Azim Premji University

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Rishi Valley

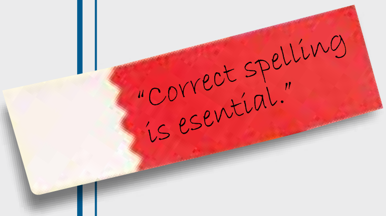
At Right Angles

A RESOURCE FOR SCHOOL MATHEMATICS

Volume 4, No.2
July 2015

PULLOUT
MEASUREMENT

THE CHARM OF PARADOXES



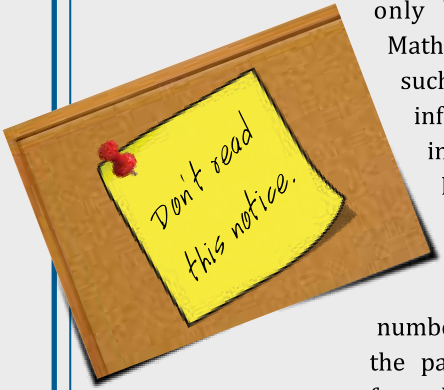
"Correct spelling is esential."

Paradoxes have been around from the most ancient of times, and yet continue to be present in our times, baffling and amusing us. At the same time, they often point to fundamental truths of life. (Perhaps the most hard-hitting of these is this one: *All men are equal, but some are more equal than others.*) You surely will have your own favourites.

"The GOLDEN rule is that there are no GOLDEN rules."

George Bernard Shaw

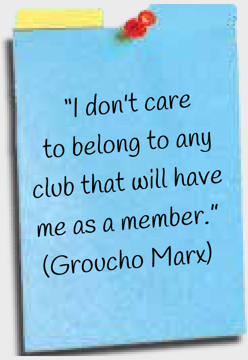
There are paradoxes of different kinds. Some lie strictly within the contours of mathematics. Everyone has seen the 'proof' that 1=2, or that every triangle is equilateral. Though these are labelled as paradoxes, they typically derive from a mathematically illegal step or an incorrectly drawn diagram. Precisely because of this, they have great value pedagogically. Then there are paradoxes in which the conclusion of a carefully constructed line of reasoning seems absurd, but this is only because our common sense is unable to accept the conclusion. Mathematically, there is nothing paradoxical about the conclusion. A well-known such Paradox is Zeno's Paradox which derives from the notion that if you add an infinite number of quantities, the sum must be infinite (that's what common sense tells us). In more recent times, we have the Banach-Tarski Paradox, in which from a given object, we produce two copies of that same object by dividing it into a large number of pieces and rearranging the pieces, and the paradoxes concerning Infinity which derive from the work of Cantor. Note the feature that all these paradoxes share: the notion of Infinity.



Don't read this notice.

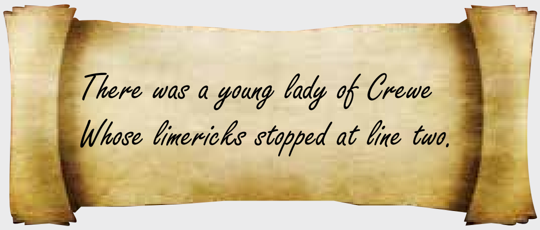
Down with bumper stickers.

And then there are paradoxes which seem genuinely impossible to resolve. Such is certainly the case with paradoxes involving self-reference. Well-known examples are the Barber's Paradox, the Russell Paradox in set theory, and the Epimenides Paradox. The underlying logic here is of an utterly simple nature, yet the conclusion frustrates us. Other such paradoxes which we find similarly frustrating are those involving time travel. (We do not possess the technology for time travel today, but imagine that human beings of the future – say a century from now – master such technology. Would they be able to move backwards in time, and come and visit us in our time and teach us the technology for time travel? If so, it would lead to a most delicious paradox!)



"I don't care to belong to any club that will have me as a member." (Groucho Marx)

All in all, paradoxes offer us a most wonderful subject. Let us treat them as gifts from the gods and delight in their study.



There was a young lady of Crewe
Whose limericks stopped at line two.

From the Editor's Desk . . .

'A popular mathematics magazine' seems to be a paradoxical phrase, but here is our tenth issue and with a growing subscriber list, we seem to be both popular and mathematical! Aptly enough, Punya Misra and Gaurav Bhatnagar wrap up their two part series on Paradoxes; selected visuals from these are featured on the cover too. Part III of Morley's Miracle by V.G. Tikekar and some stunning insights into the appearances of the 3-4-5 triangle by Shailesh Shirali complete the *Features* section.

In the *ClassRoom* is packed with goodies for the practicing teacher. Majid Sheikh describes some Tests for Divisibility by Powers of 2 and CoMaC pitches in with an article on Prime Generation and an explanation for one of the patterns reported in a post from 'AtRiUM' – our FaceBook page. Swati Sircar and Sneha Titus continue the Low Floor, High Ceiling activity series and Swati Sircar also provides an addendum to the Pullout on Division. Shailesh Shirali pulls a family of circles out of his mathematician's hat and also continues the series on 'How To Prove It'.

Paper Folding and Dynamic Geometry software have blended seamlessly in *Tech Space* where Swati Sircar begins a series on Conic sections. This section has a bonus this time – a GeoGebra investigation of the Rectangle Problem. It's time to say Open Sesame at your math lab sessions!

Prithwijit De, R. Athmaraman and CoMaC give us more fodder for *Problem Corner*. This time we feature R. Ramanujam's review of the book by George Joseph Gheverghese, 'The Crest of the Peacock'. Padmapriya Shirali wraps up the issue with a pullout on Measurement.

In the last issue, we had featured a challenge problem for readers and we are delighted to publish a solution sent in by reader Tejash Patel. A question on how we teach the rule of 'negative times negative is positive' which was thrown open to readers also sparked off several responses. This issue publishes an abbreviated version of some of these.

It's been an interesting three years and we appreciate this journey with you, our loyal readers!

— Sneha Titus
Associate Editor

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At Right Angles is a publication of Azim Premji University together with Community Mathematics Centre, Rishi Valley School and Sahyadri School (KFI). It aims to reach out to teachers, teacher educators, students & those who are passionate about mathematics. It provides a platform for the expression of varied opinions & perspectives and encourages new and informed positions, thought-provoking points of view and stories of innovation. The approach is a balance between being an 'academic' and 'practitioner' oriented magazine.

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Contents

Features

This section has articles dealing with mathematical content, in pure and applied mathematics. The scope is wide: a look at a topic through history; the life-story of some mathematician; a fresh approach to some topic; application of a topic in some area of science, engineering or medicine; an unsuspected connection between topics; a new way of solving a known problem; and so on. Paper folding is a theme we will frequently feature, for its many mathematical, aesthetic and hands-on aspects. Written by practising mathematicians, the common thread is the joy of sharing discoveries and the investigative approaches leading to them.

In the Classroom

This section gives you a 'fly on the wall' classroom experience. With articles that deal with issues of pedagogy, teaching methodology and classroom teaching, it takes you to the hot seat of mathematics education. 'In The Classroom' is meant for practising teachers and teacher educators. Articles are sometimes anecdotal; or about how to teach a topic or concept in a different way. They often take a new look at assessment or at projects; discuss how to anchor a math club or math expo; offer insights into remedial teaching etc.

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Contents contd.

Tech Space

'Tech Space' is generally the habitat of students, and teachers tend to enter it with trepidation. This section has articles dealing with math software and its use in mathematics teaching: how such software may be used for mathematical exploration, visualization and analysis, and how it may be incorporated into classroom transactions. It features software for computer algebra, dynamic geometry, spreadsheets, and so on. It will also include short reviews of new and emerging software.

Problem Corner

Reviews

Pullout

feature

Lurking within any triangle …

Morley's Miracle – Part III

…is an equilateral triangle

This article concludes the three-part series begun in the July 2014 issue, wherein we study one of the most celebrated theorems of Euclidean geometry: Morley's Miracle. In this segment we examine an unusual proof due to Professor John H Conway.

V G TIKEKAR

In Part I of this article we narrated the history of this theorem and discussed a pure geometry proof (M. T. Naraniengar's). We remarked that the proof *starts* with an equilateral triangle and then proceeds to construct a configuration similar to the original one; thus it reaches the desired conclusion.

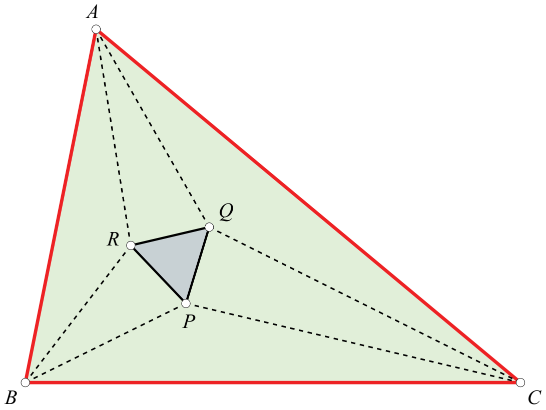


Figure 1. Morley's theorem: The angle trisectors closest to each side intersect at points which are the vertices of an equilateral triangle

Keywords: *Angle trisector, equilateral triangle, congruent, sine rule, backward proof*

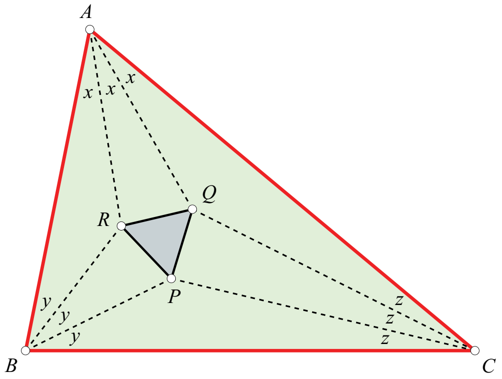


Figure 2.

We added that many of the pure geometry proofs known today proceed in just this way.

Now, in the concluding piece of this three-part series, we give another such proof; this one has sprung from the fertile mind of Professor John Conway [2]. (See https://en.wikipedia.org/wiki/John_Horton_Conway for information on this remarkable individual.) It may well be the most unusual of all the proofs of Morley's theorem. (Actually, our proof is a slight adaptation of Conway's proof.)

Given $\triangle ABC$, let angles x, y, z be defined by $A = 3x$, $B = 3y$ and $C = 3z$ (see Figure 2). We shall assume henceforth that all angles are measured in degrees, so that $x + y + z = 60$.

Conway starts by introducing the following operation for angles. Let θ be any angle (measured in degrees, of course). Then he defines θ^+ to be the angle $\theta + 60$ and θ^{++} to be the angle $\theta^+ + 60 = \theta + 120$. So a triangle exists with angles of $0^+, 0^+, 0^+$: it is an equilateral triangle. In the same way we can assert that:

- A triangle exists with angles x, y^+, z^+; for, $x + y^+ + z^+ = 180$.

  Similarly, a triangle exists with angles x^+, y, z^+, and a triangle exists with angles x^+, y^+, z.

- A triangle exists with angles x^{++}, y, z; for, $x^{++} + y + z = 180$.

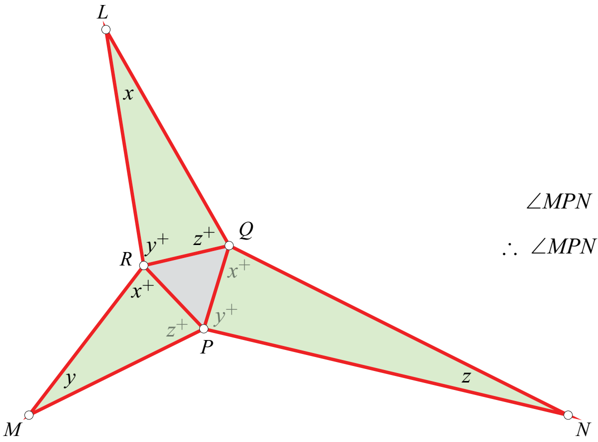
  Similarly, a triangle exists with angles x, y^{++}, z, and a triangle exists with angles x, y, z^{++}.

Conway starts by constructing an equilateral triangle PQR with side 1 unit (Figure 3). Then he constructs:

- On side PQ as base: $\triangle PQN$ with angles y^+, x^+, z at vertices P, Q, N respectively;
- On side QR as base: $\triangle QRL$ with angles z^+, y^+, x at vertices Q, R, L respectively;
- On side RP as base: $\triangle RPM$ with angles x^+, z^+, y at vertices R, P, M respectively.

Each of these is a legitimate triangle, in the sense that the prescribed angles add up to 180. Each one is uniquely fixed both in shape and size.

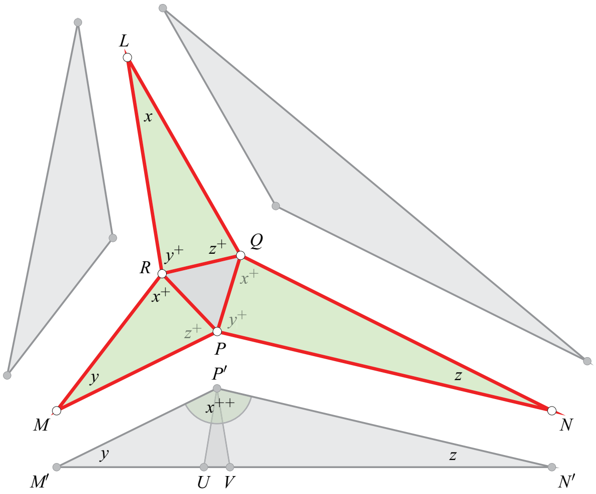
The computation in Figure 3 shows that $\angle MPN = x^{++}$. This fact allows us to insert into angle MPN a triangle $P'M'N'$ with angles x^{++}, y and z. (We have noted earlier that there does exist a triangle with these angles, as the angles do add up to 180.) But we need to fix the size of the triangle first. We do this as follows.



$$\angle MPN = 360 - 60 - y^+ - z^+$$

$$\therefore \ \angle MPN = x^{++}$$

Figure 3.



- $\angle M'P'N' = x^{++}$
- $\angle P'M'N' = y$
- $\angle P'N'M' = z$
- $\angle P'VM' = x^+$
- $\angle P'UN' = x^+$

Figure 4.

Let the candidate triangle $P'M'N'$ be drawn as shown, with its prescribed angles (see Figure 4). Next, let rays $P'U$ and $P'V$ be drawn from P' such that $\angle M'P'V = z^+$ and $\angle N'P'U = y^+$. Let these rays intersect the side $M'N'$ at U and V respectively. Then $\angle P'VM' = 180 - y - z^+ = x^+$ and $\angle P'UN' = 180 - z - y^+ = x^+$. Note that this makes $\triangle P'UV$ isosceles, with $P'U = P'V$. Now we fix the scale of the triangle so that $P'U$ and $P'V$ have the same length as the side of the equilateral triangle PQR. This is clearly possible.

With this in place, we consider $\triangle MPR$ and $\triangle M'P'V$. They are clearly congruent to each other ('ASA congruence'), as they have the same sets of angles, and the sides opposite angle y have equal length; hence $MP = M'P'$. In just the same way we have $NP = N'P'$ (consider $\triangle NPQ$ and $\triangle N'P'U$). Hence when we insert $\triangle M'P'N'$ into angle MPN, the fit is exact: $M'P'$ lines up with MP; $N'P'$ lines up with NP; and $M'N'$ lines up with MN.

The same kinds of actions can be repeated on the other two sides of the triangle: we insert into angle NQL and angle LRM triangles of suitable size, which then match up exactly with the spaces occupied by the angles. (See Figure 4. We have not named the triangles to avoid a visual clutter.) With these three triangles thus in place, the seven triangles together make up triangle LMN, whose angles at L, M and N are $3x$, $3y$ and $3z$. This means that $\triangle LMN$ is similar to the given $\triangle ABC$ (they have the same sets of angles). Moreover, the lines LQ and LR trisect $\angle MLN$; the lines MR and MP trisect $\angle LMN$; and the lines NP and NQ trisect $\angle LNM$. So the trisectors of the angles of $\triangle LMN$ give rise to an equilateral triangle, and it follows that the same must be true of $\triangle ABC$, just as Morley's theorem asserts. This proves the theorem.

Another presentation of Conway's proof

Conway's proof can be presented in a different way. See Figure 5. Consider $\triangle PMN$. Since $\angle MPN = x^{++}$, it follows that $\angle PMN + \angle PNM = y + z$.

Now we consider the ratio $PM : PN$ in $\triangle PMN$. We compute the ratio via $\triangle MPR$ and $\triangle NQP$:

$$\frac{PM}{PN} = \frac{PM/PR}{PN/PQ} = \frac{\sin\ x^+/\sin\ y}{\sin\ x^+/\sin\ z} == \frac{\sin\ z}{\sin\ y}.$$

It follows that

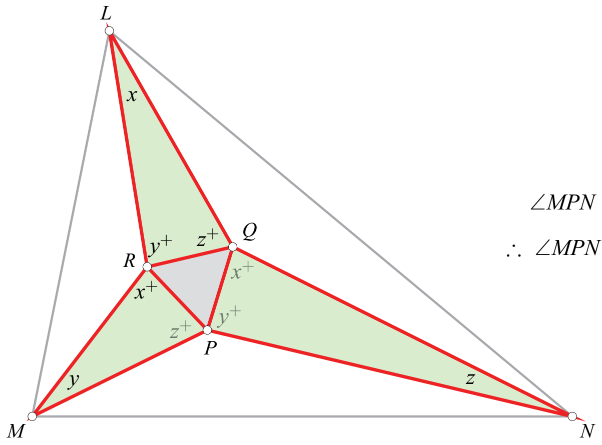
$$\frac{\sin \angle PNM}{\sin \angle PMN} = \frac{\sin\ z}{\sin\ y}.$$

We also know that $\angle PNM + \angle PMN = z + y$. From these relations we may conclude that

$$\angle PNM = z, \qquad \angle PMN = y.$$

(This may seem intuitively clear but it needs justification. Let $\angle PMN = u$, $\angle PNM = v$, and let $w = y + z$. Then $w = u + v$ too, and $\sin u : \sin v = \sin y : \sin z$. We now have:

$$\frac{\sin u}{\sin v} = \frac{\sin y}{\sin z},$$



$\angle MPN = 360 - 60 - y^+ - z^+$

$\therefore\ \angle MPN = x^{++}$

Figure 5.

$$\therefore\ \frac{\sin(w-v)}{\sin v} = \frac{\sin(w-z)}{\sin z},$$

$$\therefore\ \sin(w-v)\sin z = \sin(w-z)\sin v,$$

$$\therefore\ (\sin w\cos v - \sin v\cos w)\sin z = (\sin w\cos z - \sin z\cos w)\sin v,$$

$$\therefore\ \cos v\sin z = \cos z\sin v \quad (\text{since } \sin w \neq 0),$$

$$\therefore\ \sin(z-v) = 0,$$

which yields $v = z$ and hence $u = y$ as well. The reader could look for different ways of arguing this out.)

In just the same way we find that $\angle QNL = z$, $\angle QLN = x$, $\angle RLM = x$, $\angle RML = y$. We conclude, as earlier, that $\triangle LMN$ is similar to $\triangle ABC$; LQ and LR trisect $\angle MLN$; MR and MP trisect $\angle LMN$; and NP and NQ trisect $\angle LNM$. So the trisectors of the angles of $\triangle LMN$ give rise to an equilateral triangle, and the same must be true of $\triangle ABC$. This proves Morley's theorem.

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[1] Bankoff, L. "A simple proof of the Morley theorem", Math. Mag., **35** (1962) 223–224

[2] Conway, J. H. "Proof of Morley's Theorem", http://www.cambridge2000.com/memos/pdf/conway2.pdf

[3] Coxeter, H. S. M. "Introduction to Geometry", 2nd ed., Wiley, New York, 1969

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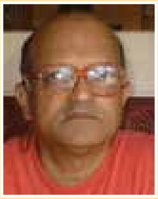

PROF. V.G. TIKEKAR retired as the Chairman of the Department of Mathematics, Indian Institute of Science, Bangalore, in 1995. He has been actively engaged in the field of mathematics research and education and has taught, served on textbook writing committees, lectured and published numerous articles and papers on the same. Prof. Tikekar may be contacted at vgtikekar@gmail.com.

Of Art and Mathematics

Paradoxes:

Part 2 of 2

PUNYA MISHRA & GAURAV BHATNAGAR

This is not the first sentence of this article.

The above sentence can be both true and false. It is clearly the first sentence of *this* article. So it is false, because it says it is not the first sentence! But because this is part 2 of our article on Paradoxes, if we regard both parts as one article, it is true! We leave it to you to resolve this paradox.

In the first part of this two-part exposition on paradoxes in mathematics, we introduced the idea of self-reference, the nature of mathematical truth, the problems with circular proofs and explored Zeno's Paradox. In this part we delve deeper into the challenges of determining the 'truth value' of pathological self-referential statements, visual paradoxes and more.

Self - Reference and Russell's Paradox

There is a class of paradoxes that arise from objects referring to themselves. The classic example is Epimenides Paradox (also called the Liar Paradox). Epimenides was a Cretan, who famously remarked "All Cretans are liars." So did Epimenides tell the truth? If he did, then he must be a liar, since he is a Cretan, and so he must be lying! If he was lying, then again it is not the case that all Cretans are liars, and so

Keywords: *Paradox, Circular proof, Zeno's paradoxes, Russell's paradox, Epimenides liar paradox, Self-reference, Contradiction, Escher, Penrose, Jourdain's paradox, Triangle, Necker's Cube*

Figure 1. An ambiguous design that can be read as both "true" and "false."

he must be telling the truth, and that cannot be! Figure 1 is an ambiguous design that can be read as both "true" and "false."

The artwork of M.C. Escher (such as his famous illustration that shows two hands painting each other) provides many visual examples of such phenomena. Another older analogy or picture is that of the *ouroboros*—an image of a snake eating its own tail (how's *that* for a vicious circle!). An ambigram of *ouroboros* was featured in our first article on paradoxes.

