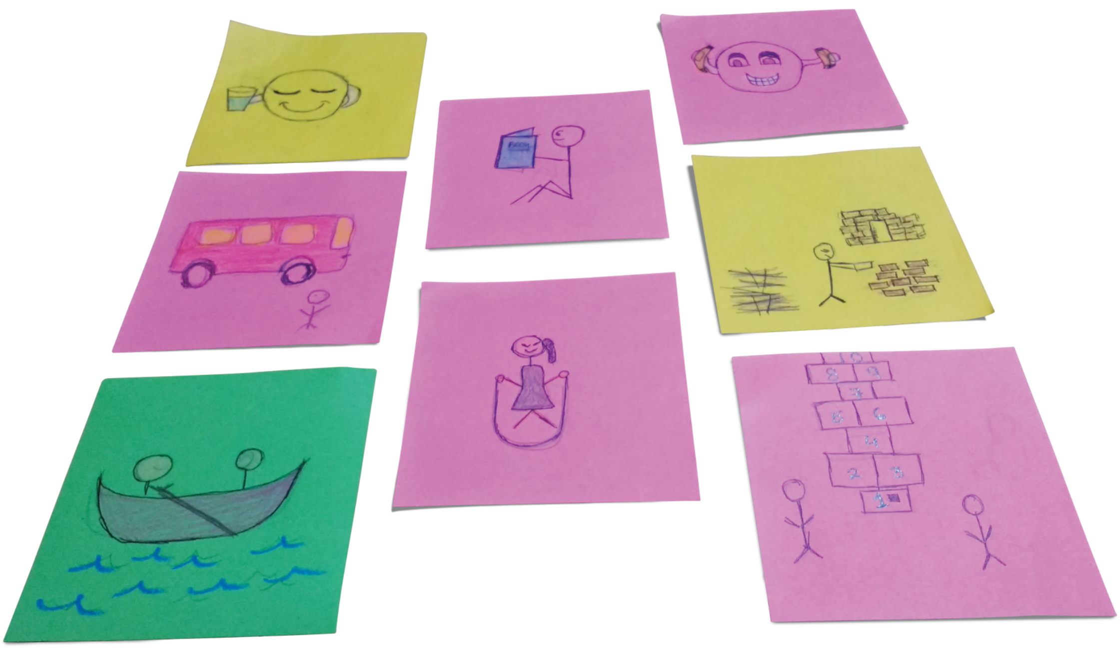
Create a set of picture cards showing activities which take a short time and which take a long time.

Prior to the activity, help children develop an understanding of spans of time (a short time, a long time).

Pose questions:

> "Which takes a longer time – brushing teeth or having a bath?"
>
> "Tell me some things you do in school which take a long time."
>
> "Can you string 100 beads in a short time?"

Prepare a set of action cards and let children sort the cards under two categories: long time and short time. Follow it up with a discussion.



WALK AND TALK ACTIVITY

Teacher can take the children for a walk in the school area. Talk about time and the passage of time while visiting places.

> "It is 9 o'clock now. Let us first go to the assembly."
>
> "We spent ten minutes at the assembly. We will now go to the Library and spend an hour there."

TEACHING TIME TO

5 to 7 Year olds

Objectives:

- Learn phrases such as 'day before yesterday' and 'day after tomorrow';
- Sequence events of one day and events over many days;
- Learn the days of the week and order them;
- Learn the months of the year and order them;
- Become familiar with the calendar;
- Read and write the time to the hour;
- Understand that durations of events can be compared;
- Learn how to measure time using non-standard units;
- Learn how to say 'half an hour' as 'half past';
- Learn how to say 'quarter past an hour' (quarter to an hour is challenging);
- Read and write the time corresponding to the half hour and quarter hour.

DISPLAY BOARD ACTIVITY

DAY BEFORE YESTERDAY	YESTERDAY	TODAY	TOMORROW	DAY AFTER TOMORROW
SUNDAY DECEMBER 18	MONDAY DECEMBER 19	TUESDAY DECEMBER 20	WEDNESDAY DECEMBER 21	THURSDAY DECEMBER 22

Create a display board in the class for illustrating phrases and words such as day before yesterday, yesterday, today, tomorrow, and day after tomorrow, to keep track of the days and the date. Children can relate this display to the calendar. They could also enter any special event of that day.