Here is another variation of the Liar Paradox. Consider the following two sentences that differ by just one word.

> This sentence is true.
>
> This sentence is false.

The first is somewhat inconsequential – apart from the apparent novelty of a sentence speaking to its own truth value.

The second, however, is pathological. The truth and falsity of such pathologically self-referential statements is hard to pin down. Trying to assign a truth value to it leads to a contradiction, just like in the Liar Paradox. Figure 2 is a rotational ambigram that reads "true" one way and "false" when rotated 180 degrees.

Figure 2: Rotational ambigram that reads "False" one way and "True" the other. (This design was inspired by a design by John Longdon.)

A variant of this (that does not employ self-reference) is also known as the Card paradox or Jourdain's paradox (named after the person who developed it). In this version, there is a card with statements printed on both sides. The front says, "The statement on the other side of this card is TRUE," while the back says, "The statement on the other side of this card is FALSE." Think through it, and you will find that trying to assign a truth value to either of them leads to a paradox!

Figure 3 combines the liar's paradox and Jourdain's paradox (in its new ambigram one-sided version) into one design.

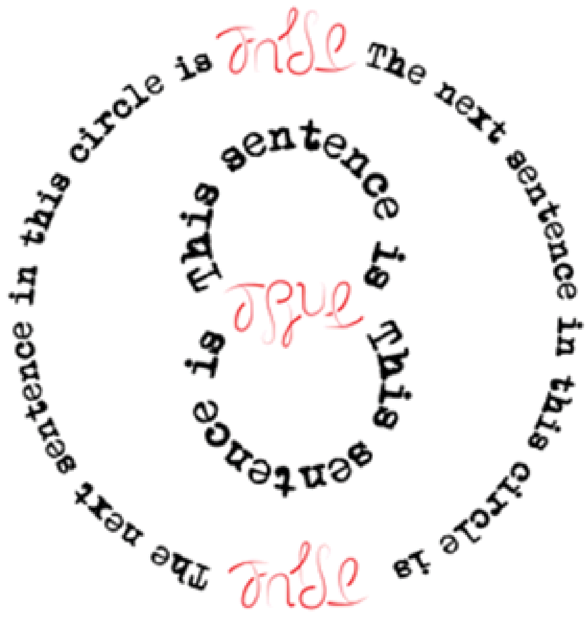

Figure 3: Two paradoxes in one. Inside the circle is the ambigram for the pair of sentences "This sentence is True/This sentence is False". The outer circle is an original design for the Jourdain two-sided-card paradox, which can, due to the magic of ambigrams, be reduced to being printed on just one side!

Another interesting example is the sentence: "This sentence has two ewrrors." Does this indeed have two errors? Is the error in counting errors itself an error? If that is the case, then does it have two errors or just one?

What is intriguing about the examples above is that they somehow arise because the sentences refer to themselves. The paradox was summarized in the mathematical context by Russell, and has come to be known as Russell's paradox. Russell's paradox concerns sets. Consider a set R of all sets that do not contain themselves. Then Russell asked, does this set R contain itself? If it does contain itself, then it is not a member of R. But if it is not a member of R, then it does contain itself.

Russell's Paradox was resolved by banning such sets from mathematics. Recall that one thinks of a set as a *well-defined* collection of objects. Here by well-defined we mean that given an element *a* and a set *A*, we should be able to determine whether *a* belongs to *A* or not. So Russell's paradox shows that a set of all sets that do not contain themselves is not well-defined. By creating a distinction between an element and a set, such situations do not arise. You could have sets whose members are other sets, but an element of a set cannot be the set itself. Thus, in some sense, self-reference is not allowed in Set Theory!

Visual contradictions

Next, we turn to graphic contradictions, where we use ambigrams to create paradoxical representations.

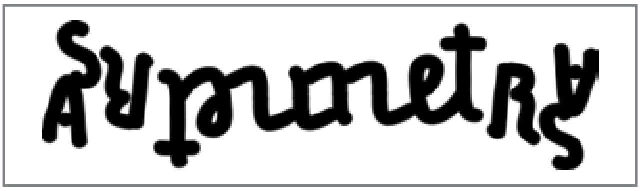

Figure 4: A somewhat inelegant design that captures a visual paradox – the word "asymmetry" written in a symmetric manner.

Figure 4 shows an ambigram for asymmetry, but it is symmetric. So in some sense, this design is a *visual contradiction*! But it is not a very elegant solution – which in some strange way is appropriate.

Recall the idea of self-similarity from our earlier column, where a part of a figure is similar to (or a scaled-down version of) the original. Here is an ambigram for similarity which is made up of small pieces of self (Figure 5). Should we consider this to be self-similarity?

Another set of visual paradoxes have to do with the problems that arise when one attempts to represent a world of 3 dimensions in 2 dimensions – such as in a painting or drawing. The Dutch artist M.C. Escher was the master at this. His amazing paintings often explore the paradoxes and impossible figures that can be created through painting. For instance, he took the mathematician and physicist Roger Penrose's image of an impossible triangle and based some of his work on it (Figure 6).

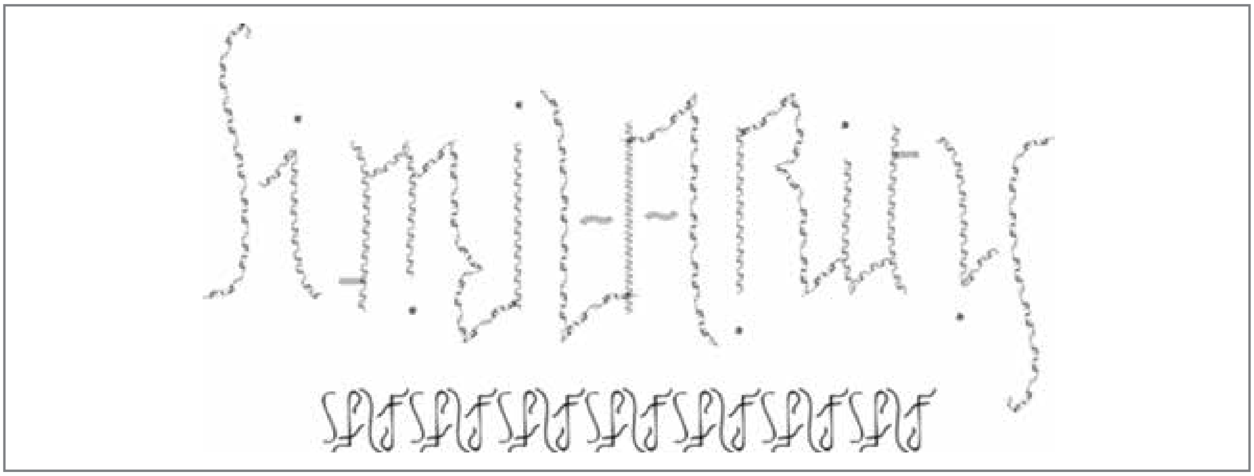

Figure 5: Here is an ambigram for Similarity which is made up of small pieces of Self. Should we consider this to be self-similarity?

Figure 6. A Penrose Triangle – a visual representation of an object that cannot exist in the real world.

As homage to M.C. Escher, we present below (Figure 7) a rotational ambigram of his name written using an impossible font!

Figure 7: Rotationally symmetric ambigram for M.C. Escher written using an impossible alphabet style.

As it turns out, the Penrose Triangle is also connected to another famous geometrical shape, the Möbius strip. A Möbius strip has many interesting properties, one of which is that it has only one side and one edge (Figure 8).

Puzzle: What is the relationship between a Penrose Triangle and a Möbius strip?

Figure 8. An unending reading of the word Möbius irrespective of how you are holding the paper!

Another famous impossible object is the "impossible cube." The impossible cube builds on the manner in which simple line drawings of 3D shapes can be quite ambiguous. For instance, see the wire-frame cube below (also known as the Necker Cube). This image usually oscillates between two different orientations. For instance, in Figure 9, is the person shown sitting *on* the cube or magically stuck to the ceiling *inside* it?

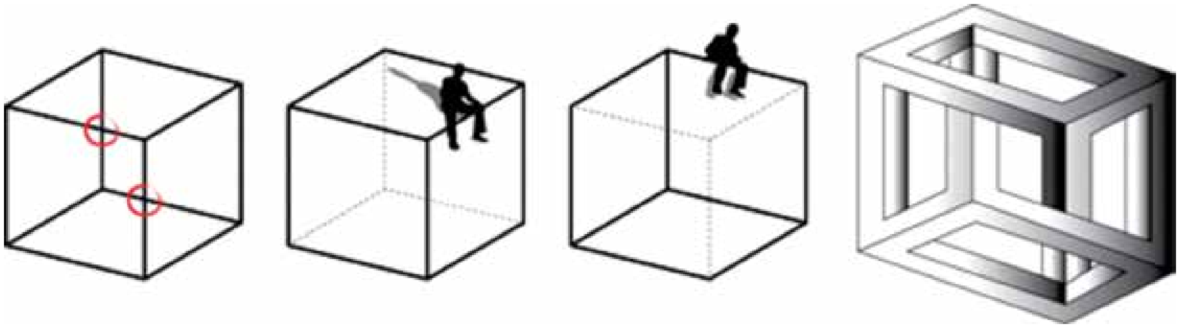

Figure 9. The Necker Cube – and how it can lead to two different 3D interpretations and through that to an impossible or paradoxical object.

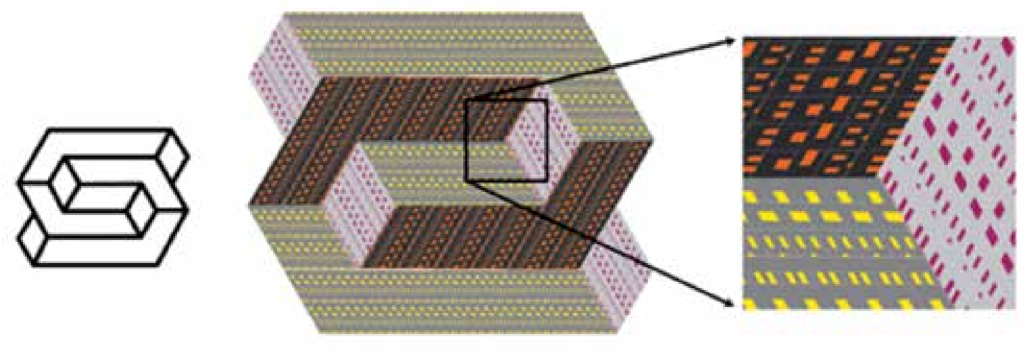

Figure 10: The impossible cube? In this design the word "cube" is used to create a series of shapes that oscillate between one reading and the other.

Figure 11: An impossible typeface based on the Necker Cube and Penrose Triangle. Spelling the word "Illusions."

These images oscillate between two opposite incommensurable interpretations, somewhat like the liar paradoxes we had described earlier. Figure 10 is another ambiguous shape that can be read two ways! What is cool about that design is that each of these shapes is built from tiny squares that read the word "cube."

These representations fool our minds to see things in ways that are strange or impossible. These are visual paradoxes, or illusions, as reflected in the design in Figure 11, which is the word "illusions" represented using an impossible font (akin to the Penrose Triangle or Necker Cube).

Mathematical Truth and the Real World

One of the most fundamental puzzles of the philosophy of mathematics has to do with the fact that though mathematical truths appear to have a compelling inevitability (from axiom to theorem via proof) and find great applicability in the world, there is little we know of why this is the case. The physicist Wigner called it the "unreasonable effectiveness of mathematics" to explain, understand and predict the phenomena in the real world. The question is how something that exists in some kind of an "ideal" world can connect to and make sense in the "real" world we live in.

Figure 12: The Ideal-Real ambigram, representing the paradoxical thought that the Real world often appears to be a reflection of the Ideal mathematical theory!

Figure 12 maps the word “ideal” to “real.” Is the ideal real – and real just a mere reflection of the ideal? Or vice versa?

Clearly this is not an issue that will be resolved anytime soon – but it is intriguing to think about.

So with that, we bid adieu, but before we depart we would like to bring you the following self-serving public announcement.

This is the last sentence of the article. No this is. This.

Answer to Puzzle:

The Möbius Strip and the Penrose Triangle have an interesting relationship to each other. If you trace a line around the Penrose Triangle, you will get a 3-loop Möbius strip. M.C. Escher used this property in some of his most famous etchings.

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Over the years, they have shared their love of art, mathematics, bad jokes, puns, nonsense verse and other forms of deep-play with all and sundry. Their talents, however, have never truly been appreciated by their family and friends.

Each of the ambigrams presented in this article is an original design created by Punya with mathematical input from Gaurav (except when mentioned otherwise). Please contact Punya if you want to use any of these designs in your own work.

To you, dear reader, we have a simple request. Do share your thoughts, comments, math poems, or any bad jokes you have made with the authors. Punya can be reached at punya@msu.edu or through his website at http://punyamishra.com and Gaurav can be reached at bhatnagarg@gmail.com and his website at http://gbhatnagar.com/.

3, 4, 5 ...

And other memorable triples – Part I

What's interesting about the triple of consecutive integers $3, 4, 5$*? Almost anyone knows the answer to that: we have the beautiful relation* $3^2 + 4^2 = 5^2$*, and therefore, as a consequence of the converse of Pythagoras' theorem, a triangle with sides* $3, 4, 5$ *is right-angled.*

Shailesh Shirali

It is easy to show that $(3, 4, 5)$ is *the only triple of consecutive integers which can serve as the sides of a right-angled triangle*. But in fact rather more can be said, which also makes the matter that much more interesting:

Theorem 1. *Let* $n > 1$ *be an integer. Then the triangle with sides* $n, n+1, n+2$ *is obtuse-angled for* $n = 2$*; right-angled for* $n = 3$*; and acute-angled for all* $n > 3$*.*

The statement is depicted in Figure 1. To see why the claim made in the theorem is true, we examine the expression $n^2 + (n+1)^2 - (n+2)^2 = n^2 - 2n - 3$, which conveniently factorizes as $(n+1)(n-3)$. From this we infer the following:

$$n^2 + (n+1)^2 - (n+2)^2 \text{ is } \begin{cases} < 0 & \text{for } n = 2, \\ = 0 & \text{for } n = 3, \\ > 0 & \text{for } n > 3. \end{cases}$$

The generalized version of Pythagoras' theorem now implies the stated result. (To refresh your memory, here is what this theorem asserts: *In* $\triangle ABC$*, the quantity* $a^2 + b^2 - c^2$ *is greater than, equal to, or less than* 0*, depending on whether* $\measuredangle C$ *is greater than, equal to, or less than a right angle.*)

Keywords: *Pythagoras, triple, acute, obtuse, consecutive integers, touching circles, trisection, in-radius*

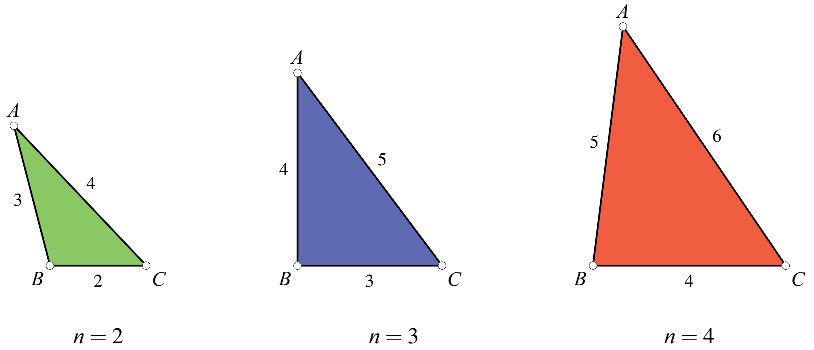


Figure 1. Triangle with sides n, $n + 1$ and $n + 2$

Remark 1. We may also express the above argument in terms of the cosine rule which states that in $\triangle ABC$, the cosine of the angle opposite side a is equal to $(b^2 + c^2 - a^2)/2bc$. Using this we find that in the triangle with sides $n, n + 1$, $n + 2$, the cosine of the largest angle (which will be opposite the largest side) is:

$$\frac{n^2 + (n + 1)^2 - (n + 2)^2}{2n(n + 1)} = \frac{n - 3}{2n}$$

(on simplification).

We see that the cosine of this angle is negative for $n = 2$, zero for $n = 3$, and positive for $n > 3$. The conclusion obtained is the same as earlier: the triangle is obtuse-angled for $n = 2$, right-angled for $n = 3$, and acute-angled for $n > 3$.

Remark 2. The condition $n > 1$ is needed so that the sides satisfy the triangle inequality: "Any two sides of a triangle are together greater than the third one." The inequality fails for $n = 1$, since $1 + 2 = 3$, and we get a 'flat' triangle with angles of 180°, 0° and 0°. (The above formula for the cosine shows that the cosines of the angles are -1, 1 and 1, corresponding to angles of 180°, 0° and 0°.) If the definition of obtuseness can be extended to cover such a triangle, then we do not need to include the condition $n > 1$; we could just say: "Let n be a positive integer."

Thus the triple $(3, 4, 5)$ has some pretty features. We now get a bit greedy and ask: *Are there other nice features that this triple has?* We find that it does, and in this article—which is the first in a multi-part series—we shall describe three such features.

In follow-up articles of the series we will ask: *Are there other triples of consecutive integers which possess geometric features of interest?* This is an open-ended question and many different kinds of results can be envisaged, depending on which "features of interest" we choose to examine. But of that, more later.

Three circles within a circle

In Figure 2 (a), we see a circle $\mathcal{C}_1$ with three circles within it, all tangent to it and also to each other. Two of them, $\mathcal{C}_2$ and $\mathcal{C}_3$, have half the size of $\mathcal{C}_1$ (and therefore pass through the centre O of $\mathcal{C}_1$). The remaining one, $\mathcal{C}_4$, fits tightly in one of the spaces enclosed by $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$.

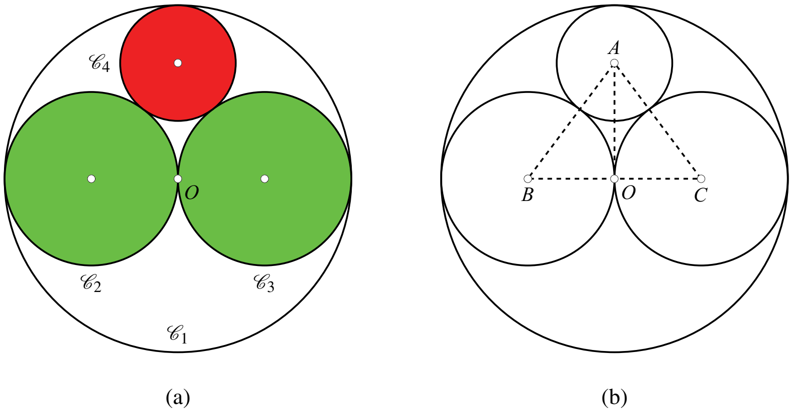


Figure 2. Finding the radius of $\mathcal{C}_4$

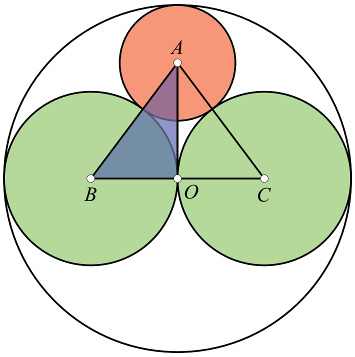


Figure 3. A 3-4-5 triangle hidden within the figure

Problem. To find the radius of $\mathcal{C}_4$.

Let A, B, C be the centres of $\mathcal{C}_4, \mathcal{C}_2, \mathcal{C}_3$, respectively. Note that B, C are collinear with O; A and O are collinear with the point of contact of $\mathcal{C}_1$ and $\mathcal{C}_4$; and A and B are collinear with the point of contact of $\mathcal{C}_2$ and $\mathcal{C}_4$. Let $\mathcal{C}_2$ and $\mathcal{C}_3$ have unit radius, and let the radius of $\mathcal{C}_4$ be x. Then, in Figure 2 (b), $\triangle ABC$ is isosceles with $AB = AC = x + 1$, and $BC = 2$. Also, the altitude $AO = 2 - x$.

Now we are in a position to apply the Pythagorean Theorem to the right-angled $\triangle AOB$; we get:

$$(x + 1)^2 = (2 - x)^2 + 1^2.$$

This simplifies to $6x = 4$, giving $x = 2/3$. Hence the radius of $\mathcal{C}_4$ is $1/3$ that of $\mathcal{C}_1$.

Now let us focus our attention on $\triangle AOB$. Its side lengths are the following:

$$AB = 1 + \frac{2}{3} = \frac{5}{3}, \quad AO = 2 - \frac{2}{3} = \frac{4}{3}, \quad BO = 1,$$

which means that $BO : AO : AB = 3 : 4 : 5$. So, lurking within this figure is a 3-4-5 triangle! We have shown this in a separate figure (Figure 3).

Folding a third

Given a square piece of paper—the kind used in origami—it is easy to produce folds corresponding to fractions such as 1/2, 1/4, 1/8, 3/4, and so on; repeated halving is involved, and nothing more. It is less clear how we can do the same for a fraction like 1/3. It would seem that we have to resort to visual estimation and/or trial-and-error. In the July 2012 issue of *At Right Angles* (the inaugural issue), Shiv Gaur described an elegant iterative procedure that will divide a rectangular strip into five equal parts; a similar method will yield three equal parts. Here we describe a paper folding method that will directly locate a point of trisection of one side of the square.

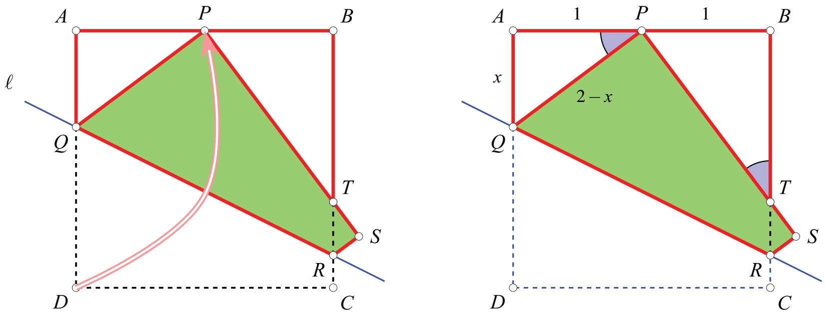
In Figure 4 (a), we see a square $ABCD$ folded so that vertex D falls upon the midpoint P of AB. The crease of the fold is QR, and the image of C under the fold is S. The point where PS cuts BC is T.

Claim. *T is a point of trisection of BC, with $BT : CT = 2 : 1$, and therefore, $BT/BC = 2/3$, $CT/BC = 1/3$.*

To prove the claim we move to Figure 4 (b). Let $AB = 2$ and $AQ = x$; then $QD = 2 - x$, hence $QP = 2 - x$ (for, QD coincides with QP after the fold). Applying the Pythagorean Theorem to $\triangle APQ$, we get $(2 - x)^2 = x^2 + 1^2$, and this yields $x = 3/4$ on solving for x.

Next we use triangle similarity: $\triangle QAP \sim \triangle PBT$ (look at their angles to see why), so $BT : 1 = 1 : x$, giving $BT = 4/3$. Since $BC = 2$, it follows that $BT : BC = 2 : 3$. Therefore, T is a point of trisection of BC.

And now for our bonus. Let us look again at $\triangle QAP$. Its side lengths are: $QA = 3/4$, $AP = 1$ and



(a) (b)

Figure 4.

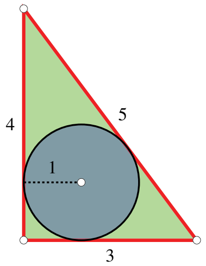


Figure 5. The in-radius of a 3-4-5 triangle is 1 unit

$QP = 2 - 3/4 = 5/4$. Hence its sides are in the ratio $3:4:5$. And since $\triangle PBT$ is similar to $\triangle QAP$, its sides too are in the ratio $3:4:5$.

So we have not one but *two* 3-4-5 triangles hidden within this figure.

The in-radius of the 3-4-5 triangle

Our last featured property focuses on what looks like a numerical oddity: the area of the 3-4-5 triangle equals its semi-perimeter. For, its area equals $\frac{1}{2}(3 \times 4) = 6$, and its semi-perimeter equals $\frac{1}{2}(3 + 4 + 5) = 6$. So both have the same value. Those of you who are "physics-minded" may give a cry of outrage here. "This is nonsense! How can area *ever* equal semi-perimeter? Area and semi-perimeter have different dimensions, and one can never equal the other!" That is of course perfectly right, and we shall not make that error here. But the same observation can be translated into a perfectly acceptable form to which no one can object, via this simple formula which connects the in-radius r of a triangle, its area Δ and its semi-perimeter s:

$$rs = \Delta, \quad \text{or} \quad r = \frac{\Delta}{s}.$$

This tells us that for a 3-4-5 triangle, the in-radius is 1 unit. (See Figure 5.) Now we see the source of the dimensionality problem and its resolution at the same time: namely, that the correct relationship is "area equals semi-perimeter times in-radius which equals 1 unit."

The mathematician within us is now provoked to ask the following question: *Are there other integer-sided right-angled triangles whose in-radius is* 1 *unit?* We shall show that the answer is ***No***.

Let ABC be an integer-sided right-angled triangle with $\measuredangle C = 90°$. Let its sides be a, b, c. Then we have:

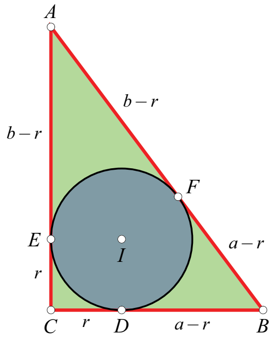


Figure 6. The in-radius of a right triangle: $r = \frac{1}{2}(a + b - c)$

$$\Delta = \frac{ab}{2}, \quad s = \frac{a+b+c}{2}, \quad \therefore r = \frac{\Delta}{s} = \frac{ab}{a+b+c}.$$

Since $r = 1$ we get:

$$ab = a + b + c, \qquad \therefore c = ab - a - b.$$

As the triangle is right-angled, we also have $c^2 = a^2 + b^2$. It follows that

$$(ab - a - b)^2 = a^2 + b^2.$$

We must find pairs of integers that solve the above equation. To avoid duplication of solutions we may assume that $a \leq b$. Note that this actually means $a < b$ as we cannot have $a = b$. (We cannot have an integer-sided right-angled triangle which is also isosceles. This is the same as asserting that $\sqrt{2}$ is not a rational number.) Let us now write the above equation as

$$(ab - a - b)^2 - b^2 = a^2.$$

The expression on the left side factorizes as $(ab - a)(ab - a - 2b) = a(b - 1)(ab - a - 2b)$. Hence we have:

$$(b - 1)(ab - a - 2b) = a.$$

Since $a < b$ and a, b are integers, we have $a \leq b - 1$. Hence the above equality can hold only if we have $a = b - 1$ and $ab - a - 2b = 1$. These conditions yield:

$$(b - 1)b - (b - 1) - 2b = 1, \qquad \therefore b^2 - 4b = 0,$$

which yields $b = 4$ (obviously $b \neq 0$) and hence $a = 3$ and $c = 5$. Thus the 3-4-5 triangle is the only integer-sided right-angled triangle whose in-radius is 1 unit.

Alternate solution. Here is another way of reaching the same conclusion. It may be preferred by some, and it also generalizes more easily. It starts by establishing a neat geometrical result: *If* $\triangle ABC$ *is right-angled with* $\measuredangle C = 90°$*, then the radius of the incircle of the triangle is* $(a + b - c)/2$. The result has value and interest in itself (mathematicians would call it a 'lemma').

Let the incircle touch the sides BC, CA, AB at points D, E, F respectively. The triangle being right-angled at C, points I, E, C, D form the vertices of a square of side r, hence $CD = r = CE$. From this it follows that $DB = a - r$ and $EA = b - r$. Next, drawing on the fact that the two tangents to a circle from a point outside the circle have equal length, it follows that $AF = b - r$ and $BF = a - r$. From this we get:

$$(a - r) + (b - r) = c, \qquad \therefore\ r = \frac{a + b - c}{2},$$

as claimed.

Now we apply this result to the problem at hand. Let a, b, c be the sides of an integer-sided right-angled triangle whose in-radius is 1 unit. We may assume with no loss of generality that $a < b < c$. The result just proved implies that $a + b - c = 2$, giving $c = a + b - 2$. Invoking the Pythagorean relation we get:

$$a^2 + b^2 = (a + b - 2)^2.$$

This yields: $2ab - 4a - 4b + 4 = 0$, i.e., $ab - 2a - 2b = -2$. Adding 4 to both sides and factorizing, we get:

$$ab - 2a - 2b + 4 = 2, \qquad \therefore\ (a - 2)(b - 2) = 2.$$

The only way of expressing 2 as a product of two positive integers is $2 = 1 \times 2$, so we must have $a - 2 = 1$ and $b - 2 = 2$ (remember that $a < b$), giving $a = 3$ and $b = 4$ and hence $c = 5$. We reach the same conclusion as earlier.

The advantage of this approach is that it can easily be extended. For example, we may want to list the Pythagorean triples which correspond to triangles with in-radius 2 units. Since the 3-4-5 triangle has in-radius 1 unit, it follows by scaling that the 6-8-10 triangle has in-radius 2 units. Are there any others? Let's see .... Let a, b, c be the sides of an integer-sided right-angled triangle whose in-radius is 2 units; assume that $a < b < c$. Then we have $a + b - c = 4$, giving $c = a + b - 4$. Hence we have:

$$a^2 + b^2 = (a + b - 4)^2.$$

This yields: $2ab - 8a - 8b + 16 = 0$, i.e., $ab - 4a - 4b = -8$. Adding 16 to both sides and factorizing, we get:

$$ab - 4a - 4b + 16 = 8, \qquad \therefore\ (a - 4)(b - 4) = 8.$$

The ways of expressing 8 as a product of two positive integers are $8 = 1 \times 8 = 2 \times 4$, so the possibilities are:

$$(a - 4, b - 4) = (1, 8), \qquad \therefore\ (a, b) = (5, 12);$$
$$(a - 4, b - 4) = (2, 4), \qquad \therefore\ (a, b) = (6, 8).$$

So there are two such triangles—the 5-12-13 triangle and the 6-8-10 triangle—which have in-radius equal to 2 units.

Readers may wish to continue the exploration and search for r-values which give rise to large numbers of candidate triangles.

Closing remark. We have attempted to list some features of the right-angled triangle with sides 3-4-5, and to highlight configurations where this triangle occurs naturally. Without doubt, there are many more such features and many more such configurations. We invite you to design investigations for your students which add to the above list. In the process, students could learn how to make conjectures and then test them and prove them using valid mathematical procedures. May the list grow, and may the conjectures outnumber the theorems!

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Divisibility Tests by Powers of 2

MAJID SHAIKH

There is a well-known test for divisibility by powers of 2: to check the divisibility of a number M by 2^n, we form a new number M' using only the last n digits of M and then examine the divisibility of *that* number (i.e., M') by 2^n. The test works because of the easily-proved fact that M is divisible by 2^n if and only if M' is divisible by 2^n. All that is required for this is the observation that 10^n is divisible by 2^n.

In the traditional implementation of this test, checking whether M' is divisible by 2^n is accomplished by actual division, i.e., actually working out $M' \div 2^n$; there are no further shortcuts. In this note, we show that checking the divisibility of M' by 2^n can be done with less effort than actual division.

A new procedure

Purpose: To check the divisibility of a given number M by 2^n. We execute the following steps.

- Form the number M' using only the last n digits of M.
- Take one digit at a time of M', starting from the *left* side. (The 'right' side is the units side.)
- Divide these digits by the increasing powers of 2, starting with $2^1 = 2$.
- After each division, retain only the remainder and combine it with the next digit of M', appending it from the left side to form a new number. Use that number for the next division.

Keywords: *Divisibility, digits, place value, powers of 10, divisor, quotient, remainder*

- Repeat the procedure till the last digit.
- If the final remainder obtained is 0, then M is divisible by 2^n, else not.

The procedure is best illustrated using a few concrete examples.

Example 1. Let us check whether $M = 123456$ is divisible by 8.

Here the divisor is $8 = 2^3$, so we consider only the last three digits, giving $M' = 456$. Following the steps given above:

Step 1: Divide 4 by $2^1 = 2$. The remainder is 0 so there is no 'carry'. We move to the next digit.
Step 2: Divide 5 by $2^2 = 4$. The remainder is 1. So the 'carry' is 1; this is combined with the next digit (6) to form the number 16.
Step 3: Divide 16 by $2^3 = 8$. There is no remainder.

Hence 123456 is divisible by 8.

Example 2. Let us check whether $M = 123456$ is divisible by 16.

Here the divisor is $16 = 2^4$, so $M' = 3456$.

Step 1: Divide 3 by 2. The remainder is 1.
Step 2: Divide 14 by 4. The remainder is 2.
Step 3: Divide 25 by 8. The remainder is 1.
Step 4: Divide 16 by 16. There is no remainder.

Hence 123456 is divisible by 16.

Example 3. Let us check whether $M = 110640$ is divisible by 32.

Here the divisor is $32 = 2^5$, so $M' = 10640$.

Step 1: Divide 1 by 2. The remainder is 1.
Step 2: Divide 10 by 4. The remainder is 2.
Step 3: Divide 26 by 8. The remainder is 2.
Step 4: Divide 24 by 16. The remainder is 8.
Step 5: Divide 80 by 32. The remainder is 16 which is non-zero.

Hence 110640 is not divisible by 32. Note that the remainder (16) obtained in the last step is also the remainder left when 110640 is divided by 32.

The mechanics of the algorithm should be clear from these examples. The computations are best done with the digits arranged in a tabular form, but this is easier done in handwritten work than in printed form, which is why we have described the algorithm the way we have done.

We will prove the correctness of the algorithm later.

The quotient

It is interesting that the quotient too can be worked out by this method, but with a slight modification: we retain the quotient at each stage. Then, from this sequence of partial quotients, we can recover the desired quotient. All we need to do is multiply each partial quotient by 5 and then add the next partial quotient, and so on till the end.

To start with we only show how to compute the quotient in the division $M' \div 2^n$. Remember that M' has only n digits.

Example 4. Let us compute the quotient in the division $3456 \div 16$.

Step 1: Divide 3 by 2. The quotient is 1 and the remainder is 1.
Step 2: Divide 14 by 4. The quotient is 3 and the remainder is 2.
Step 3: Divide 25 by 8. The quotient is 3 and the remainder is 1.
Step 4: Divide 16 by 16. The quotient is 1 and there is no remainder.

The sequence of quotients, starting from the first one, is 1, 3, 3, 1. So the computations are:

$$1 \longmapsto (1 \times 5) + 3 = 8 \longmapsto (8 \times 5) + 3 = 43 \longmapsto (43 \times 5) + 1 = 216.$$

Hence the quotient is 216.

Example 5. Let us compute the quotient in the division $23456 \div 32$.

Step 1: Divide 2 by 2. The quotient is 1 and the remainder is 0.
Step 2: Divide 3 by 4. The quotient is 0 and the remainder is 3.
Step 3: Divide 34 by 8. The quotient is 4 and the remainder is 2.
Step 4: Divide 25 by 16. The quotient is 1 and the remainder is 9.
Step 5: Divide 96 by 32. The quotient is 3 and there is no remainder.

The sequence of quotients, starting from the first one, is 1, 0, 4, 1, 3. The computations:

$$\begin{aligned}1 \longmapsto (1 \times 5) + 0 = 5 &\longmapsto (5 \times 5) + 4\\ &= 29 \longmapsto (29 \times 5) + 1\\ &= 146 \longmapsto (146 \times 5) + 3 = 733.\end{aligned}$$

Hence the quotient is 733.

Explaining the divisibility test

Now we explain why the divisibility procedure works. We shall show how it works because of the place value system. We start by noting that:

- 10 is divisible by 2 but not by 4. However, 20 is divisible by 4.
- 100 is divisible by 4 but not by 8. However, 200 is divisible by 8.
- 1000 is divisible by 8 but not by 16. However, 2000 is divisible by 16.

And so on. In general, 10^k is divisible by 2^k but not by 2^{k+1}. However, 2×10^k is divisible by 2^{k+1}. (When stated in that form, the reason should be obvious, for $10^k = 2^k \times 5^k$ and $2 \times 10^k = 2^{k+1} \times 5^k$.)

Consider the divisibility of (say) 3456 by 16. We follow a theme commonly seen in divisibility studies: if we subtract multiples of the divisor from the dividend, divisibility will not be affected. In other words, in checking, say, the divisibility of M by d, we can equally well check the divisibility of $M - qd$ by d for any convenient value of q; the subtracted portion qd is then 'washed' away and need not be looked at again. Combining this observation with the one made above, we may write:

$$\begin{aligned}3456 &= 2000 + 1456\\ &\quad \text{(here, 2000 is a multiple of 16)}\\ &= 2000 + 1200 + 256\\ &\quad \text{(here, 1200 is a multiple of 16)}\\ &= 2000 + 1200 + 240 + 16\\ &\quad \text{(here, 240 is a multiple of 16)}\\ &= \text{a multiple of 16.}\end{aligned}$$

Now compare these steps with the ones made when we checked the divisibility of 3456 by 16. We have put the two actions side by side for ease of understanding. In each line we have used a bold font for the relevant digit.

Step 1	$3456 = 2000+\mathbf{1}456$	Divide 3 by 2. The remainder is **1**.
Step 2	$1456 = 1200 + \mathbf{2}56$	Divide 14 by 4. The remainder is **2**.
Step 3	$256 = 240 + \mathbf{16}$	Divide 25 by 8. The remainder is **1**.
Step 4	$16 = 1 \times 16$	Divide 16 by 16. There is no remainder.

Another example: checking whether $M = 10640$ is divisible by 32. We have:

Step 1	$10640 = 0 + \mathbf{1}0640$	Divide 1 by 2. The remainder is **1**.
Step 2	$10640 = 8000 + \mathbf{2}640$	Divide 10 by 4. The remainder is **2**.
Step 3	$2640 = 2400 + \mathbf{2}40$	Divide 26 by 8. The remainder is **2**.
Step 4	$240 = 160 + \mathbf{8}0$	Divide 24 by 16. The remainder is **8**.
Step 5	$80 = 2 \times 32 + \mathbf{16}$	Divide 80 by 32. The remainder is **16**.

We shall not try to explain the working any more as we feel that these examples carry enough of a suggestion that one can mentally construct the explanation or proof for oneself.

Explaining the recovery of the quotient

Now we explain why the method described for recovering the quotient works. Once again, we shall work through two examples in a suggestive manner and leave it at that. We use the instances 3456 ÷ 16 and 23456 ÷ 32.

Example 6. Let us compute the quotient in the division 3456 ÷ 16. Here is the working.

Action	Quotient	Remainder
Divide 3 by 2	1	1
Divide 14 by 4	3	2
Divide 25 by 8	3	1
Divide 16 by 16	1	0

The sequence of partial quotients is 1, 3, 3, 1. Now consider:

$$\begin{aligned} 3456 &= 2000 + 1200 + 240 + 16 \\ &= (125 + 75 + 15 + 1) \times 16 \\ &= \left(\mathbf{1} \times 5^3 \ + \ \mathbf{3} \times 5^2 \ + \ \mathbf{3} \times 5^1 \ + \ \mathbf{1}\right) \times 16. \end{aligned}$$

We see the role played by the string of digits 1, 3, 3, 1.

Example 7. Let us compute the quotient in the division 23456 ÷ 32.

Action	Quotient	Remainder
Divide 2 by 2	1	0
Divide 3 by 4	0	3
Divide 34 by 8	4	2
Divide 25 by 16	1	9
Divide 96 by 32	3	0

The sequence of quotients, starting from the first one, is 1, 0, 4, 1, 3. Now consider:

$$\begin{aligned} 23456 &= 20000 + 0 + 3200 + 160 + 96 \\ &= (625 + 0 + 100 + 5 + 3) \times 32 \\ &= \left(\mathbf{1} \times 5^4 \ + \ \mathbf{0} \times 5^3 \ + \ \mathbf{4} \times 5^2 \ + \ \mathbf{1} \times 5^1 \right. \\ &\quad \left. + \ \mathbf{3}\right) \times 32. \end{aligned}$$

We see the role played by the string of digits 1, 0, 4, 1, 3.

Thus the procedure, which looks mysterious at first encounter, is simply a manifestation of the fact that $10^n = 2^n \times 5^n$.

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in the classroom

1 5 3 and so on and on and on …

On the AtRiUM FaceBook page we came across this striking set of arithmetical relations (posted by a reader, Ms Paromita Roy):

SHAILESH SHIRALI

Amazing Mathematical Fact!

$$\begin{aligned} 1^3+5^3+3^3 &= 153,\\ 16^3+50^3+33^3 &= 165033,\\ 166^3+500^3+333^3 &= 166500333,\\ 1666^3+5000^3+3333^3 &= 166650003333, \end{aligned}$$

and so on and on and on and on!

The relations are true and can be checked using a calculator. The "and so on and on and on …" invites us to state and prove a valid mathematical generalization of the relations. The obvious candidate is this statement:

$$(166\ldots66)^3+(500\ldots00)^3+(333\ldots33)^3 = 166\ldots66\,500\ldots00\,333\ldots33, \qquad (1)$$

where the strings 166 … 66, 500 … 00 and 333 … 33 all have the same number of digits. We shall prove that relation (1) is true.

To do so we note that if the numbers $x = 166\ldots66$, $y = 500\ldots00$ and $z = 333\ldots33$ have n digits each, then

$$x=\frac{10^n-4}{6},\qquad y=\frac{10^n}{2},\qquad z=\frac{10^n-1}{3}.$$

With this notation, relation (1) can be stated as follows:

$$x^3+y^3+z^3 = x\cdot 10^{2n}+y\cdot 10^n+z. \qquad (2)$$

Keywords: *Pattern, string, digits, powers of ten, sum of cubes*

We shall prove relation (2). Write $a = 10^n$. Then the relation can be restated as:

$$\left(\frac{a-4}{6}\right)^3 + \left(\frac{a}{2}\right)^3 + \left(\frac{a-1}{3}\right)^3 = \frac{a-4}{6} \cdot a^2 + \frac{a}{2} \cdot a + \frac{a-1}{3}. \tag{3}$$

It is easily checked that relation (3) is an identity, true for all a. Indeed, both sides simplify to the following after a routine computation:

$$\frac{a^3}{6} - \frac{a^2}{6} + \frac{a}{3} - \frac{1}{3} = \frac{(a-1)(a^2+2)}{6}. \tag{4}$$

Hence relation (1) is true.

Postscript I. The fact that the two sides result in an easily factorized expression allows us to extend the FaceBook post. Now we can write the following:

$$\begin{aligned}
1^3 + 5^3 + 3^3 &= 153 &&= \frac{9 \times 102}{6},\\
16^3 + 50^3 + 33^3 &= 165033 &&= \frac{99 \times 10002}{6},\\
166^3 + 500^3 + 333^3 &= 166500333 &&= \frac{999 \times 1000002}{6},\\
1666^3 + 5000^3 + 3333^3 &= 166650003333 &&= \frac{9999 \times 100000002}{6},
\end{aligned}$$

and so on and on and on and on!