GROWING UP ACTIVITY

Ask children to bring photographs of their baby days at various years. Let them sequence the photographs and put them up on the bulletin board. Have a discussion about growing up.

"Find out at what age you walked, at what age you talked." As a side activity it is great fun to play a guessing game using the photos. "Who is this baby?"

DIARY ACTIVITY

Help children create a small diary for the month. Each day they can enter the date and the day at the top by referring to the calendar. They could record any special events (classmates' birthdays, festivals, school functions, and interesting classes) that happened on that day.

CLAP! ACTIVITY

Choose some actions which children perform routinely – like tying one's shoe laces, solving a puzzle, building a tower. Children can work in pairs. As one child performs the action, the other claps and counts the number of claps it takes to time the action.

There will be variations in the way children clap and that can give rise to a good discussion on finding other ways of measuring time.

CLASS MADE CLOCKS ACTIVITY

Children can create non-standard time measuring units like sand timers using plastic bottles, or make candle clocks with equally spaced notches. These can be used to measure the durations of many activities.

Children of this age love to jump, to hop, do flips. They will have great fun trying to see how many flips they can perform during the time the sand runs out.

WHICH TAKES A SHORTER TIME? ACTIVITY

Choose any three activities which can be performed in the class. Examples: solving a puzzle, erasing the board, sharpening a pencil. Let children start all three activities at the same time. Using measuring devices like the sand timer, they can find out which activity takes the least time, and which one takes the longest time.

12 HOUR CLOCK FOR DEMONSTRATION ACTIVITY

The teacher should use a demonstration clock (preferably one that is large) and make the children familiar with the hour hand and minute hand. In the initial stages, let the focus be only on the hour hand, with the minute hand fixed at the '12' position. Let the children know that the hour hand tells us the hour of the day and that when it is an exact hour, the minute hand points to '12'. Without going into the way the minute hand works, one can tell children the purpose of the minute hand – that it indicates the time between two hours.

LIVE CLOCK! ACTIVITY

It is great fun for children to act out a clock. A group of 14 children can demonstrate a live clock. A large circle is drawn on the floor. Mark the centre. Children sit around the circle in places corresponding to the hours 1 to 12. Each child holds up a placard to show the number. The remaining two children, preferably one taller than the other, act out the hour hand and the minute hand. The child demonstrating the minute hand lies flat from the centre pointing to '12'. This may need some explanation on the part of the teacher. When the teacher calls out 'morning break time' the hour hand child lies flat from the centre pointing to '10'. Onlookers (other children) can verify whether the clock is working properly!

By age seven, as the lessons progress, children can also demonstrate half past 5, quarter past 4, quarter to 10, etc. The child acting out the minute hand will need to point to the appropriate place. While introducing half past an hour, it is important to repeatedly stress on the position of the minute hand. In a similar manner, children can later be shown quarter past and quarter to an hour.

PLATE CLOCKS ACTIVITY

While learning how to read time, it is good for children to have their own paper clocks. (Thermocol plates are available in the market but they are not environmentally friendly; it is best to avoid them.) Teachers can guide children to prepare clocks with paper plates. A paper clip and strips made from stiff card paper can be used to show the hour hand and minute hand. The strips should be movable. Help them to first mark 12, 3, 6 and 9 on the clock. They should be able to write the other numbers after that. At this point the clock will have marks only for the hours. It will not have the finer lines drawn between consecutive hours.

The teacher can use a real clock to show how the short hand points to the hours. She can name some activity, say breakfast time, and let the children turn the hour hand to the appropriate hour. Many activities can be tried out to give practice to the children.

It would be also interesting to do this in the reverse way. One child can turn the hour hand to a certain time. Other children can name the activity that they do at that time.

As the teacher slowly turns the hour hand from one hour to the other, children can call out the sequence of the day's activities.

HOW OLD ARE OUR PLANTS? ACTIVITY

This activity will help build a sense of how long a week is.