Postscript II. A bit of judicious experimentation allows us to discover a second set of such relations, just as pleasing:

$$\begin{aligned}
3^3 + 7^3 + 1^3 &= 371,\\
33^3 + 67^3 + 01^3 &= 336701,\\
333^3 + 667^3 + 001^3 &= 333667001,\\
3333^3 + 6667^3 + 0001^3 &= 333366670001,
\end{aligned}$$

and so on and on and on and on!

We leave the verification and proof to the reader.

Thanks to Ms Paromita Roy for the post!

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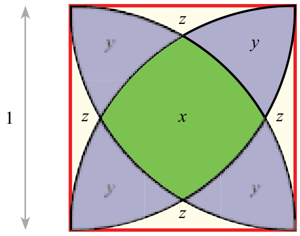
A Flower with Four Petals

Shown below is a square with side 1 unit, with four circular arcs drawn within it, each with radius 1 unit and centred at the four vertices of the square. The four arcs demarcate a region in the centre of the square, shown coloured green. The problem we pose is to find the area x of this region.



To find x we use a method that may remind you of Venn diagram computations. We start by assigning symbols to the areas of the other regions. The four regions marked y are clearly congruent to each other, as are the four regions marked z. Let their areas be denoted by the same symbols (y and z). Then the following relations are immediate:

$$y + 2z = 1 - \frac{\pi}{4}, \tag{1}$$

$$x + 2y = 1 - 2\left(1 - \frac{\pi}{4}\right) = \frac{\pi}{2} - 1, \tag{2}$$

$$x + 2y + z = \frac{\pi}{6} + \left(\frac{\pi}{6} - \frac{\sqrt{3}}{4}\right) = \frac{\pi}{3} - \frac{\sqrt{3}}{4}. \tag{3}$$

Keywords: *Unit circle, unit square, vertices, arcs, area, sector, segment, triangle, angle*

The three equations are readily solved for z, y, x (in that order). From (2) and (3) we get, by subtraction:

$$z = 1 - \frac{\sqrt{3}}{4} - \frac{\pi}{6}. \tag{4}$$

Next from (1) and (4) we get:

$$y = 1 - \frac{\pi}{4} - 2 + \frac{\sqrt{3}}{2} + \frac{\pi}{3} = \frac{\pi}{12} + \frac{\sqrt{3}}{2} - 1. \tag{5}$$

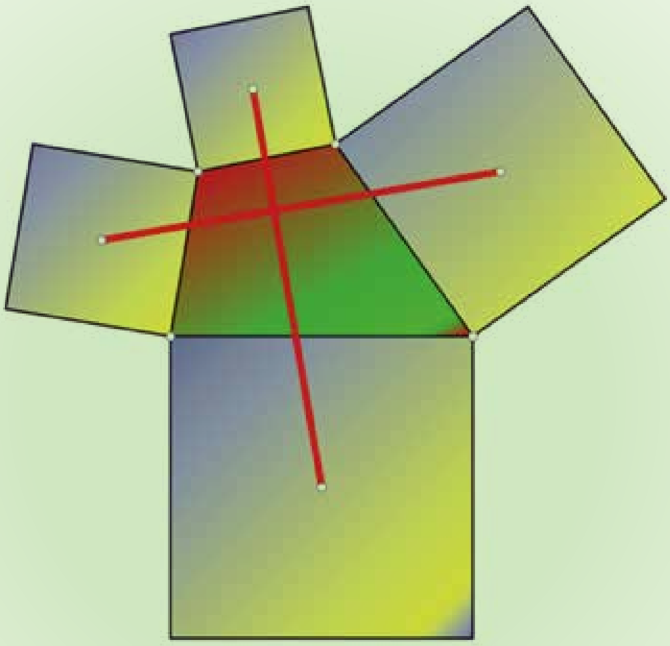
Finally from (2) we get:

$$x = \frac{\pi}{3} - \sqrt{3} + 1. \tag{6}$$

— *Thanks to Shri Bharat Karmarkar for suggesting the problem.*

The **COMMUNITY MATHEMATICS CENTRE** (CoMaC) is an outreach arm of Rishi Valley Education Centre (AP) and Sahyadri School (KFI). It holds workshops in the teaching of mathematics and undertakes preparation of teaching materials for State Governments and NGOs. CoMaC may be contacted at shailesh.shirali@gmail.com.

THEOREM CONCERNING A QUADRILATERAL



The picture above demonstrates a beautiful result which is true of every quadrilateral. It should be self-explanatory: On each side of the quadrilateral, we draw a square facing outwards. Next, we join the centres of opposite pairs of squares (see the thick red lines). Then the two line segments thus drawn have equal length, and they are perpendicular to each other. Isn't that a beautiful result? Try finding a proof of it on your own.

Generating the n-th Prime

Here is an unusual way of generating the prime numbers. It is taken from a letter written by Ronald Skurnick of Nassau Community College (New York, USA) to Mathematics Teacher (National Council of Teachers of Mathematics) and published in the November 2009 issue of the journal.

C⊗MαC

What it does is this: given the first n primes $p_1 = 2$, $p_2 = 3, p_3 = 5, \ldots, p_n$, it works out the next prime p_{n+1}. To avoid triviality we assume $n \geq 3$. Here's how it works:

Step 1: List the first $n - 2$ odd primes: $p_2 = 3, p_3 = 5, \ldots, p_{n-1}$.

Step 2: List the *odd multiples* of the primes listed in Step 1, as many as are required (the number required will become clear from the example).

Step 3: Subtract p_n from all the numbers listed in Step 2. (The resulting numbers are naturally all even.) Discard all the *negative numbers* so obtained.

Step 4: Identify *the smallest even number* $2k$ *that does not occur in any of the lists of numbers that remain in Step 3*.

Step 5: The desired prime number is then given by $p_{n+1} = p_n + 2k$.

What a strange procedure! Before we try to justify it, let us demonstrate this using a few examples.

Example 1. Let us find p_6 given the first five prime numbers: $2, 3, 5, 7, 11$.

Step 1: The odd primes till p_4 are $3, 5, 7$.

Keywords: *Odd, even, prime, algorithm, Eratosthenes sieve*

Step 2: Their odd multiples, listed row-wise:

3	9	15	21	27	33	...
5	15	25	35	45	55	...
7	21	35	49	63	77	...

Step 3: Subtract 11 from the numbers listed in Step 2. Then discard all the negative numbers so generated. We get:

−8	−2	4	10	16	22	...
−6	4	14	24	34	44	...
−4	10	24	38	52	66	...

The negative numbers (regarded as 'discarded') are shown highlighted in green.

Step 4: The smallest even number that does not occur in the lists of numbers that remain is 2.

Step 5: Hence the next prime number p_6 is $11 + 2 = 13$.

Example 2. Let us find p_{10} given the first nine prime numbers: 2, 3, 5, 7, 11, 13, 17, 19, 23.

Step 1: The odd primes till p_8 are 3, 5, 7, 11, 13, 17, 19.

Step 2: Their odd multiples, listed row-wise:

3	9	15	21	27	33	...
5	15	25	35	45	55	...
7	21	35	49	63	77	...
11	33	55	77	99	121	...
13	39	65	91	117	143	...
17	51	85	119	153	187	...
19	57	95	133	171	209	...

Step 3: Subtract 23 from the numbers listed in Step 2. Then discard all the negative numbers so generated. We get:

−20	−14	−8	−2	4	10	...
−18	−8	2	12	22	32	...
−16	−2	12	26	40	54	...
−12	10	32	54	76	98	...
−10	16	42	68	94	120	...
−6	28	62	96	130	164	...
−4	34	72	110	148	186	...

The negative numbers are shown highlighted in green. They are discarded.

Step 4: The smallest even number that does not occur in the lists of numbers that remain is 6.

Step 5: Hence the next prime number p_{10} is $23 + 6 = 29$.

Remarks.

- Given the significant role played by oddness in this algorithm (odd multiples of the odd primes), it seems justified to describe this as a very odd algorithm!
- The algorithm is to be regarded as a pedagogical curiosity rather than a practical way of generating the primes. Its surprise value is that it actually does yield the next prime!
- But is it really all that mysterious? Or is it simply a disguised way of applying the very definition of a prime number? Is it simply a disguised form of the well-known Eratosthenes sieve? Perhaps! We'll leave it to you to work it out.

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Low Floor High Ceiling Tasks

Keeping Things in Proportion

The Midas Touch

In the last issue, we began a new series which was a compilation of 'Low Floor High Ceiling' activities. A brief recap: an activity is chosen which starts by assigning simple age-appropriate tasks which can be attempted by all the students in the classroom. The complexity of the tasks builds up as the activity proceeds so that each student is pushed to his or her maximum as they attempt their work. There is enough work for all, but as the level gets higher, fewer students are able to complete the tasks. The point, however, is that all students are engaged and all of them are able to accomplish at least a part of the whole task.

SWATI SIRCAR & SNEHA TITUS

Our activity this time is an investigation which begins with the Fibonacci sequence. Throughout this activity, students are called upon to exercise the skills of observation, pattern recognition, mathematical notation and communication and visualization. Along with this, there is an opportunity to apply their understanding by using an algorithm to generate the spreadsheet version of the sequence – though this last is an optional addition. Their prior knowledge of arithmetic, algebra and geometry is exercised and students are also able to appreciate the connection between these areas. Students who are more adept in one or the other of the three can work in their comfort zone and gain confidence to work on the others, thus the teacher is able to assign work on exercising strengths and addressing weaknesses. This task can be comfortably attempted by students in grade 11 although the mathematically able in grades 9 or 10, can also give it a shot. Intuitive pattern recognition is the starting point of task 1. To attempt the tasks students will need to know how to form algebraic expressions (including the use of suffixes to denote the term in a particular position). They should be familiar with the Pythagoras theorem

Keywords: *pattern, algebra, Pythagoras, irrational, quadratic, roots, angle, triangle, pentagon, ratio*

and comfortable with the use of irrational numbers. The formula for the roots of a quadratic equation and the properties of the angles of a triangle are also necessary prerequisites.

Each card (or set of cards) is a task which features a series of questions which build up in complexity.

Task 1

Consider the sequence 1, 1, 2, 3, 5, 8, 13, ……………..

- What is the next term of this sequence?
- Generate the next 10 terms of this sequence.
- State in words how each new term is generated.
- Find an algebraic expression that expresses how each new term is generated.
- Find the ratio of each term to the preceding term. State your finding.
- Do a similar investigation for the sequence 8, 10, 18, 28, 46, ……….
- What is common between your findings about the ratio in both these sequences?
- Choose any two natural numbers and generate the sequence in the same way. Do your findings change?

Teacher's Note: Spreadsheets such as Excel can easily be used to generate the Fibonacci sequence. For a complete description please refer to http://teachersofindia.org/en/article/exploring-fibonacci-numbers-using-spreadsheet. This is also a great place to introduce students to a recursive formula for generating a sequence.

Task 2

The golden ratio

- Consider the expression: $F_{n+1} = F_{n-1} + F_n$……………………(1)
- We shall assume $F_0 = 0$ and $F_1 = 1$

  If this expression is used to generate the Fibonacci sequence, state in words:

  (i) What is F_2?

  (ii) Which equivalent expression describes the fifth term?

  (iii) What is meant by F_0?

  (iv) What does expression (1) mean?
- Using your observations regarding the Fibonacci sequence from task 1, what can you conclude about the ratios $\frac{F_{n+1}}{F_n}$ and $\frac{F_n}{F_{n-1}}$ in the long run? Can we assume that after a certain stage, $\frac{F_{n+1}}{F_n}$ is nearly the same as $\frac{F_n}{F_{n-1}}$?………(2)
- Using equations (1) and (2), we can conclude that $\frac{F_{n-1}+F_n}{F_n}$ and $\frac{F_n}{F_{n-1}}$ are 'nearly the same' after a certain stage. Let us now replace 'nearly the same' by an equality sign. If $\frac{F_n}{F_{n-1}} = x$, then show how this expression yields the equation $\frac{1}{x} + 1 = x$.
- Reduce this to the quadratic equation $x^2 - x - 1 = 0$.

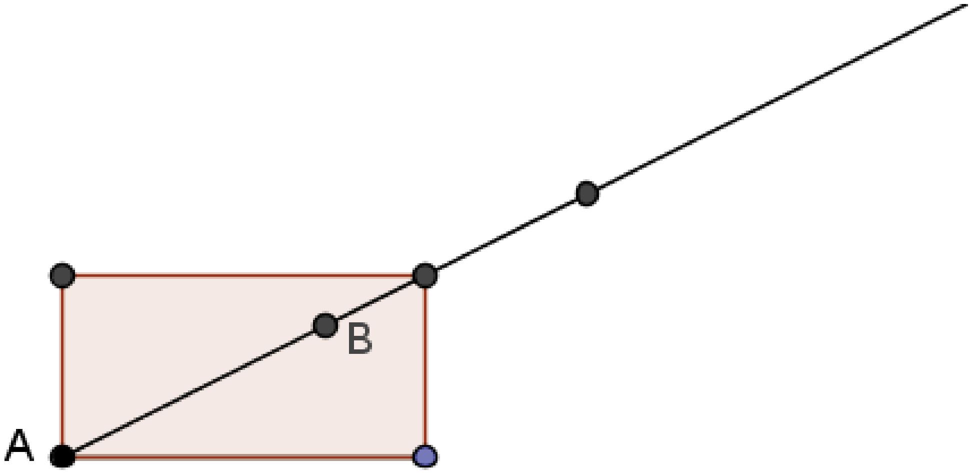
- Find the solutions of this quadratic equation.
- What is the positive root of this equation?
- How is this root related to the ratios from the three different sequences in task 1?

Teacher's Note: This task can be quite challenging for students who are not familiar with notations for recursive relations. Careful facilitation by the teacher (particularly in helping them to express that the $(n+1)^{\text{th}}$ term is the sum of the n^{th} and $(n-1)^{\text{th}}$ terms) will give the student confidence to negotiate the climb in this task.

Task 3

Constructing the number $\frac{\sqrt{5}+1}{2}$

- On a sheet of cardboard, construct a rectangle of length 2 inches and breadth 1 inch.
- Join one diagonal of this rectangle. What is the length of this diagonal? Show your calculation and verify by measurement.
- Extend the diagonal by 1 inch outside the rectangle. Mark the mid-point B of the extended diagonal and call this segment AB. Measure AB.
- From the construction, what is the exact measure of AB in inches?

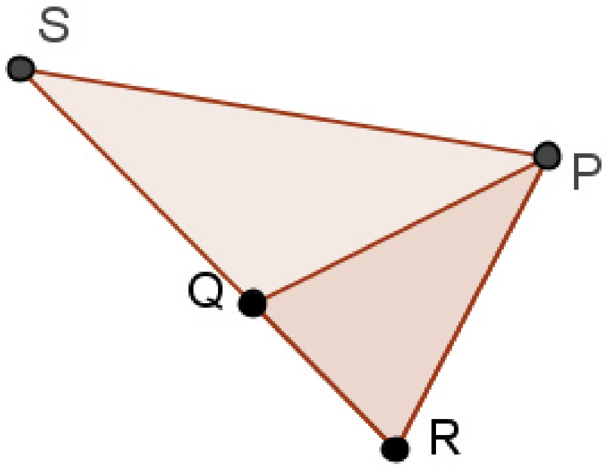


Teacher's Note: There are benefits to doing this construction either with compass and ruler or with dynamic geometry software such as GeoGebra. The teacher should in either case encourage students to investigate and validate their findings with careful reasoning. The teacher may need to explain to the student that exact measurements may involve square roots and fractions. Also, whenever the measure of AB is used the teacher must ensure that the student uses the constructed length from the figure and not the rounded-off approximation.

Task 4

Constructing and investigating the specified triangle.

- Construct ΔRPQ with sides $QR = 1, PQ = RP = \frac{\sqrt{5}+1}{2}$ (Remember $AB = \frac{\sqrt{5}+1}{2} = x$ from Task 2).
- Extend RQ to S such that QS = AB. Join PS.

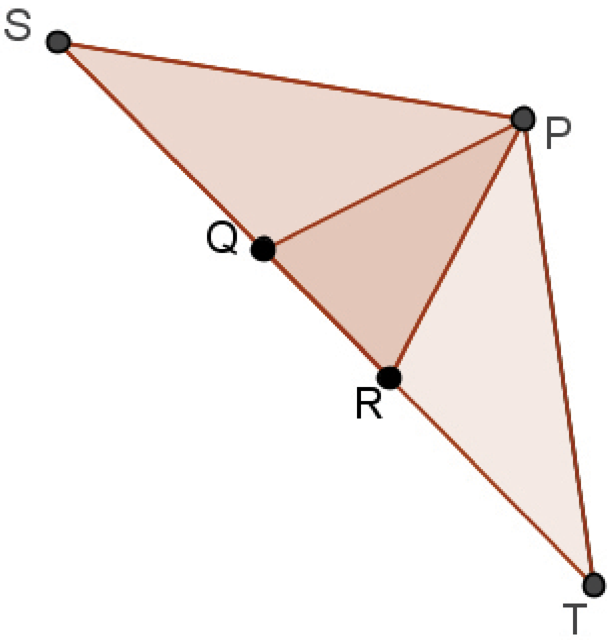


- Prove that $RS = x^2$.
- Find the ratios $\frac{RS}{RP}$ and $\frac{RP}{RQ}$.
- Investigate ΔRPS and ΔRQP and note down your findings, stating clearly all relationships between sides and angles.
- Find ∡PSR and ∡RQP.
- If $RS = 1 + x$, explain in two different ways why $PS = x^2$.

Teacher's Note: Proving that $RS = x^2$ can be done either by using the exact value for x or by using the fact that it is the root of $1 + x = x^2$. The SAS axiom is used to prove the similarity of the two triangles. Once students note that both triangles PSQ and PQR are isosceles triangles, they can easily use the properties of angles of a triangle (including exterior angle of a triangle) to find the required angles. They will need to refer to the quadratic equation in Task 2 to justify their answer to the last question – the teacher is strongly advised to give the students time to arrive at the result of the last question using both properties of triangles as well as similarity. In doing this task, students are able to appreciate the implications of results they have arrived at previously.

Task 5

- On PR, construct ΔPRT congruent to ΔPQS as shown in the diagram.
- Calculate ∡TPS.
- Cut out the triangle TPS and trace its outline in your notebook. Now, place your outline over the trace so that triangle PSQ is covered exactly by triangle PTR with P and T (of ΔPTR) directly over S and Q (of ΔSQP) respectively. Extend your diagram by outlining your cutout. Repeat this step until you get a closed figure. You will notice that the acute angled triangle alternates with the obtuse angled triangle.
- Identify the final shape.
- How is ∡TPS connected to this shape?
- What is the ratio of diagonal to side of this polygon?
- Observe a similar smaller polygon within the larger outer one. Express its diagonal in terms of x.



Teacher's Note: Once the students complete the polygon, they should be able to see that it is a regular pentagon and that there is a smaller regular pentagon created by its diagonals.

This investigation, which culminates in the creation of a regular pentagon, begins with a seemingly unrelated investigation of number patterns. Using the strategy of guided discovery, students can investigate numbers, algebra and geometry while practicing the skills of visualization, representation and communication. It is precisely this route that enhances the construction of the pentagon – clearly the focus is on the process and not on the product as the pentagon could just as well have been constructed in a more direct manner. Here is a thought – it would be interesting to motivate students to design an investigation which starts with a study of the pentagon and works backward to arrive at the golden triangle!

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THE JOYS OF COMMUTATIVITY

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Here are two additions that yield the same sum:

$$\begin{array}{rr} & 987654321 \\ & 087654321 \\ & 007654321 \\ & 000654321 \\ & 000054321 \\ & 000004321 \\ & 000000321 \\ & 000000021 \\ + & 000000001 \\ \hline & 1083676269 \\ \hline \end{array} \qquad\qquad \begin{array}{rr} & 123456789 \\ & 123456780 \\ & 123456700 \\ & 123456000 \\ & 123450000 \\ & 123400000 \\ & 123000000 \\ & 120000000 \\ + & 100000000 \\ \hline & 1083676269 \\ \hline \end{array}$$

The equality vividly illustrates the commutativity of multiplication: $a \times b = b \times a$. Do you see how?

The example is taken from `http://www.futilitycloset.com/2015/05/05/math-notes-111/`, where it is credited to Raymond F. Lausmann's *Fun With Figures*, 1965.

in the classroom

Thoughts on the Division Operation

Notes from a Reader

SWATI SIRCAR

Among the four basic arithmetical operations, division is easily the most complex, and it is the one with which children have the most difficulty. In this regard, I would like to share the following thoughts with readers on how division can be taught. They may be regarded as alternatives to the way division was presented in the pullout of an earlier issue. In particular, the following questions have been addressed:

1. Why do we proceed from left to right in division whereas all the other operations proceed from right to left?
2. What is the role of place value in the standard division algorithm?
3. How do we estimate the quotient when the divisor has two or more digits?

Points 1 and 2 are connected. The argument I use with students is that it is just a matter of efficiency. In the case of addition and subtraction, a left to right algorithm will not be efficient, as illustrated below.

Left to right					Right to left				
Step 1		T	U	We start by adding the tens.	Step 1		T	U	We start by adding the units and write the units digit of the sum and the tens digit of the sum separately.
		5	7				1		
	+	3	9				5	7	
		8				+	3	9	
								6	
Step 2		T	U	Next we add the units and write the unit digit of their sum and the tens digit of the sum separately.	Step 2		T	U	In the 2nd step we add up all the tens.
		1					1		
		5	7				5	7	
	+	3	9			+	3	9	
		8	6				9	6	
Step 3		T	U	So a 3rd step requires **changing** the already written digit in the tens place.	Here we **didn't have to change** any digit in the answer.				
		1							
		5	7						
	+	3	9						
		9	6						

A similar situation arises in subtraction as well. But because of the variety of cases which arise, I have doubts on whether to teach or even demonstrate this to children.

For division a right to left algorithm is not the most efficient one, as shown below.

Right to left						Left to right					
Step 1			T	U	Let's start with the units, i.e. we divide 6 by 2 	Step 1			T	U	 We start with the tens, i.e. we divide 3 by 2 and we are left with 1 ten
				3					1		
	2	)	3	6			2	)	3	6	
				6					2		
									1		
Step 2			T	U	 Now we try to divide the 3 tens We are left with 1 ten	Step 2			T	U	 Now we exchange this ten to 10 units and add the 6 units from the dividend to get 16 units
			1	3					1		
	2	)	3	6			2	)	3	6	
				6					2		
			3						1	6	
			2								
			1								

Right to left						Left to right					
Step 3			T	U	 So we have to exchange this remaining ten into 10 units Then we distribute these 10 units  ∴ we have to increase the unit's digit by 5 i.e. **change** it from 3 to 3 + 5 = 8	Step 3			T	U	 Now we divide these 16 units by 2 Here again we **didn't have to change** any digit in the answer (or quotient).
			1	8					1	8	
	2	)	3	6			2	)	3	6	
				6					2		
			3						1	6	
			2						1	6	
			1	0							
			1	0							

Note: The exchange of units is not obvious in the algorithm and students benefit if the teacher demonstrates the conversion from tens to units by using material manipulatives such as flats and longs or bundles of ten.

So if we show these to students or, even better, get them to work out both ways – left to right and right to left – and then reflect on the efficiency of each method, they will have a much better understanding of why the standard algorithms are the way they are.

I have tried to document how place value is invoked in the standard division algorithm with a detailed step by step demo of both long and short methods simultaneously showing 8643 ÷ 7 at http://teachersofindia.org/en/video/division-pay-attention.

When the division algorithm is taught, we generally use the shorter version of the division algorithm that violates the subtraction rules that a child has learnt.

Longer version						Shorter version					
Step 1			T	U	 Writing the complete 20 doesn't violate the subtraction rule and we are actually subtracting 2 tens or 20*	Step 1			T	U	Writing 20 as just 2 and then subtracting it from 36 is confusing since children may not realize they are subtracting 2 tens from 3 tens
			1	0					1		
	2	)	3	6			2	)	3	6	
			2	0					2		
			1	6					1		
Step 2			T	U	 Here we need to **change** the unit from 0 to 8 However, if this is shown with arrow cards, then this problem can be resolved.	Step 2			T	U	Here we **don't need to change** the unit. This is why we don't write the unit in step 1.
			1	8					1	8	
	2	)	3	6			2	)	3	6	
			2	0					2		
			1	6					1	6	
			1	6					1	6	
				0						0	

Point 3 is important since division is the only place where the standard algorithm requires estimation. For addition, subtraction and multiplication, no matter how large the numbers are, there is no need for estimation.

The strategy is multi-pronged as rounding can go either way. So we need to start with 2 digit divisors and round them to the nearest tens in order to estimate. Naturally, it is a good idea to give children enough practice with divisors that are multiples of 10, e.g. 40, 70 etc., before going into general 2 digit divisors. Next, if the divisor is 62, we can round it to 60 and look for the highest multiples less than the dividend in each step. Similarly, if the divisor is 37, it should be rounded to 40. Of course, numbers ending in 5 like 85 are always tricky – you can use either 80 or 90. It is also important to highlight here that the difference at each step should be less than the divisor. The estimation steps are as follows:

Estimation steps	**Example: 256 ÷ 36**	**Example: 256 ÷ 33**
1. Round off divisor to the nearest multiple of 10	36 rounded off to 40	33 rounded off to 30
2. Estimate quotient (or quotient digit) at that step using the estimate	256 ÷ 40 (or 25 ÷ 4) ≈ 6	256 ÷ 30 (or 25 ÷ 3) ≈ 8
3. Calculate quotient digit × actual divisor	6 × 36 = 216	8 × 33 = 264
4. Check a. if quotient digit × divisor > dividend: decrease quotient digit by 1 and repeat step 3 b. if not, proceed to check dividend – quotient digit × divisor > divisor: increase quotient digit by 1 and repeat step 3 c. If dividend – quotient digit × divisor ≤ divisor proceed to step 5	Check: a. 216 < 256 b. 256 – 216 = 40 > 36 ⇒ quotient = 6 + 1 = 7 Go back to step 3 using 7 as the quotient digit	Check: a. 264 > 256 ⇒ quotient digit= 8 – 1 = 7 Go back to step 3 using 7 as the quotient digit
5. Complete division step with the (modified) quotient	7 × 36 = 252, i.e. 256 ÷ 36 = 7 remainder 4	7 × 33 = 231, i.e. 256 ÷ 33 = 7 remainder 25

I welcome suggestions from readers.

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in the classroom

A Baby One Quarter the Size of its Parents

CoMaC

In Figure 1 we see two 'parent' circles $\mathcal{K}_1$ and $\mathcal{K}_2$ of equal radius tangent to a line ℓ, and a 'baby' circle $\mathcal{K}_3$ tangent to $\mathcal{K}_1, \mathcal{K}_2$ and ℓ; the baby has been held tight by its parents! We shall show that the baby has one quarter the radius of its parents. And the main result needed to prove this? It is an old friend, the Pythagorean theorem.

Let the common radius of $\mathcal{K}_1$ and $\mathcal{K}_2$ be taken as 1 unit, and let the radius of $\mathcal{K}_3$ be x units. Let A, B and C denote the centres of $\mathcal{K}_1, \mathcal{K}_2$ and $\mathcal{K}_3$, respectively (see Figure 2). Drawing the segments connecting these points, we see that $\triangle ABC$ is isosceles; the base AB has length $1 + 1 = 2$ units, while AC and BC have length $1 + x$ units each. (For, when two circles are

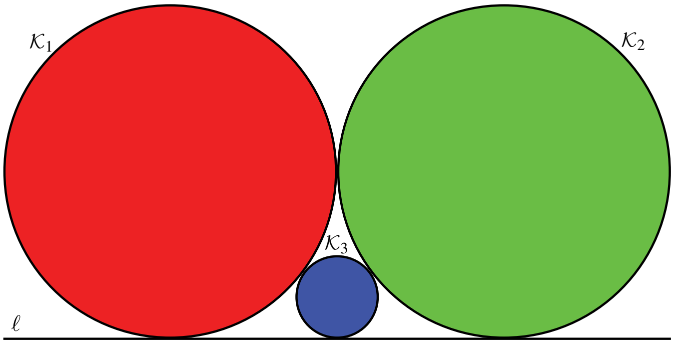


Figure 1.

Keywords: *Circle, radius, Pythagoras, touching circles*

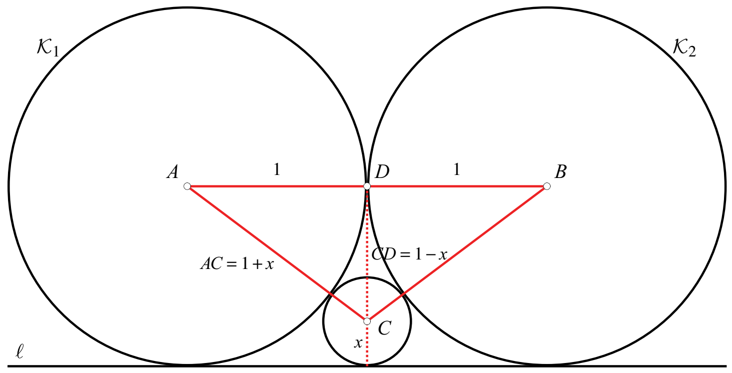


Figure 2.

externally tangent to each other, the distance between their centres equals the sum of their radii.) Segment CD is both a median and an altitude of $\triangle CAB$, and has length $1 - x$ units (because the perpendicular distance of D from ℓ is 1 unit, and the perpendicular distance of C from ℓ is x units).

Now we apply the Pythagorean theorem to $\triangle CAD$, which is right angled at D. We get:

$$1^2 + (1 - x)^2 = (1 + x)^2$$
$$\therefore\ x^2 - 2x + 2 = x^2 + 2x + 1,$$
$$\therefore\ 4x = 1,$$

which yields $x = 1/4$. Thus, the baby has $1/4$ the radius of the parents, as claimed.

The case of unequal radii

What happens if the two parent circles have unequal radii? Let the parents have radii a and b, respectively (see Figure 3). Denote the radius of the baby circle by c. We shall show that c may be found using the following elegant and symmetric relationship:

$$\frac{1}{\sqrt{c}} = \frac{1}{\sqrt{a}} + \frac{1}{\sqrt{b}}.$$

What we proved above is a particular case of this formula; for if $a = 1 = b$, then the formula gives $1/\sqrt{c} = 2$, so $c = 1/4$.

To prove this we first solve an auxiliary problem (see Figure 4): *What is the length of the tangent segment PQ on ℓ?* We answer this by drawing the segment $BR \parallel PQ$. Then we have: $BR = QP$, $AB = a + b$, $AR = |a - b|$, and now by the theorem of Pythagoras:

$$BR^2 = (a + b)^2 - (|a - b|)^2 = 4ab,$$

which yields $BR = 2\sqrt{ab}$. Therefore, $PQ = 2\sqrt{ab}$.

If we apply this result to the pairs of circles $\{\mathcal{K}_1, \mathcal{K}_3\}$ and $\{\mathcal{K}_2, \mathcal{K}_3\}$, we get:

$$PT = 2\sqrt{ac}, \qquad TQ = 2\sqrt{bc}.$$

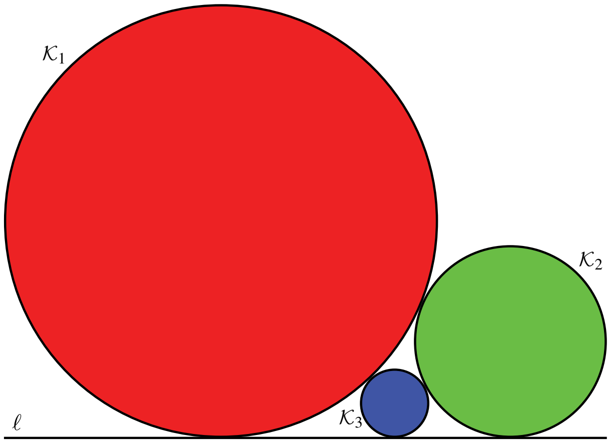


Figure 3.

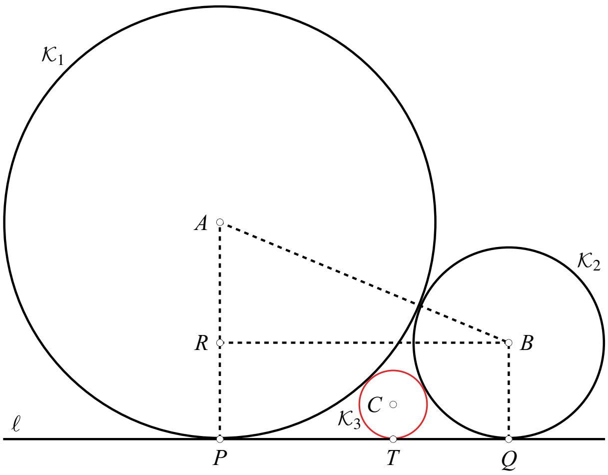


Figure 4.

Since $PT + TQ = PQ$, we obtain:

$$2\sqrt{ab} = 2\sqrt{ac} + 2\sqrt{bc}. \tag{1}$$

On dividing through by $2\sqrt{abc}$ we immediately get the desired relation:

$$\frac{1}{\sqrt{c}} = \frac{1}{\sqrt{a}} + \frac{1}{\sqrt{b}}. \tag{2}$$

For example, if $a = 1/4$ and $b = 1/9$, then $c = 1/25$.

One can visualize an unending sequence of circles being constructed in this way:

- a circle $\mathcal{K}_4$ enclosed by $\mathcal{K}_1, \mathcal{K}_3$ and ℓ;
- a circle $\mathcal{K}_5$ enclosed by $\mathcal{K}_2, \mathcal{K}_3$ and ℓ;
- a circle $\mathcal{K}_6$ enclosed by $\mathcal{K}_3, \mathcal{K}_4$ and ℓ;

and so on.

As a special case of formula (2), we have the following:

$$\text{If } a = \frac{1}{m^2} \text{ and } b = \frac{1}{n^2}, \text{ then } c = \frac{1}{(m+n)^2}. \tag{3}$$

And here is a lovely consequence of (3) for which we invite you to provide the complete justification:

> *If the radii of the initial two circles $\mathcal{K}_1$ and $\mathcal{K}_2$ are $1/m^2$ and $1/n^2$ for some two integers m and n, then every circle in this infinite chain has a radius of the form $1/p^2$ for some integer p.*

Figure 5 shows a few such circles. The configuration makes for colourful pictures!

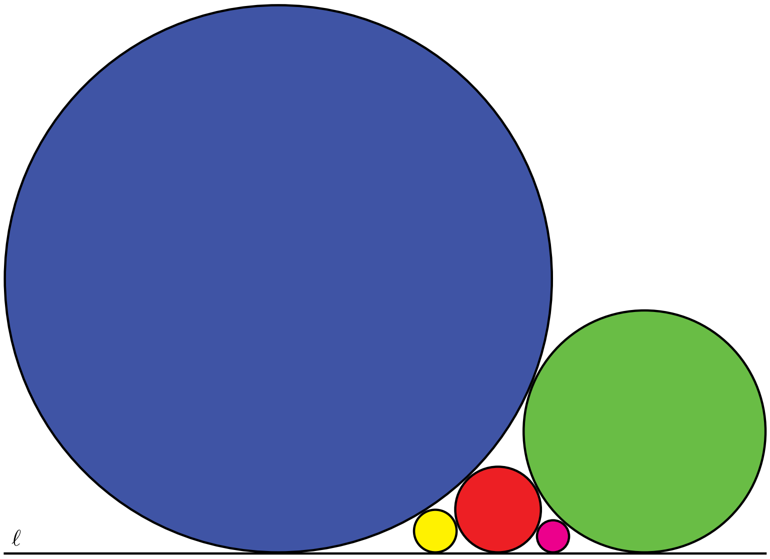
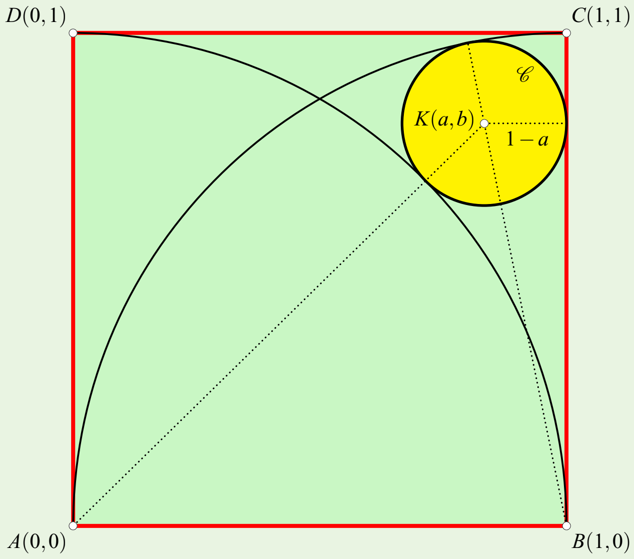


Figure 5. Circles galore

The **COMMUNITY MATHEMATICS CENTRE** (CoMaC) is an outreach arm of Rishi Valley Education Centre (AP) and Sahyadri School (KFI). It holds workshops in the teaching of mathematics and undertakes preparation of teaching materials for State Governments and NGOs. CoMaC may be contacted at shailesh.shirali@gmail.com.

SOLUTION TO THE "CIRCLE CHALLENGE"



Let the plane be coordinatized so that the coordinates of the vertices A,B,C,D of the square and the centre K of circle $\mathscr{C}$ are $(0,0)$, $(1,0)$, $(1,1)$, $(0,1)$ and (a,b), respectively. Then the perpendicular distance from K to side BC is $1-a$, hence the radius of $\mathscr{C}$ is $1-a$. Now we invoke the result that if two circles touch each other, the distance between their centres is equal to the *sum of their radii* in the case of external contact, and the *difference between their radii* in the case of internal contact. This yields $KA = 1+(1-a) = 2-a$ and $KB = 1-(1-a) = a$. Hence, using the distance formula:

$$a^2+b^2 = (2-a)^2,$$

$$(a-1)^2+b^2 = a^2.$$

Subtraction yields: $a^2-(a-1)^2 = (2-a)^2-a^2$, giving $2a-1 = 4-4a$, and $6a = 5$. Hence $a = 5/6$. It follows that the radius of the circle is $1/6$.

- Adapted from solution submitted by ***Tejash Patel of Patan, Gujarat.***
A similar solution was sent in by ***Adithya of BGS National Public School.*** *Thanks to both our solvers!*

How to Prove It

In this article we examine how to prove a result obtained after careful GeoGebra experimentation. It was featured in the March 2015 issue of At Right Angles, in the 'Tech Space' section.

SHAILESH A SHIRALI

In the 'Tech Space' article in the March 2015 issue of *At Right Angles*, Thomas Lingefjärd had considered the problem of a triangle drawn within a given triangle in a specified manner, and had wondered what could be said about the ratio of their areas. We study this problem in depth here.

Triangle in a triangle

We are given an arbitrary $\triangle ABC$. Let t be any number between 0 and 1. Locate points D, E, F on sides BC, CA, AB respectively, dividing them in the ratio $t : 1-t$. This is the same as saying that

$$\frac{BD}{BC} = \frac{CE}{CA} = \frac{AF}{AB} = t.$$

Let segments AD, BE, CF be drawn. The three lines intersect and demarcate a triangle PQR within the larger triangle ABC. The question now asked is: What is the ratio of the area of $\triangle PQR$ to that of $\triangle ABC$? In what way does this ratio depend on t? (See Figure 1.)

Note that in asking for a formula for 'the' ratio, we seem to be assuming implicitly that *the ratio of areas does not depend in any way on the shape of $\triangle ABC$; it depends only on t*. In fact we shall find that this is actually the case.

In Thomas Lingefjärd's original article, each side had been divided into $2n + 1$ equal parts (for a variable positive integer n), and the points D, E, F were the n-th points on their respective

Keywords: *Vector, linear independence, triangle, ratio, pattern, area*

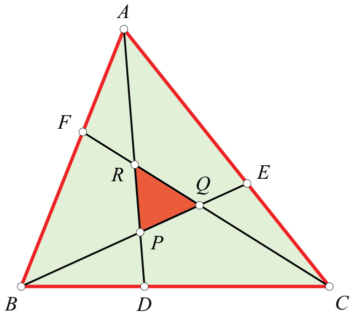


Figure 1.

edges, thus making $t = n/(2n+1)$. It had been conjectured as a result of careful GeoGebra-based experimentation that the ratio of areas is $1/(3n^2 + 3n + 1)$. But the formula had *not* been proved. We must check after completing our study whether the formula we obtain reduces to the one above for the case when $t = n/(2n+1)$.

Let $f(t)$ denote the ratio Area $(\triangle PQR)$: Area $(\triangle ABC)$. We certainly expect the following of f:

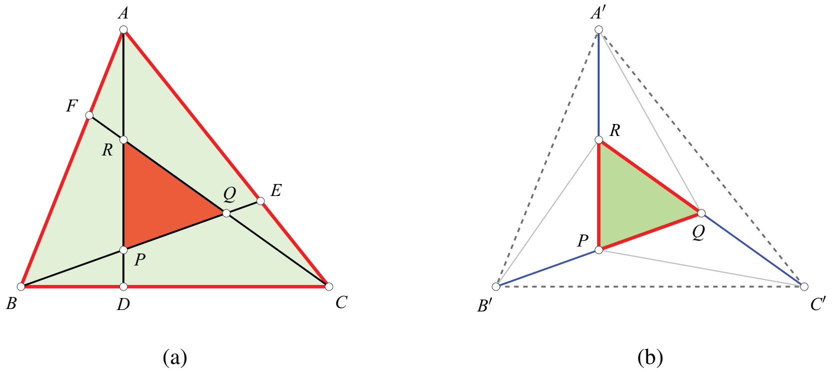
- $f(0) = 1$; for if $t = 0$ then D, E, F coincide with B, C, A respectively, so P, Q, R coincide with B, C, A respectively, and $\triangle PQR$ is identical with $\triangle BCA$.
- $f(1) = 1$; for if $t = 1$ then D, E, F coincide with C, A, B respectively, so P, Q, R coincide with C, A, B respectively, and $\triangle PQR$ is identical with $\triangle CAB$.
- $f(1/2) = 0$; for if $t = 1/2$, then D, E, F lie at the midpoints of BC, CA, AB respectively, which means that AD, BE, CF concur. Hence the points P, Q, R coincide, and $\triangle PQR$ has zero area.
- $f(t) = f(1-t)$. This is because replacing t by $1 - t$ is essentially the same as replacing $\triangle ABC$ by $\triangle ACB$ and retaining the original value of t.

The case when $t = 1/3$. We start by studying the case $t = 1/3$ in an ad hoc manner. Recall that the starting point of Thomas Lingefjärd's investigation was this case; GeoGebra had revealed the ratio of areas to be $1 : 7$. We will now show how the result can be obtained. We shall draw inspiration from some of the 'backward' proofs of Morley's theorem (one such—due to John Conway—is given elsewhere in this very issue of *At Right Angles*). What we shall do is to start with the 'inner' $\triangle PQR$, extend the figure in an appropriate way and construct a $\triangle A'B'C'$ 'around' it in a way that makes it visually obvious that the area of $\triangle A'B'C'$ is 7 times that of $\triangle PQR$. Then we shall show that $\triangle A'B'C'$ is congruent to the given $\triangle ABC$. This will complete the proof.

Figure 2 (a) shows the given configuration, and Figure 2 (b) shows our construction: sides QP, RQ and PR are extended in a cyclic manner through their own length to points B', C' and A' respectively; that is, $PB' = QP$, $QC' = RQ$ and $RA' = PR$. Then the segments $B'C'$, $C'A'$, $A'B'$ are drawn. Let us first show that $\triangle A'B'C'$ has 7 times the area of $\triangle PQR$.