Children can plant some seeds in a pot and keep track of their growth in terms of time. They can make drawings of the plant to show how it looks in the first week, the second week, the third week, etc. They can record the week in which the first bud appears, the day it flowers, the day it sheds a leaf, etc.

TEACHING TIME TO

7 to 9 Year olds

Objectives:

- Read and write the time to the minute;
- Understand the purpose of the seconds hand;
- Understand the relationship between analogue and digital clock;
- Read time from a 12 hour clock and a 24 hour clock;
- Understand the need to use AM and PM;
- Understand that the day is divided into hours;
- Develop a sense of durations such as one minute, one second, one hour;
- Understand the relationship of hour to minutes and minutes to seconds;
- Understand the usage of calendar and leap year;
- Register the number of days in each month;
- Understand the sequencing of dates on a number line.

READING HALF HOURS ACTIVITY

Teacher should use a real clock for teaching half hours and quarter hours. Children should be able to see the complementary action of the hour and minute hands. Help the children to notice and verbalise the observation that when the minute hand makes a half turn of the circle from 12 and reaches 6, the hour hand is halfway between two hours. Repeat this several times resetting the clock each time.

Let children guess at some activities that take about half an hour to complete. Children can now use their paper clocks to show timings specified by the teacher. Check that they have turned the minute hand to the correct place.

READING QUARTER HOURS AND THREE QUARTER HOURS ACTIVITY

In just the same way, demonstrate 'quarter turn' and let children observe and verbalise the observation that when the minute hand makes a quarter turn from 12 and reaches 3, the hour hand is one quarter the way between two hours. When the minute hand makes a three quarter turn from 12 and reaches 9, the hour hand is three quarter the way between two hours.

SKIP COUNTING IN 5'S ACTIVITY

To develop understanding of the minute hand, let the children count time in steps of 5 (0, 5, 10, 60) as the teacher turns the demonstration clock from 12 back to 12.

This can be repeated in various ways for practice. Help the children to read from 3 to 9 (15 to 45) or between any other pair of numbers so that they begin to associate each number with a certain multiple of 5. This can be repeated in the reverse way. 60, 55, 50…

Once the children have mastered this, the teacher can use a real clock to read time.

It is important not to rush children while they are learning these skills. It is a slow process and needs repeated reinforcement and practice over the year.

READING TIME TO THE MINUTE ACTIVITY

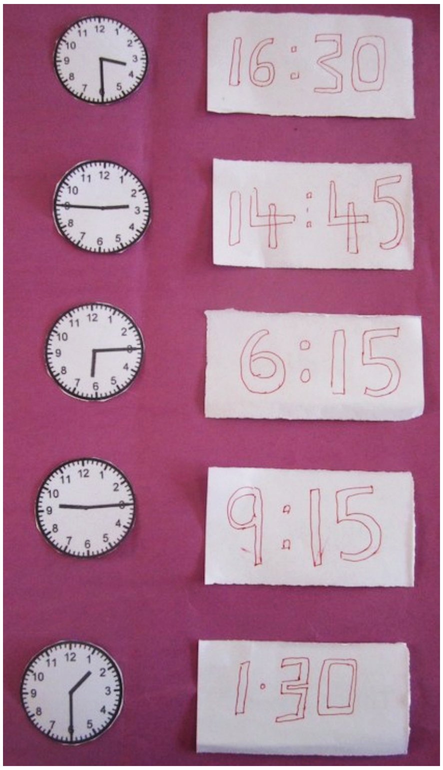
The teacher uses a real clock to draw children's attention to the small markings (minutes) between 12 and 1, between 1 and 2, etc. Practice can now be given to help them read to the minute.

ANALOGUE AND DIGITAL CLOCKS ACTIVITY

At this point it will be necessary to have both digital and analogue clocks in the class to show how time is displayed in digital clocks as compared to analogue clocks. In particular children need to observe how the display changes in the digital clock when the minute advances to 59. Also while reading analogue clocks children need to understand that 0 is used as a placeholder, e.g., when the time is 2:05. It is read as 'two-oh-five'. Students must be given regular practice both with analogue and digital clocks.