Join $B'R$, $C'P$ and $A'Q$. It is easy to see that the seven triangles thus created all have exactly the same area (we merely have to make repeated use of the fact that a median of a triangle divides it into two parts with equal area). It follows immediately that the area of $\triangle A'B'C'$ is 7 times that of $\triangle PQR$.

Next, we extend sides RP, PQ, QR to meet the sides $B'C'$, $C'A'$, $A'B'$ at points D', E', F' respectively (see Figure 3). We must show that D',



(a) (b)

Figure 2.

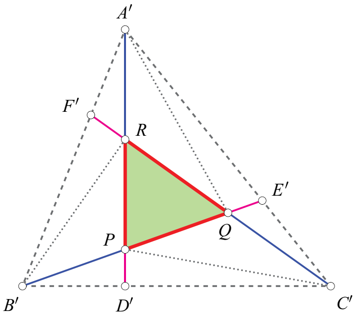


Figure 3.

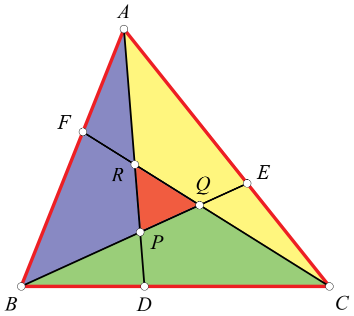


Figure 4.

E', F' are points of trisection of the sides $B'C'$, $C'A'$, $A'B'$ respectively, i.e., $B'D'/B'C' = C'E'/C'A' = A'F'/A'B' = 1/3$. If we do this, then the proof will be complete, for we will have simply reproduced the original configuration—except that we will have started from the 'inside' rather than the 'outside'.

This will follow from a comparison of areas. Let $B'D'/D'C' = k$. Then the ratio of areas of $\triangle PB'D'$ and $\triangle PC'D'$ is also k, as is the ratio of areas of $\triangle A'B'D'$ and $\triangle A'C'D'$. Hence, by subtraction, so also is the ratio of areas of $\triangle A'B'P$ and $\triangle A'C'P$. But a glance at Figure 2 (b) shows that the ratio of areas of $\triangle A'B'P$ and $\triangle A'C'P$ is $2 : 4 = 1 : 2$. Hence $k = 1/2$, implying that $B'D'/B'C' = 1/3$. In the same way we show that $C'E'/C'A' = 1/3$ and $A'F'/A'B' = 1/3$. This is just what we wished to prove.

For another treatment of this problem, please refer to the article *Feynman's Triangle: Some Feedback and More* by Prof Michael de Villiers, available online at:

http://mysite.mweb.co.za/residents/profmd/feynman.pdf.

The configuration we study here is referred to by de Villiers as 'Feynman's Triangle.'

Finding a formula for f(t) in the general case. We now consider the general case and derive a formula for $f(t)$; we use vectors in our derivation. We shall use a 'subtraction logic': we shall *subtract* the areas of $\triangle ABP$, $\triangle BCQ$ and $\triangle CAR$ from that of $\triangle ABC$ and thus obtain the area of $\triangle PQR$. (See Figure 4.)

Let B be treated as the origin, and let

$$\overrightarrow{BC} = \mathbf{c}, \qquad \overrightarrow{BA} = \mathbf{a}.$$

By construction we have

$$\overrightarrow{BD} = t\,\overrightarrow{BC} = t\mathbf{c}, \qquad \overrightarrow{CE} = t\,\overrightarrow{CA} = t\,(\mathbf{a} - \mathbf{c})\,,$$
$$\overrightarrow{AF} = t\,\overrightarrow{AB} = -t\mathbf{a}.$$

Let $AP/AD = k$. To find the unknown quantity k, we argue as follows:

$$\overrightarrow{AD} = \overrightarrow{AB} + \overrightarrow{BD} = -\mathbf{a} + t\mathbf{c},$$
$$\therefore\ \overrightarrow{AP} = k\,\overrightarrow{AD} = -k\mathbf{a} + kt\mathbf{c},$$
$$\therefore\ \overrightarrow{BP} = \overrightarrow{BA} + \overrightarrow{AP} = (1-k)\mathbf{a} + kt\mathbf{c}.$$

We also have:

$$\overrightarrow{BE} = t\mathbf{a} + (1-t)\mathbf{c}.$$

This is a consequence of the 'section formula'. Now consider the last two results we have obtained:

$$\overrightarrow{BP} = (1-k)\mathbf{a} + kt\mathbf{c}, \qquad (1)$$
$$\overrightarrow{BE} = t\mathbf{a} + (1-t)\mathbf{c}. \qquad (2)$$

To proceed further we make use of an important yet simple result from vector algebra.

> *Suppose that* $\mathbf{u}$ *and* $\mathbf{v}$ *are two non-zero, non-parallel vectors. Suppose further that for some choice of non-zero real numbers* a, b, c, d *it happens that* $a\mathbf{u} + b\mathbf{v}$ *is parallel to* $c\mathbf{u} + d\mathbf{v}$. *Then it must be that* $a : b = c : d$. *In other words,* $\mathbf{u}$ *and* $\mathbf{v}$ *are 'mixed' in the same proportions in the two vectors.*

The result holds provided that $\mathbf{u}$ and $\mathbf{v}$ are non-zero and non-parallel (i.e., they 'point in different directions'; in linear algebra we say that they are 'linearly independent'). The proof is based on the fact that a non-zero multiple of $\mathbf{u}$ can never be equal to a non-zero multiple of $\mathbf{v}$.

Now consider the vectors $\overrightarrow{BP}$ and $\overrightarrow{BE}$. They are parallel, and in expressions (1) and (2) they have been expressed in terms of the non-zero, non-parallel vectors **a** and **c**. Hence the above principle applies (i.e., **a** and **c** must be mixed in the same proportions in $\overrightarrow{BP}$ and $\overrightarrow{BE}$), and we have:

$$\frac{1-k}{kt} = \frac{t}{1-t}. \tag{3}$$

This allows us to find the unknown quantity k. Cross-multiplying and solving for k, we get:

$$k = \frac{1-t}{1-t+t^2}. \tag{4}$$

We have thus found the ratio $AP : AD$. Having found this ratio, we easily deduce that

$$\frac{\text{Area of } \triangle ABP}{\text{Area of } \triangle ABD} = \frac{1-t}{1-t+t^2}.$$

We also know that $BD/BC = t$. From this it follows that:

$$\frac{\text{Area of } \triangle ABD}{\text{Area of } \triangle ABC} = t.$$

By multiplication we get:

$$\frac{\text{Area of } \triangle ABP}{\text{Area of } \triangle ABC} = \frac{t(1-t)}{1-t+t^2}.$$

Observe that in the formula the only independent variable is t; there is no dependence on the shape of the triangle! It follows that the very same formula also gives the ratio of areas of $\triangle BCQ$ and $\triangle CAR$ to that of $\triangle ABC$. From this we deduce a formula for $f(t)$:

$$\frac{\text{Area of } \triangle PQR}{\text{Area of } \triangle ABC} = 1 - 3 \times \frac{t(1-t)}{1-t+t^2}.$$

This simplifies after a couple of steps to:

$$f(t) = \frac{(2t-1)^2}{1-t+t^2}. \tag{5}$$

We have obtained the desired formula! We may easily verify that it passes all the tests we had listed: $f(0) = 1 = f(1)$, $f(1/2) = 0$ and $f(t) = f(1-t)$.

Let us also study whether our newly discovered formula yields correct results. Let $t = 1/3$ (the case with which Thomas Lingefjärd had begun his investigation). Let's see what our formula gives:

$$f\left(\tfrac{1}{3}\right) = \frac{\left(1-\frac{2}{3}\right)^2}{1-\frac{1}{3}+\frac{1}{9}} = \frac{\frac{1}{9}}{\frac{7}{9}} = \frac{1}{7}.$$

It has given the right result! More generally, for the case $t = n/(2n+1)$ we find, after some simplification, that

$$f\left(\frac{n}{2n+1}\right) = \frac{1}{3n^2+3n+1}.$$

We have proved the experimentally discovered formula. The reach of the vector approach is indeed very impressive.

Remark. We remark in closing that other treatments are possible, including those that use nothing more sophisticated than the geometry of similar triangles. We will feature one such approach in the next issue.

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Of Paper Folding, Geogebra and Conics

It all began with two activities mentioned on page 19 of At Right Angles Vol.1, no. 2 November 2013, in the article "Axioms of Paper Folding". Needless to say, I gave them a try and was rewarded by the first of the three conics that are part of every higher secondary syllabus – parabola, ellipse and hyperbola. With the help of "Mathematics through Paper Folding" by Alton T. Olson, I could connect a lot of properties of each with my paper models.

Swati Sircar

The conics include circle, ellipse, parabola, hyperbola and a pair of intersecting lines. The first and the last can be readily drawn on paper, but there are no constructions for the remaining three using compasses and straight edge. If one has to befriend (i.e., learn about, understand and ultimately develop some intuition based on practical knowledge) a shape, it is important to work with that shape. Paper folding (to be described presently) provided one such way to generate the three remaining conics on paper without much difficulty.

Each of the three conics is generated by one simple fold carried out repeatedly. We fold a point on a straight line or a circle onto to a fixed point (not on the line or circle). As we vary the point along the line (or circle) we get different fold lines which form the envelope of a curve; this turns out to be a conic. By studying the way the folds are constructed, we can derive the equation of the conic. This repeated folding as a point varies along a line (or a circle) is a simple 'low cost' way of generating a locus without resorting to technology. It enables a student to get a glimpse of how a curve can be generated dynamically.

Keywords: *paper folding, GeoGebra, conics, parabola, envelope, locus, focus, directrix.*

The analysis used to derive the formula of the generated conic can be extended to determine its properties too. And everyone knows how "doing" makes a better case for understanding and remembering than "reading" from a book (or board).

Generating the parabola

Part I of this two-part article focuses on the parabola, the simplest of the three conics.

On a rectangular sheet of paper, consider one edge to be a line *l*. Mark a point *P* anywhere on the sheet, but not on *l*. Next, mentally select a point on *l*, fold that point to *P*, and neatly crease the paper along the fold. Repeat this for a new point on *l*, not far from the point just used. Repeat for a large number of points on *l*, moving only a small distance at each step. The resulting crease lines visibly give rise to a curve (which is best seen by holding up the sheet against the light). See Figure 1.

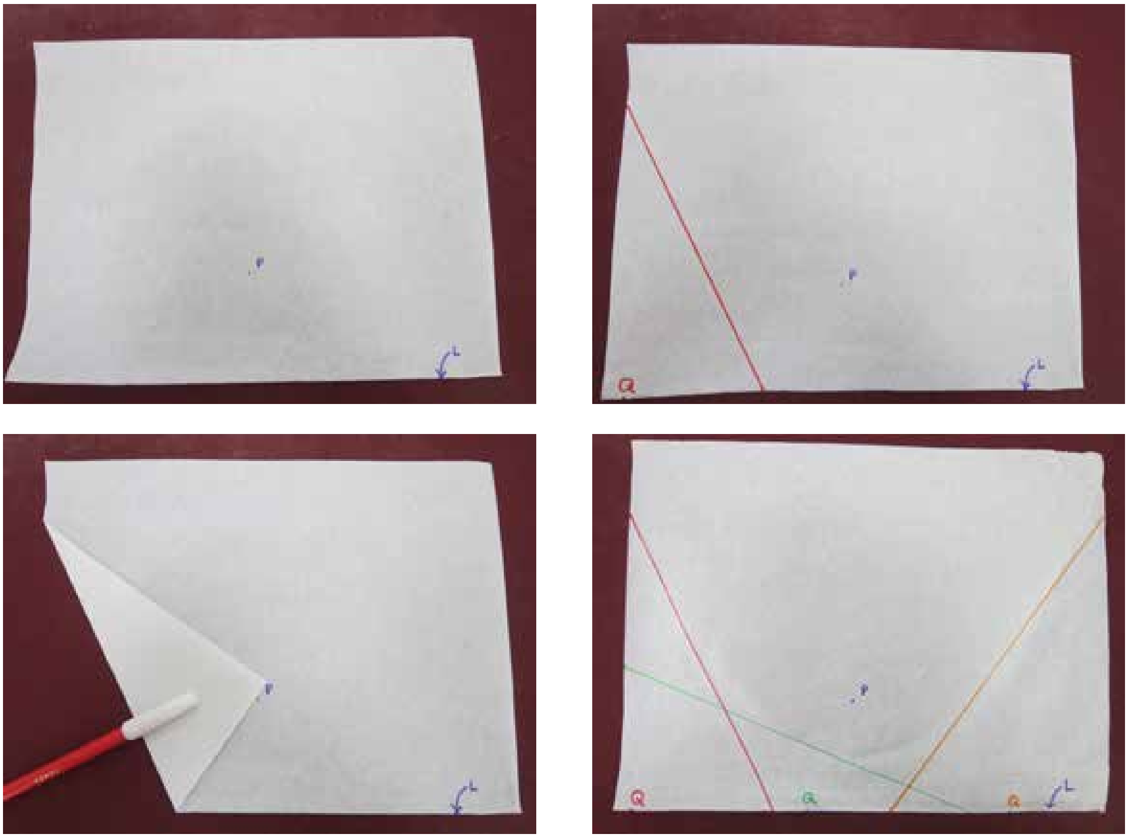


Figure 1 - Photo of paper being folded

What shape emerges from doing this? What is this curve?

Figure 2 - Photo of paper with creases, curve visible

Clearly a parabola!

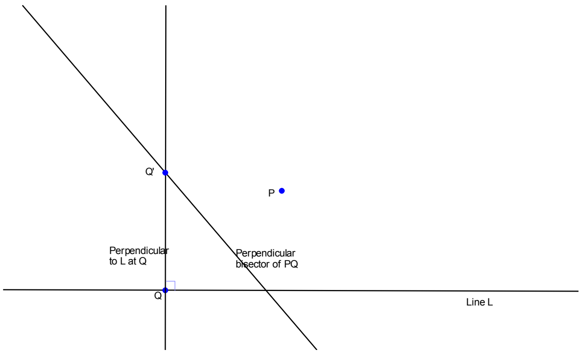
To do this experiment you can use any rectangular sheet of paper. We strongly recommend using discarded one-side-used print outs! The activity now proceeds from paper folding as shown above to get the curve (or rather the envelope of the curve) to the following:

1. Locating the exact point in each fold-line (or tangent to the curve) which belongs to (or lies on) the curve.
2. Cross-checking the construction with GeoGebra, plotting the actual conic and finding its formula.
3. Exploring the properties of the conic.
4. Deriving the calculations and generating the respective conic formula.

We start with the fold lines or creases. How are these fold lines related to the curve? They are tangents to the curve. Next we observe what the fold line is in terms of P and the point (say Q) on l: it is nothing but the perpendicular bisector of PQ. This is a simple application of the laws of reflection!

Next we try to guess the point on each tangent which is a point on the curve as well. Pick any point Q on l and draw the corresponding fold line. Draw a line perpendicular to l, through Q. Observe where this line intersects the fold line corresponding to Q; call this point Q'.

The curve is the locus of Q' as Q varies along l!



Why is this?

$PQ' = QQ'$ by symmetry and the parabola is defined as the locus of all points equidistant from a given point and a given line. If we therefore define the given point P as the focus and the given line l as the directrix, then clearly Q' is equidistant from P and l and the locus of Q' is the parabola which emerges from the creases.

Now we will try to get the formula as follows: Note that Q' is the midpoint of PQ when Q is directly below P, take this point as the origin. Choose P to be $(0, a)$ and line l to be $y = -a$. A variable point on l can be taken as $(q, -a)$ with q varying. Next, we calculate the equation of the fold line using the midpoint of PQ, the slope of PQ and the relation between slopes of perpendicular lines. This gives: $y = (q/2a)\,(x - q/2)$.

The equation of the line through Q and perpendicular to l is: $x = q$.

Hence the point of intersection is $Q'(x, y)$ where:

$$x = q, y = q^2/4a.$$

Hence the locus of the point of intersection is $y = x^2/4a$ or $x^2 = 4ay$. This is the equation of a parabola the standard form obtained by choosing P as $(0, a)$ and l as $y = -a$. There are three other standard forms of the parabola: $x^2 = -4ay$, $y^2 = 4ax$ and $y^2 = -4ax$. These can be obtained by making specific choices of the point P and line. For example, choosing P as $(a, 0)$ and l as $x = -a$ leads to the standard form $y^2 = 4ax$.

Working with GeoGebra

GeoGebra is one of the best known Dynamic Geometry Software (DGS) packages available currently; it can be freely downloaded from http://www.geogebra.org/download. It proves to be invaluable for mathematical investigations.

While most students are comfortable with hands-on activities, the one being studied here involves repeated and careful folding which can get tedious. This can be eliminated with the use of technology. Patterns emerge quickly and can easily be viewed with the help of the 'Trace' button and the judicious use of colour. The student is thus able to focus on the mathematics of the investigation rather than the technicalities of the activity. (Of course, some degree of familiarity with the software is mandatory.) In this instance, even working out the software equivalent of the physical act of paper folding is an instructive exercise. To give the command, the student needs to ask the questions:

- What is the outcome? That is, what "should" happen?
- What is the mathematical aspect to this physical activity?
- How can I give this command?

For example, in order to replicate the steps *Next, mentally select a point on l, fold that point to P and crease the paper along the fold,* the student should arrive at the following answers:

- What is the outcome? **The point on *l* should coincide with P after the folding.**
- What is the mathematical aspect to this physical activity? **P should be the image of the chosen point under reflection in the crease line. The crease on the paper is the mirror for the reflection of the point on *l* so that it coincides with P.**
- How can I give this command? **The crease is the perpendicular bisector of the line joining the point on *l* to P.**

The tedium of shifting the point on *l* on the other hand is smooth sailing in GeoGebra with the use of sliders and the arrow key. Similarly, investigating the three other cases of the parabola becomes very simple with dynamic geometry software since it merely involves locating the line *l* differently (constructing it first above *P* and then to its right or left).

The activity can be replicated in GeoGebra with the following steps:

1. Take any point *P* on the positive *y*-axis
2. Reflect *P* in the *x*-axis to get *P′*
3. Get *l* parallel to the *x*-axis (or perpendicular to the *y*-axis) through *P′*

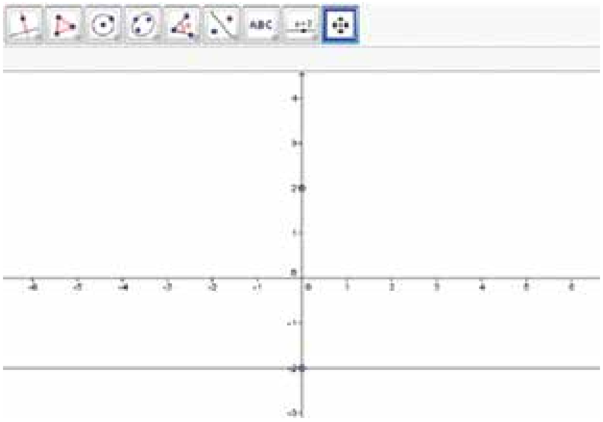


Figure 3: Steps 1-3: Marking the origin, the focus and the directrix

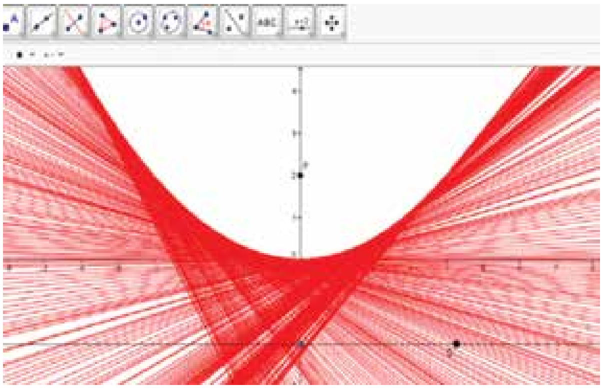


Figure 4: Steps 4–7: Getting the envelope of the parabola

4. Take any point *Q* on *l*
5. Get the perpendicular bisector of *PQ*
6. Use the Trace feature to generate the envelope similar to the folds
7. Move *Q* along *l* and observe the envelope obtained
8. Undo Trace
9. Construct the line perpendicular to *l* through *Q*
10. Find the intersection between this line and the perpendicular bisector of *PQ*, i.e., *Q′*
11. Use the Trace feature on *Q′* to verify that *Q′* indeed is a point on the parabola

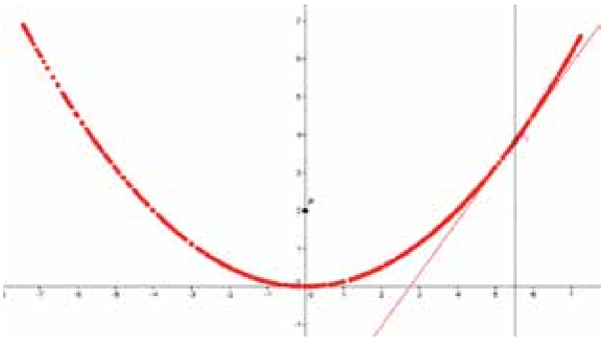


Figure 5: Steps 8-11: Finding the locus of the point of intersection

12. Using the Conics tool, generate the parabola using P and l
13. Verify that Q' really does move along the parabola
14. Now vary P along the y-axis and thereby verify the formula $x^2 = 4ay$

Conclusion

Parabolas are encountered in many everyday contexts. For example, think of the path taken by a cricket ball when it has been hit for a 'six'. Another example is the parabolic reflectors used in satellite dishes and car head lights. If you study the geometry of the folding process described above, you should be able prove the reflective property for yourself.