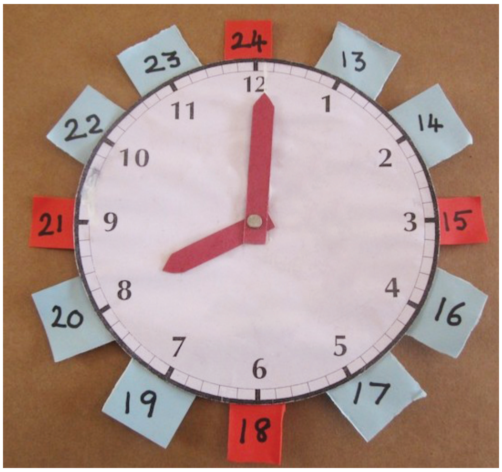
Let children observe both the clocks to see how the change in the digits of the digital clock corresponds to the change in the minute hand of the analogue clock.

GAME : SNAP! (CAN BE PLAYED BY A GROUP OF STUDENTS)



Prepare a set of matching cards (10 pairs) which show analogue clocks with time and digital representation of these. Children should open two cards at a time. If they match, whoever says 'Snap!' first gets a point. If they are wrong, they are out of the game. The winner is the one who gets maximum points.

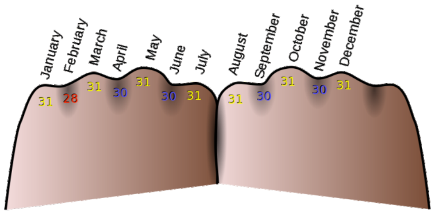
24 HOUR CLOCK ACTIVITY



Build another ring of numbers around a clock face to show the p.m. hours in 24 hour time.

Help children to understand that the inner numbers are used to read time from midnight to noon and the outer numbers are used to read from noon to midnight.

REGULAR CALENDAR ACTIVITY



The classroom should have a real calendar which shows all the months on one page. Discussion can happen around the calendar itself. Calendars have many interesting patterns which are worth exploring.

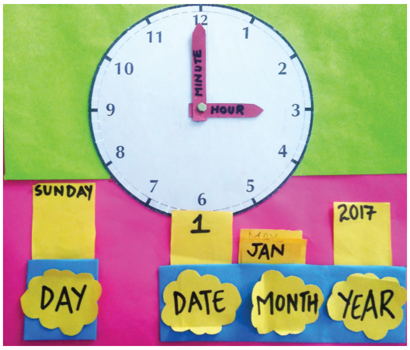
Ideally, calendars should be used throughout the year. While teaching months of the year one should ensure that they should be related to the children's world. Children carry vivid memories of summer holidays, rainy season, festivals like Diwali and kite flying. Association between these festivals and months should be made.

Number of days in the month can be taught by making children use the two fists together. From left to right, the knuckles and valleys (leaving out the thumbs) are matched with months. Those which match with knuckles have 31 days while the others have 30 days, except for February. The children will also need to be taught that in a leap year, February has 29 days.

SCHOOL CALENDAR! ACTIVITY

Creating a school calendar at the beginning of each month can be an exciting educational activity for children. Children can draw the grid on a chart paper, label the columns, and enter the dates in the correct space using a ready-made calendar as reference. They can now mark the birthdays of their classmates, any school events, and favourite activity days.

POCKET CALENDAR ACTIVITY



Let children design a pocket calendar for the class bulletin board which can hold information about the date and day. This can be changed every day. It is much more readable than a regular calendar and helps to build an understanding of the sequence of days, months and dates.

SENSE OF MINUTE, SENSE OF A SECOND AND TEN SECONDS, SENSE OF AN HOUR ACTIVITY

Ask students: 'How long is a minute?'

'What can you do in a minute?'

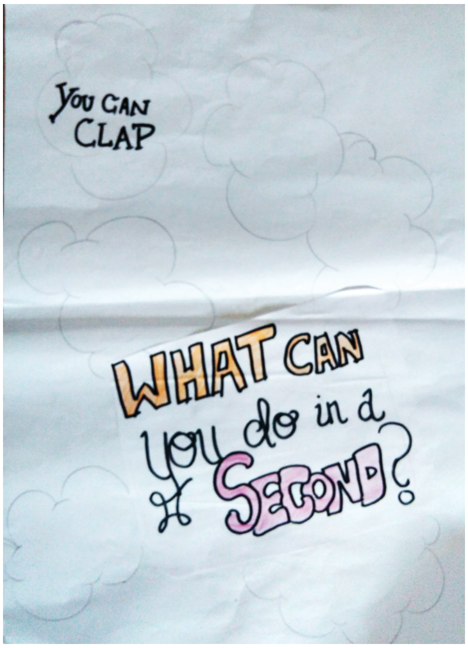
"Can you jump ten times in a minute?'

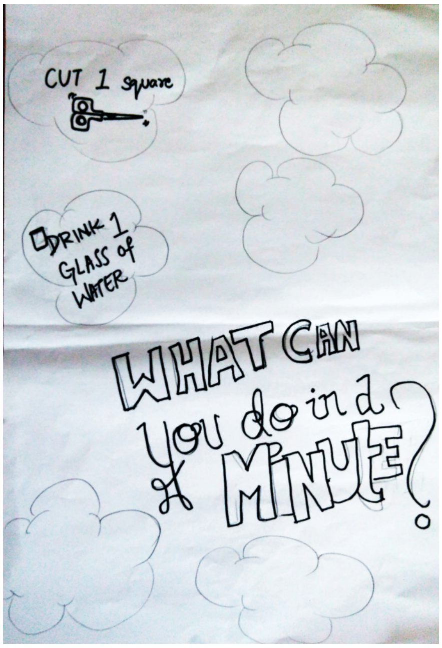
'Can you run 50 metres in a minute?'

In a similar manner, let children check the number of seconds it takes to count numbers, say from hundred and one to hundred and ten. They can try to guess how many actions (e.g., push ups) they can do in ten seconds. Students can make a guess first and then try out the action.

By repeatedly doing such activities children will begin to develop a sense of a minute, half a minute, ten seconds, etc. Most teaching periods vary between 40 minutes to one hour. Associating an hour with a class period helps children to build a sense of an hour. They can be asked to guess the time taken to travel from home to school, the length of their favourite program on TV, the time they spend on games, etc.

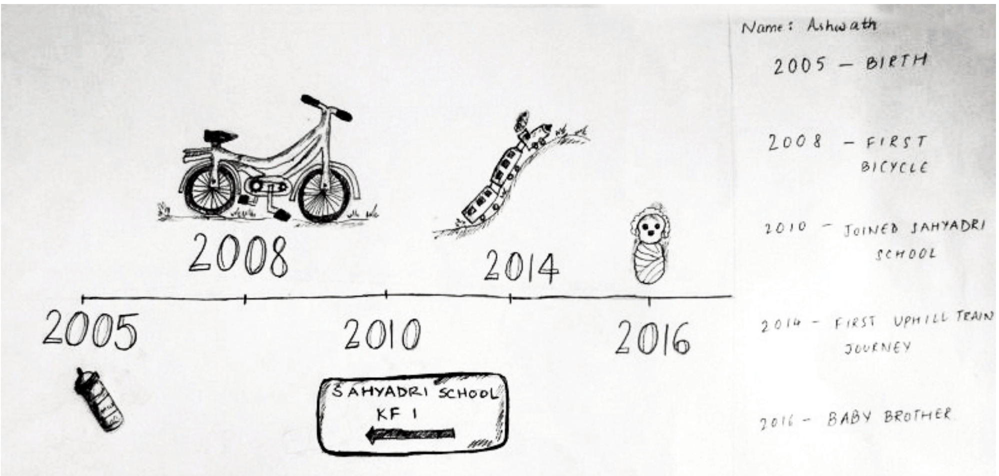
CHARTS FOR SECOND, MINUTE, HOUR ACTIVITY





Children can now make charts for each of these and write down the different things they can do in a second, in a minute. They can make a chart for an hour with three categories: things that can be done in less than an hour, that take an hour and that take more than an hour.