By actually generating a parabola using paper-folding and verifying its formula, students can understand its properties better and appreciate its multiple uses. The use of dynamic geometry software provides a second window for the study of the parabola and the use of two different media highlights different aspects of the same concept. Best of all, it caters to a variety of learners by playing to their strengths. What could be more enabling?

References

1. *Mathematics Through Paper Folding*, Alton T. Olson, University of Alberta

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Rectangle in a Triangle ...

When does it have Maximum Area?

A GeoGebra Exploration

SHAILESH A SHIRALI

Problem. *Let ABC be a triangle in which $\measuredangle B$ and $\measuredangle C$ are acute, and let $PQRS$ be a rectangle inscribed in the triangle, with vertex P on side AB, vertices Q and R on side BC, and vertex S on side AC (so $PQ \perp BC$ and $PS \parallel BC$). Find the maximum possible value of the ratio of the area of rectangle $PQRS$ to that of triangle ABC.*

Obviously, infinitely many possibilities exist for the inscribed rectangle, as Figure 1 suggests. Which of them has maximum area?

Solution. The problem is tailor-made for a "tech investigation"! We invite you to use the applet available at

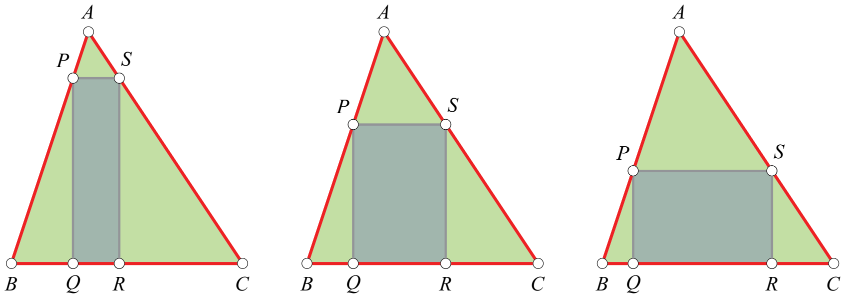

Figure 1. Many rectangles inscribed in a triangle: which has the largest area?

Keywords: triangle, rectangle, inscribed, area, maximum, ratio, investigation, GeoGebra

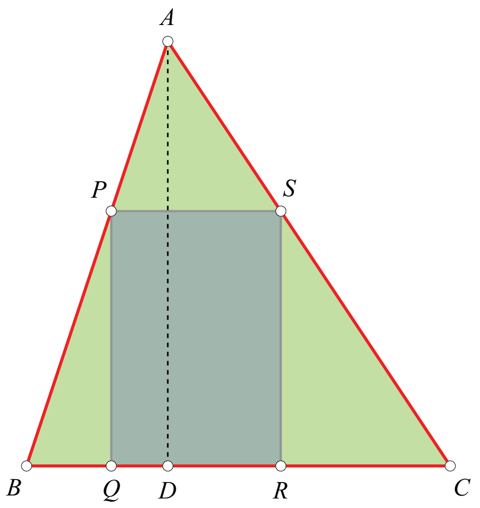

Figure 2. A 'typical' rectangle inscribed in a triangle

https://www.geogebratube.org/student/m140986 (or to write an applet of such a kind for yourself) and arrive at a plausible answer.

In Figure 2, let $BC = a$, $CA = b$, $AB = c$. Let AD be the altitude through A of $\triangle ABC$, and let its length be h; then the area of triangle ABC is $\frac{1}{2}ah$.

Now, note that $\triangle APS$ is similar to $\triangle ABC$. Let $AP/AB = t$ (so t is the coefficient of similarity); then $AS/AC = t$ and $PS/BC = t$ as well. Thus, the dimensions of $\triangle APS$ are ta, tb, tc. Hence, by similarity, the altitude through A of $\triangle APS$ has length th. It follows that $PQ = h - th = (1-t)h$.

Hence the area of rectangle $PQRS$ is $t(1-t)ah$, and the ratio of the area of rectangle $PQRS$ to that of triangle ABC is $2t(1-t)$.

The function $2t(1-t)$ is quadratic, and it is easy to find its maximum value by simple algebra (no calculus is needed). We find that for $0 \le t \le 1$, the maximum value attained by $2t(1-t)$ is $\frac{1}{2}$, attained when $t = \frac{1}{2}$. Here is a proof:

$$\begin{aligned} t(1-t) = t - t^2 &= \frac{1}{4} - \left(\frac{1}{4} - t + t^2\right) \\ &= \frac{1}{4} - \left(\frac{1}{2} - t\right)^2 \le \frac{1}{4}, \end{aligned}$$

with equality just when $t = \frac{1}{2}$. Hence $2t(1-t) \le \frac{1}{2}$, with equality just when $t = \frac{1}{2}$.

So the maximum possible value of the ratio of the area of rectangle $PQRS$ to that of triangle ABC is $\frac{1}{2}$.

Is this what your GeoGebra exploration revealed to you?

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A Potpourri of Problems

C⊗MαC

We present once again a miscellaneous collection of nice problems, followed by their solutions. We state the problems first so you have a chance to try them out on your own.

Problems

(1) Find all positive integers which can be written as the sum of the squares of some two consecutive non-negative integers and also as the sum of the fourth powers of some two consecutive non-negative integers. In other words, solve the equation

$$m^2 + (m+1)^2 = n^4 + (n+1)^4$$

over the non-negative integers $\mathbb{N}_0$.

(2) Is there any positive rational number r other than 1 such that $r + \frac{1}{r}$ is an integer?

(3) Determine the smallest prime that does not divide any five-digit number whose digits are in strictly increasing order.

[Regional Math Olympiad 2013]

(4) A "three-sum" integer n is one that can be expressed in the form $n = a + b + c$, where a, b, c are positive integers such that $a < b < c$ and a divides b and b divides c. For example, 7 is a three-sum by virtue of the equality $1 + 2 + 4 = 7$. It is easy to see that $1, 2, 3$ are not three-sums. How many non-three-sums are there?

[Adapted from the Indian National Math Olympiad 2011]

Solutions

(1) *Solve the equation* $m^2 + (m+1)^2 = n^4 + (n+1)^4$ *over the non-negative integers.*

Solution: As with so many such problems, the tools that come handy here are the "completing the square" technique and the "difference of two squares" factorization. (The lesson for you therefore is: always be prepared to use these two humble tools.)

Keywords: integer, square, fourth power, rational, prime, multiple, composite, difference of squares, completing the square, Pythagoras, triple

The equation yields on simplification:

$$m^2 + m = n^4 + 2n^3 + 3n^2 + 2n,$$

$$\therefore \left(m + \frac{1}{2}\right)^2 - \frac{1}{4} = \left(n^2 + n + 1\right)^2 - 1$$

(on completing the square on both sides)

$$\therefore X^2 - 1 = Y^2 - 4,$$

where $X = 2m + 1$ and $Y = 2\left(n^2 + n + 1\right)$.

Hence $Y^2 - X^2 = 3$.

Here X and Y are positive integers. The only expression for 3 as a difference of two squares of positive integers is $3 = 2^2 - 1^2$ (this draws from the fact that the only expression for 3 as a product of two positive integers is $3 = 3 \times 1$), therefore $(Y, X) = (2, 1)$, giving $m = n = 0$. So the equation $m^2 + (m+1)^2 = n^4 + (n+1)^4$ has only the trivial solution in which both m and n are 0. Hence the only positive integer expressible in the form described is 1.

(2) *Is there any positive rational number r other than 1 such that $r + \frac{1}{r}$ is an integer?*

Solution: The answer to this is **No**. But in the process of getting to the answer, we find an unexpected and nice link between this question and primitive Pythagorean triples!

Let $r = \frac{a}{b}$ where a, b are coprime positive integers, and suppose that $r + \frac{1}{r} = n$, a positive integer. Then we have:

$$r + \frac{1}{r} = \frac{a}{b} + \frac{b}{a} = \frac{a^2 + b^2}{ab} = n,$$
$$\therefore \quad a^2 + b^2 - nab = 0.$$

If $b > 1$ then there exists a prime p dividing b. The equality $a^2 + b^2 - nab = 0$ now shows that p divides a^2 and hence that p divides a. (This statement can be made precisely because p is prime.) But this means that a, b are *not* coprime. Hence there does not exist such a prime p. What kind of value can b take if it is to be not divisible by any prime number? Clearly we must have $b = 1$.

The same reasoning applied to a shows that $a = 1$. Hence $a = b = 1$, which means that $r = 1$. This yields $r + \frac{1}{r} = 2$, an integer. So there is just one positive rational number r for which $r + \frac{1}{r}$ is an integer, namely: $r = 1$.

The connection with Pythagorean triples is discussed below.

(3) *Determine the smallest prime that does not divide any five-digit number whose digits are in strictly increasing order.*

Solution: The total number of such numbers is finite; it is equal to $\binom{9}{5} = 126$. For any collection of 126 integers, there must exist infinitely many primes that do not divide any of the integers and hence there must exist a smallest such prime.

Let's see what this prime might be for the set of five digit numbers whose digits are in strictly increasing order. It cannot be 2, since 2 divides 12346. Nor can it be 3 or 5 since 3 and 5 divide 12345. How about 7? Experimentation reveals that 7 divides 12348. So the answer is not any of 2, 3, 5, 7. How about 11? Trials reveal that 11 does not divide any of 12345, 12346, 23456, 12347, 12348. We begin to suspect: maybe 11 is the answer? It turns out to be, quite contrary to our intuition. Here is the proof.

Let $N = \overline{abcde}$ be a five-digit base ten number with $0 < a < b < c < d < e < 10$. We shall show that 11 does not divide N. For this we must show that 11 does not divide $a - b + c - d + e$. Now we have:

$$a - b + c - d + e = a + (c - b) + (e - d) > a > 0.$$

On the other hand:

$$a - b + c - d + e = e - (d - c) - (b - a) < e.$$

So we have:

$$a < a - b + c - d + e < e.$$

Hence $a - b + c - d + e$ is a single-digit non-zero number and therefore is not a multiple of 11.

It follows that N too is not a multiple of 11. So 11 is the sought-after prime number.

(4) *A "three-sum" integer n is one that can be expressed in the form $n = a + b + c$, where a, b, c are positive integers such that $a < b < c$ and a divides b and b divides c. How many non-three-sums are there?*

The definition may be re-stated as follows: a positive integer n is a three-sum if it can be expressed in the form $n = a + ax + axy$, where a, x, y are positive integers with $x > 1, y > 1$. How many non-three-sum integers are there?

Solution: We claim that all numbers are three-sums except 1, 2, 3, 4, 5, 6, 8, 12, 24. The proof is as follows.

- If $n = a + ax + axy$ is a three-sum, then the numbers $n - a = ax(1 + y)$ and $n/a - 1 = x(1 + y)$ are composite. It follows that if $n - a$ or $n/a - 1$ is prime for all feasible values of a, then n is a non-three-sum.
- Suppose $n = a + ax + axy$ is a three-sum. Then $a \geq 1, ax \geq 2, axy \geq 4$, hence $n \geq 7$. It follows that 1, 2, 3, 4, 5, 6 are non-three-sums.
- If n is a three-sum, so is any multiple of n.
- If $n > 6$ is an odd number, then we can write $n = 1 + 2 + (n - 3)$. Hence all odd numbers exceeding 6 are three-sums, as are all multiples of these numbers.
- 8 is a non-three-sum. For, if $8 = a + ax + axy$, then $a = 1$ or 2. The primality of $8 - 1 = 7$ contradicts the former, while the primality of $8/2 - 1 = 3$ contradicts the latter.
- The equality $10 = 1 + 3 + 6$ shows that 10 is a three-sum. Hence all multiples of 10 are three-sums.
- 12 is a non-three-sum. For, if $12 = a + ax + axy$, then a is one of 1, 2, 3, 4. The primality of $12 - 1 = 11$ contradicts the possibility $a = 1$, the primality of $12/2 - 1 = 5$ contradicts the possibility $a = 2$, the primality of $12/3 - 1 = 3$ contradicts the possibility $a = 3$, and similarly for $a = 4$.
- The equality $16 = 1 + 5 + 10$ shows that 16 is a three-sum. Hence all multiples of 16 are three-sums.
- 24 is a non-three-sum. For, if $24 = a + ax + axy$, then a is one of 1, 2, 3, 4, 6. The primality of the numbers $24-1 = 23, 24/2-1 = 11, 24/3-1 = 7$, $24/4 - 1 = 5$ and $24/6 - 1 = 3$ contradict respectively the possibilities $a = 1, a = 2, a = 3, a = 4$ and $a = 6$.
- With the exception of 24, every number from 15 to 30 is a three-sum, hence so is every even number from 30 till 60 except possibly for 48. But this case is settled by $48 = 3 + 9 + 36$. Hence every number from 30 till 60 is a three-sum. By repeated doubling it follows that every even number beyond 24 is a three-sum.
- It follows that the only non-three-sums are 1, 2, 3, 4, 5, 6, 8, 12, 24.

A Pythagorean connection

Now as promised we describe how while examining the expression $r + \frac{1}{r}$ we are led to a way for generating PPTs. We will only point out the connection and leave the proof to you. In the expression $r + \frac{1}{r}$, let us assign different rational values to r and then let us examine the resulting value of the expression. Here are a few such instances:

r	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{2}{3}$	$\frac{3}{4}$	$\frac{3}{5}$	$\frac{4}{5}$
$r + \frac{1}{r}$	$\frac{5}{2}$	$\frac{10}{3}$	$\frac{17}{4}$	$\frac{26}{5}$	$\frac{13}{6}$	$\frac{25}{12}$	$\frac{34}{15}$	$\frac{41}{20}$

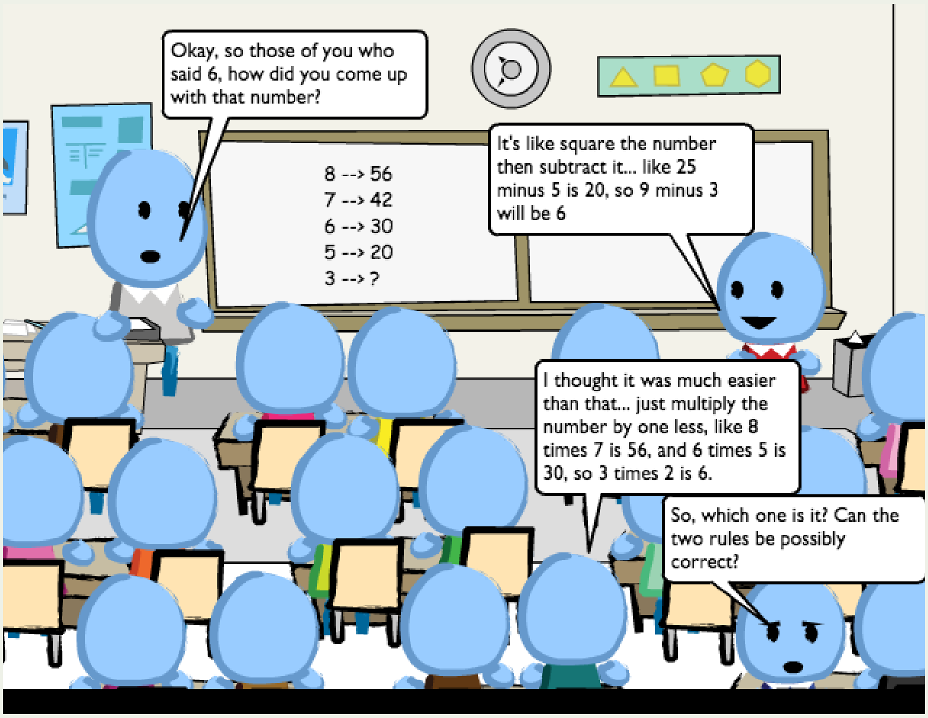
Do you see the connection? Here it is: If we halve each fraction in the last row, we obtain the hypotenuse and one leg of an integer-sided right-angled triangle! We have copied the above array afresh with two extra rows to make this clear.

r	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{2}{3}$	$\frac{3}{4}$	$\frac{3}{5}$	$\frac{4}{5}$
$r+\frac{1}{r}$	$\frac{5}{2}$	$\frac{10}{3}$	$\frac{17}{4}$	$\frac{26}{5}$	$\frac{13}{6}$	$\frac{25}{12}$	$\frac{34}{15}$	$\frac{41}{20}$
$\frac{1}{2}\left(r+\frac{1}{r}\right)$	$\frac{5}{4}$	$\frac{5}{3}$	$\frac{17}{8}$	$\frac{13}{5}$	$\frac{13}{12}$	$\frac{25}{24}$	$\frac{17}{15}$	$\frac{41}{40}$
PPT	$(3,4,5)$	$(3,4,5)$	$(8,15,17)$	$(5,12,13)$	$(5,12,13)$	$(7,24,25)$	$(8,15,17)$	$(9,40,41)$

Many questions arise from this display which could serve as the starting point of further investigations. For example, we see that in some cases, two different r-values yield the same PPT: $\frac{1}{2}$ and $\frac{1}{3}$; $\frac{1}{5}$ and $\frac{2}{3}$; and so on. What is the explanation governing this? We leave the investigation to the reader.

The **COMMUNITY MATHEMATICS CENTRE** (CoMaC) is an outreach arm of Rishi Valley Education Centre (AP) and Sahyadri School (KFI). It holds workshops in the teaching of mathematics and undertakes preparation of teaching materials for State Governments and NGOs. CoMaC may be contacted at shailesh.shirali@gmail.com.

THE EXPRESSION SAYS IT ALL!



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What expressions did your students come up with to generalise the pattern shown?

Did it spark off a good discussion?

Write in at AtRiA.editor@apu.edu.in and share your story for other teachers to engage with.

Problems for the Middle School

Problem Editor : R. ATHMARAMAN

Problems

Problem IV-2-M.1
(a) Find the sum of the prime divisors of 2015.

(b) Find another number for which the sum of the prime divisors is the same.

Problem IV-2-M.2
The sum of the digits of a natural number n is 2015. Can n be a perfect square?

Problem IV-2-M.3
Is there any five-digit perfect square such that when 1 is added to each digit, the answer is again a perfect square? (You may assume that the addition of 1 to each digit starts from the units end and proceeds 'leftwards'. If the addition of 1 results in a 'carry', the 'carry' is added to the digit on the left.)

Problem IV-2-M.4
The sum of three integers is 0. Show that the sum of their fourth powers when doubled yields a perfect square.

Problem IV-2-M.5
Consider the following two relations:

$$a - b - c = 0, \qquad (1)$$

$$a^4 + b^4 + c^4 = 2\left(b^2c^2 + c^2a^2 + a^2b^2\right). \qquad (2)$$

It is easy to prove (2) from (1) by simple manipulation. Now the interesting thing is: while identity (2) is symmetrical in a, b, c, condition (1) is not so. How do you explain this?

Problem IV-2-M.6
Let a, b be two positive real numbers. Denote their product ab by P, and their sum $a + b$ by S. The following facts are known:

- If the sum S is a constant, then the maximum value of the product P is $\frac{1}{4}S^2$.
- If the product P is a constant, then the minimum value of the sum S is $2\sqrt{P}$.

Use these results to find the maximum and minimum values taken by $\dfrac{x^2}{1+x^4}$.

Problem IV-2-M.7
Given a parallelogram $ABCD$ and a point P inside the parallelogram such that $\measuredangle APB$ and $\measuredangle CPD$ are supplementary. Show that $\measuredangle PBC = \measuredangle PDC$.

Keywords: *digit, parity, integer, product, sum, multiple, divisibility, maximum, minimum, parallelogram, supplementary, circle, chord*

Solutions of Problems in Issue-IV-1 (March 2015)

Solution to problem IV-1-M.1 *If the sum of the reciprocals of three non-zero real numbers is zero, can the sum of the three numbers be zero?*

The answer is: **No**. Let the numbers be a, b, c (all non-zero). Then $1/a + 1/b + 1/c = 0$, hence $abc(1/a + 1/b + 1/c) = 0$. This leads to: $bc + ca + ab = 0$ and then to: $(a + b + c)^2 = a^2 + b^2 + c^2$. Since the right-hand side of the last equality must be positive, so must be the left-hand side, hence $a + b + c \neq 0$.

Solution to problem IV-1-M.2 *If a and b are integers such that $a + 2b$ and $b + 2a$ are square numbers, show that each of a and b is divisible by 3.*

Let $a + 2b = c^2$ and $2a + b = d^2$ where c, d are integers. Then $c^2 + d^2 = 3(a + b)$ which is a multiple of 3. We first show that this implies that both c and d are multiples of 3. If either one of them is a multiple of 3, then so is the other one too, clearly. If both c and d are non-multiples of 3, then both c^2 and d^2 leave remainder 1 on division by 3, hence $c^2 + d^2$ cannot be a multiple of 3. This shows that both c and d are multiples of 3. Let $c = 3u$ and $d = 3v$ where u, v are integers. Then $a + 2b = 9u^2$ and $2a + b = 9v^2$. Solving for a, b we get: $a = 3\left(2v^2 - u^2\right)$ and $b = 3\left(2u^2 - v^2\right)$. This shows that both a and b are multiples of 3.

Solution to problem IV-1-M.3 *Show that a power of 2 cannot be represented as a sum of two or more consecutive positive integers.*

Suppose that $2^n = a + (a + 1) + \cdots + (a + b)$ where n, a, b are positive integers. Here we have written 2^n as a sum of $b + 1$ consecutive positive integers. Using the formula for the sum of an arithmetic progression we get:

$$\begin{aligned} 2^n &= \text{number of terms} \times \frac{\text{first term} + \text{last term}}{2} \\ &= \frac{(b+1)(2a+b)}{2}, \end{aligned}$$

hence $2^{n+1} = (b + 1)(2a + b)$. Now consider the integers $b + 1$ and $2a + b$. Both exceed 1. Their sum is $2a + 2b + 1$, which is an odd number. Hence one of them is odd. This means that 2^{n+1} has an odd divisor exceeding 1. But this is not possible as 2^{n+1} has no odd prime divisors. Hence the equality is not possible.