MY STORY! ACTIVITY

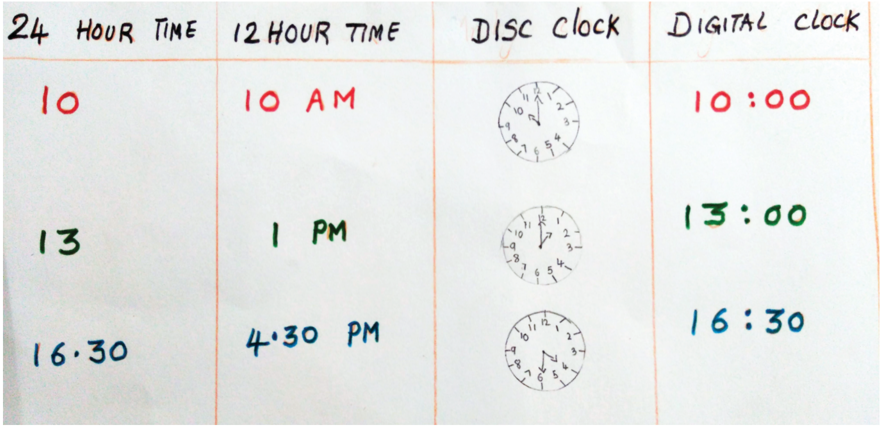


Children can build a personal timeline from their birth till now. They can record when they joined school, other schools they attended or any other major event in their lives (birth of a younger sibling, or a prize won)

CONVERSIONS ACTIVITY

Students can now be taught the relationship between hour and minutes, minute and seconds, day and hours.

Discuss usage of AM and PM as before midday and post midday and the need for using it. Often children get confused about writing AM and PM for noon and midnight. It is best to teach children at this age to write 12 as '12 noon' or '12 midnight'.



Create a table with four columns as shown, to convert time from one form into another:

24 hour clock, 12 hour clock, analogue clock, digital clock

Time estimations can be done for various activities and then measured.

COLLECTING CLIPS FROM NEWSPAPERS SHOWING USAGE OF TIME ACTIVITY

Ask children to bring cuttings of items in newspapers which show time. (TV shows, weather charts, bus timings).

Use them in the class to talk about time interval, comparing time intervals, longest program, and shortest program). While calculating time interval between two timings children can use the counting forward approach.

Example: 12:15 to 2:30 can be calculated as 12:15 to 1:00 (45 minutes), 1:00 to 2:00 (one hour) and 2:00 to 2.30 (thirty minutes). That makes two hours and fifteen minutes.

CLASS TIMETABLE ACTIVITY

Children can enter the timings for the classes on the class timetable. They can work out the times spent on various classes, hobbies, games, etc. How many minutes of physical education do we have each week? How long does our music lesson last?

DAILY TIME CHART ACTIVITY

Let the children create a personal timetable of their day. Each one can be asked to total the time they spend in studies, on games, on watching TV, in reading books, in taking food, in cleanliness, etc. Discussion about this can lead to efficient planning of activities to make better usage of time.

REFLECTIONS!

While teaching this topic, it is necessary and worthwhile to help children to reflect on different aspects of time in their lives. Here are some possible questions they can engage in:

- What is my favourite time of the day? Why?
- Why is it important to put events in the correct sequence?
- When was I late? Why? Why are some people always late?
- Why is it important to be on time? When do I have to be on time?
- Why is it important to measure time accurately?
- How do I spend my free time? What else can I do in my free time?
- How can I use my time in a better way?

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Padmapriya Shirali is part of the Community Math Centre based in Sahyadri School (Pune) and Rishi Valley (AP), where she has worked since 1983, teaching a variety of subjects – mathematics, computer applications, geography, economics, environmental studies and Telugu. For the past few years she has been involved in teacher outreach work. At present she is working with the SCERT (AP) on curricular reform and primary level math textbooks. In the 1990s, she worked closely with the late Shri P K Srinivasan, famed mathematics educator from Chennai. She was part of the team that created the multigrade elementary learning programme of the Rishi Valley Rural Centre, known as 'School in a Box'. Padmapriya may be contacted at padmapriya.shirali@gmail.com