Solution to problem IV-1-M.4 *In $\triangle ABC$, one of the mid-segments is longer than one of its medians. Show that $\triangle ABC$ is obtuse-angled.* (A mid-segment of a triangle is a segment joining the midpoints of two sides of a triangle.)

It suffices to prove the following: *If $\triangle ABC$ is acute-angled, then its shortest median is longer than its longest mid-segment*. Suppose that $\triangle ABC$ is acute-angled with BC its longest side (see Figure 1). This means that $\measuredangle A$ is the largest angle of the triangle, but $\measuredangle A < 90^\circ$. Let D be the mid-point of BC. Then AD is its shortest median. We must show that $AD > \frac{1}{2}BC$, i.e., $2AD > BC$. Complete the parallelogram $ABEC$. We must show that $AE > BC$, i.e., AE is the longer diagonal. Note that $\measuredangle ABE > \measuredangle BAC$; this is so because the two angles are supplementary but $\measuredangle BAC < 90^\circ$ implying that $\measuredangle ABE > 90^\circ$. Now consider $\triangle ABC$ and $\triangle BAE$. We have: $AB = BA$, $AC = BE$ but $\measuredangle ABE > 90^\circ > \measuredangle BAC$. The cosine rule applied to the two triangles now shows that $AE > BC$. (It also follows from the inequality form of the SAS congruence theorem.)

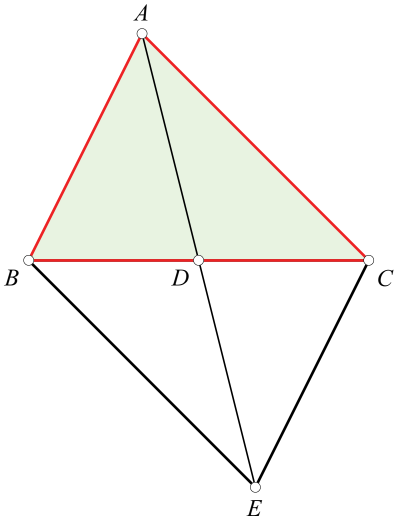


Figure 1.

Solution to problem IV-1-M.5 *Show that in any circle, two non-diametrical chords cannot both bisect each other.*

Suppose that AB and CD are chords of a circle, intersecting at a point M which is their common midpoint (see Figure 2). Then $ACBD$ is a

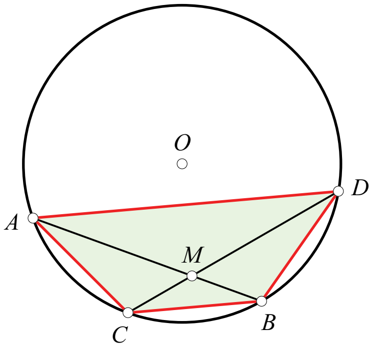


Figure 2.

parallelogram. But a cyclic parallelogram is a rectangle, as its opposite angles are equal and add up to 180°. Hence AB and CD are diameters of the circle. So if AB and CD are *not* diameters, the stated situation is not possible. (Another proof: Join the centre O to M. Then we have $OM \perp AB$ and also $OM \perp CD$, an impossibility if $O \neq M$.)

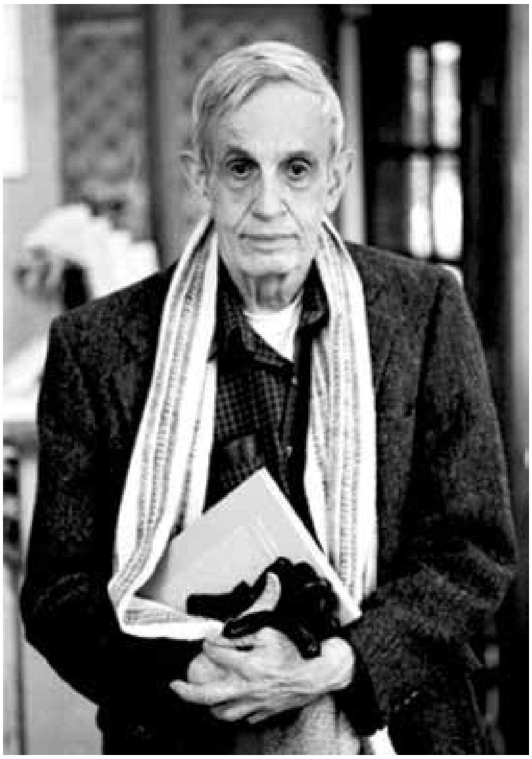
Solution to problem IV-1-M.6 *A and B are two boxes. Box A contains* 100 *white marbles, while box B contains* 100 *black marbles. We take out* 10 *marbles at random from box A and put them into box B. After this we take out* 10 *marbles at random from box B and put them in box A. Which is now larger: the number of black marbles in box A, or the number of white marbles in box B?*

The two numbers are equal.

Solution to problem IV-1-M.7 *Let* $a_1, a_2, a_3, \ldots, a_n$ *represent the numbers* $1, 2, 3, \ldots, n$ *subjected to an arbitrary arrangement. Assume that* n *is odd. Consider the number* $X = (a_1 - 1)(a_2 - 2)(a_3 - 3) \ldots (a_n - n)$. *Is X even or odd?*

The *sum* of the numbers $a_1 - 1, a_2 - 2, \ldots, a_n - n$ is clearly 0, since the string $(a_1, a_2, a_3, \ldots, a_n)$ is a permutation of the string $(1, 2, 3, \ldots, n)$. The sum of an odd number of odd numbers cannot be 0; hence at least one of the numbers $a_1 - 1, a_2 - 2, \ldots, a_n - n$ is even. Hence X is even.

JOHN NASH (1928-2015)



Mathematician-economist John Nash, known for his work in a variety of fields including game theory, differential geometry and partial differential equations, died in May this year, along with his wife Alice Nash, in a freak car accident in New Jersey, USA.

John Nash received the Nobel Prize for economics in 1994 for his work in game theory which has subsequently had a profound impact in economics. (He shared the prize with two other game theorists. His work focused on the study of non-cooperative games and resulted in an important concept now known as *Nash equilibrium*.) This very year (2015), he was awarded the prestigious Abel prize for his work in nonlinear partial differential equations.

Nash is best known not only for his work in game theory, but also for the fact that he began to show signs of severe mental illness when he was about 30 years old (1959). The following decade was a period of intense struggle for him as he passed in and out of psychiatric hospitals, receiving numerous treatments. His symptoms began to abate as he grew older, and much later in his life he ascribed his recovery more to the natural process of ageing than to any treatment. His suffering during that period led to the writing of a Pulitzer prize-winning and bestselling book, *A Beautiful Mind*, by Sylvia Nasar, and later an award-winning Hollywood film by the same name, starring Russell Crowe.

Source: https://en.wikipedia.org/wiki/File:John_Forbes_Nash,_Jr._by_Peter_Badge.jpg

Problems for the Senior School

Problem Editors : PRITHWIJIT DE & SHAILESH SHIRALI

Problems

Problem IV-2-S.1

Starting with any three-digit number n we obtain a new number $f(n)$ which is equal to the sum of the three digits of n, their three products in pairs and the product of all three digits. (Example: $f(325) = (3+2+5)+(6+15+10)+30 = 71$.) Find all three-digit numbers such that $f(n) = n$. [Adapted from British Mathematical Olympiad, 1994]

Problem IV-2-S.2

Solve in integers the equation: $x+y = x^2-xy+y^2$.

Problem IV-2-S.3

Let a, b, c be the lengths of the sides of a scalene triangle and A, B, C be the opposite angles. Prove that

$$2(Aa+Bb+Cc) > Ab+Ac+Ba+Bc+Ca+Cb.$$

Problem IV-2-S.4

Three positive real numbers a, b, c are such that

$$a^2+5b^2+4c^2-4ab-4bc = 0.$$

Can a, b, c be the lengths of the sides of a triangle? Justify your answer. [Regional Mathematical Olympiad, 2014]

Problem IV-2-S.5

Let D, E, F be the points of contact of the incircle of an acute-angled triangle ABC with the sides BC, CA, AB respectively. Let I_1, I_2, I_3 be the incentres of the triangles AFE, BDF, CED, respectively. Prove that the lines I_1D, I_2E, I_3F concur. [Adapted from the Regional Mathematical Olympiad, 2014]

Solutions of Problems in Issue-IV-1 (March 2015)

Solution to problem IV-1-S.1 *Let* $A = \{1, 3, 3^2, 3^3, \ldots, 3^{2014}\}$. *A partition of* A *is a union of non-empty disjoint subsets of* A.

(a) *Prove that there is no partition of* A *such that the product of all the elements in each subset is a square.*

Assume that such a partition exists. Then the product of all elements of A must be a square as well. But the product of all elements is equal to $3^{2015\times1007}$, which is not a square.

(b) *Does there exist a partition of* A *such that the sum of elements in each subset is a square?*

Keywords: digit, sum, product, triangle, sides, angles, incircle, partition

Observe that $1 + 3 = 2^2$ and therefore $3^{2n} + 3^{2n+1} = (3^n \times 2)^2$. Hence a possible partition is:

$$A = \{1, 3\} \cup \{3^2, 3^3\} \cup \cdots \cup \{3^{2012}, 3^{2013}\} \cup \{3^{2014}\}.$$

Solution to problem IV-1-S.2 Let ABC be a triangle in which $\measuredangle A = 135^\circ$. The perpendicular to line AB at A intersects side BC at D, and the bisector of $\measuredangle B$ intersects side AC at E. Find the measure of $\measuredangle BED$ (see Figure 1).

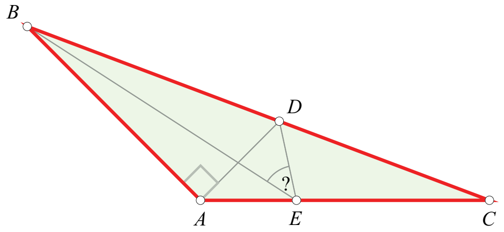


Figure 1.

Let I on BE be such that IA bisects $\measuredangle DAB$. Then $\measuredangle DIB = 90^\circ + \frac{1}{2}\measuredangle DAB = 135^\circ$, hence $\triangle ABE \sim \triangle IBD$. It follows that $\frac{AB}{EB} = \frac{BI}{BD}$. Therefore $\triangle ABI \sim \triangle DBE$ as well. Since $\measuredangle BED = \measuredangle BAI$, we infer that $\measuredangle BED = 45^\circ$.

Solution to problem IV-1-S.3 *Determine all pairs* (n, p) *of positive integers such that*

$$\left(n^2 + 1\right)\left(p^2 + 1\right) + 45 = 2(2n + 1)(3p + 1).$$

The given expression simplifies to

$$(np - 6)^2 + (n - 2)^2 + (p - 3)^2 = 5,$$

hence $(np - 6)^2$, $(n - 2)^2$ and $(p - 3)^2$ are equal to 0, 1 and 4, in some order. By inspection we find $(n, p) = (2, 4), (2, 2)$.

Solution to problem IV-1-S.4 *Determine all irrational numbers* x *such that both* $x^2 + x$ *and* $x^3 + 2x^2$ *are integers.*

Let $a = x^2 + x$ and $b = x^3 + 2x^2$. Then $b - ax = x^2 = a - x$, hence $x(a - 1) = b - a$. Since x is an irrational number and a, b are integers, we deduce that $a = b = 1$, and therefore

$$x = \frac{-1 \pm \sqrt{5}}{2}.$$

Solution to problem IV-1-S.5 *Find all pairs* (p, q) *of prime numbers, with* $p \leq q$, *such that* $p(2q + 1) + q(2p + 1) = 2\left(p^2 + q^2\right)$.

The equality can be written as $(q + p) = 2(q - p)^2$, which shows that p, q are unequal (if not, the right-hand side would be 0 while the left-hand side is positive). The same equation also shows that both p, q are odd, for the right-hand side is even and so therefore must be the left-hand side, i.e., $q + p$. Hence $3 \leq p < q$.

Suppose that $5 \leq p$. Since $p, q \geq 5$, both p and q leave remainder 1 or 2 when divided by 3. If p and q leave the same remainder when divided by 3, then 3 divides the right-hand side, i.e., $2(q - p)^2$, but not the left-hand side, i.e., $q + p$. If p and q leave different remainders when divided by 3, then 3 divides the left-hand side but not the right-hand side. Hence it cannot be that $5 \leq p$. Therefore $p < 5$. The only odd prime less than 5 is 3, so $p = 3$. The equation now yields $q + 3 = 2(q - 3)^2$, which simplifies to $2q^2 - 13q + 15 = 0$, or $(q - 5)(2q - 3) = 0$. This yields $q = 5$. Hence $p = 3$ and $q = 5$.

review

Review of

"The Crest of the Peacock: Non-European Roots of Mathematics"

By George Gheverghese Joseph

R Ramanujam

Background

If we made contact with an alien civilization, what would we communicate with them, and how? Lacking a common language, how can we expect to make sense at all? Many thinkers have pondered this and among all their divergent views, one thread is common: the language of mathematics presents the best possibility for such communication. For instance, sending the sequence of primes expressed in base one (unary) may be a good idea.

What is the conviction that underlies such a suggestion? We believe that mathematical pursuit of some kind is generic to intelligence and sapience, therefore we expect and look for mathematical pursuits in all civilizations. But then, each civilization and cultural tradition may develop mathematics independently and in its own way, so while communication of what is basic and fundamental may yet be possible, that of sophisticated technique may be hard.

This is an easy argument to accept, and yet hard to internalize in a way that informs our practice, especially of the teaching of mathematics. We have a picture of mathematics as a modern discipline, and trace its roots to ancient Greece and the European Renaissance, absorbing the vacuum in between, but fail to ask what other trajectories of mathematical development might have taken place in other cultures, especially during these "dark ages". The 'we' here includes historians and practitioners of mathematics, all over the world.

Such ignorance and intellectual laziness has come under question in the last couple of decades, and at least in the corners inhabited by historians of mathematics, the Euro-centric account is being interrogated now. The book under review, first published in 1991, contributed in no small way to this change of perspective and was tremendously influential in causing the shift. By now it has seen three editions and is yet relevant, especially in the Indian context, where it has become fashionable to talk of deep mathematical knowledge in 'ancient India' without factual basis. Keeping a clear view of history while yet not buying in entirely to the Euro-centric account, is important for opening our minds to historical progressions of ideas, an integral component of learning.

The central aims of the book (from Chapter 1) are, to highlight:

1. "the global nature of mathematical pursuits of one kind or another,
2. the possibility of independent mathematical development within each cultural tradition followed or not followed by cross-fertilization,
3. the crucial importance of diverse transmissions of mathematics across cultures, culminating in the creation of the unified discipline of modern mathematics".

George Joseph does a remarkable job of addressing these aims, and takes us through a fascinating journey of mathematics in non-European cultures, principally the Asian ones.

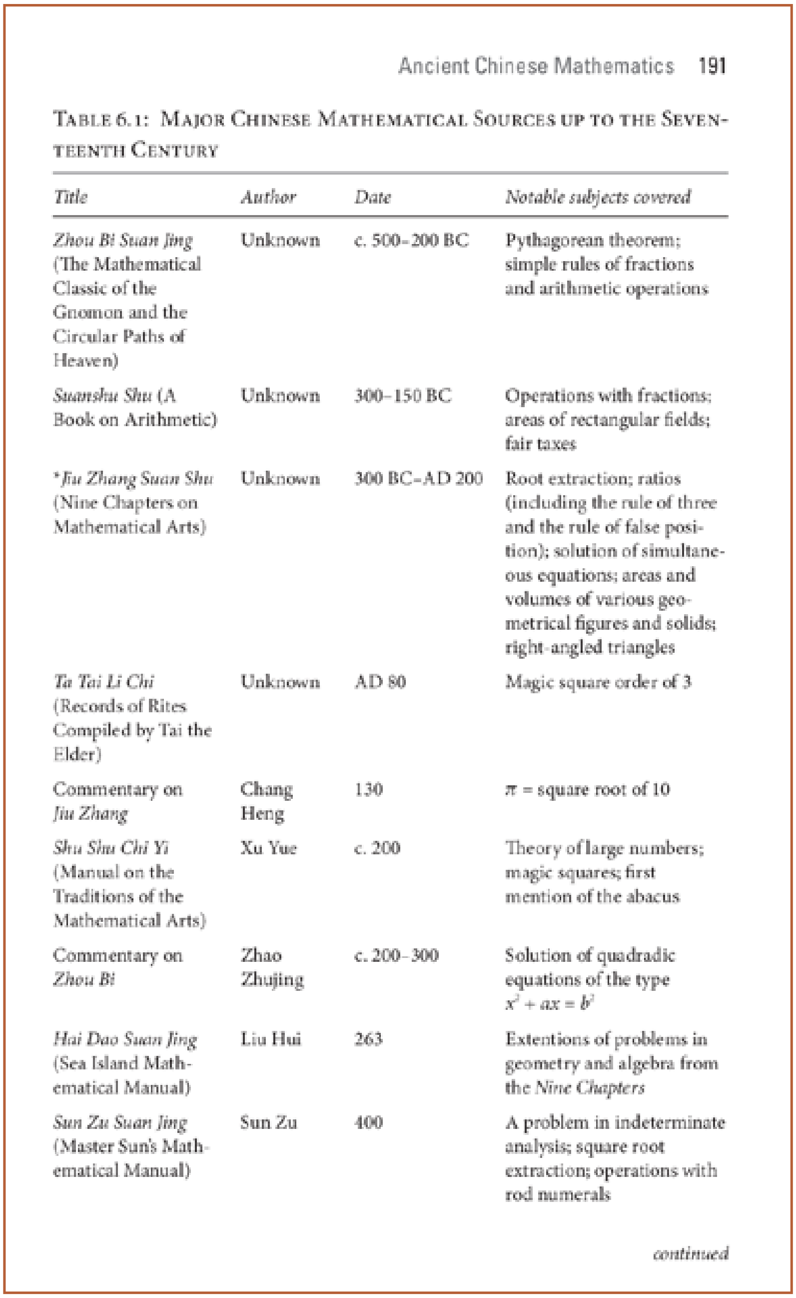
Content

The book opens with a discussion of the Euro-centric picture dominant in the history of mathematics, its critique and the necessity to view mathematical development in many cultures of the world, and transmissions between them. Then the author gives a brief account of pre-historic mathematics, such as those found in the *Ishinga Bone*, the mathematics of the Incas and the Mayans, and the development of number systems. (Interestingly, some authors have raised serious questions on mathematical inference from Ishinga bones, Asolom's wolf bones and such. See *The fables of Ishango, or the irresistible temptation of mathematical fiction*, Olivier Keller, Préhistoire de l'arithmetique, Feb 2015.) An important discussion here is on mathematics in Africa, especially geometric designs: very brief, but pointing to an area not generally discussed.

Chapters 3 and 4 discuss mathematics from Egypt and Mesopotamia and Chapter 5 is an 'assessment' of these two. Joseph presents an illuminating picture of the empiricist and algebraic tradition prevalent in Egypt and Babylonia. Since the Greeks had extensive interaction with these societies, Joseph makes a case for how a synthesis of the deductive and geometric tradition of Greece with this algebraic approach might have led to the powerful mathematics that emerged, especially in the works of Archimedes, Ptolemy and Diophantus.

Many of the examples presented of Egyptian and Babylonian mathematics have their origins in people's work, on the empirical need for calculation. For instance, calculate the number of persons needed to move an obelisk. The modeling needs of such tasks and their subsequent abstraction, seems to have led to interesting mathematical constructions. Studying these can be inspiring for today's students, relating to similar tasks in today's world. The tasks are largely arithmetical and measurement oriented, and involving basic algebra, all accessible to a child in middle school. There are also algorithms from Mesopotamia like the one for extraction of square roots, but it is not clear how different and enriching they are.

For shock value, consider the following problem (Example 4.5, Chapter 4): *Calculate how long it would take for a certain amount of money to double if it has been loaned at a compound annual rate of* 20%. You expect to see this in current day high school texts. This is from the Louvre tablets of the Old Babylonian Period, approximately 1500 BC. Here is another, from the Susa tablets of the Old Babylonian Period (Example 4.11, Chapter 4): *Find the circumradius of a triangle whose sides are* 50, 50, *and* 60. It is this problem that leads Joseph to assert: "there can be little doubt that the Mesopotamians knew and used the Pythagorean theorem." Be that as it may, it would be instructive



Ancient Chinese Mathematics 191

TABLE 6.1: MAJOR CHINESE MATHEMATICAL SOURCES UP TO THE SEVENTEENTH CENTURY

Title	*Author*	*Date*	*Notable subjects covered*
Zhou Bi Suan Jing (The Mathematical Classic of the Gnomon and the Circular Paths of Heaven)	Unknown	c. 500–200 BC	Pythagorean theorem; simple rules of fractions and arithmetic operations
Suanshu Shu (A Book on Arithmetic)	Unknown	300–150 BC	Operations with fractions; areas of rectangular fields; fair taxes
**Jiu Zhang Suan Shu* (Nine Chapters on Mathematical Arts)	Unknown	300 BC–AD 200	Root extraction; ratios (including the rule of three and the rule of false position); solution of simultaneous equations; areas and volumes of various geometrical figures and solids; right-angled triangles
Ta Tai Li Chi (Records of Rites Compiled by Tai the Elder)	Unknown	AD 80	Magic square order of 3
Commentary on *Jiu Zhang*	Chang Heng	130	π = square root of 10
Shu Shu Chi Yi (Manual on the Traditions of the Mathematical Arts)	Xu Yue	c. 200	Theory of large numbers; magic squares; first mention of the abacus
Commentary on *Zhou Bi*	Zhao Zhujing	c. 200–300	Solution of quadradic equations of the type $x^2 + ax = b^2$
Hai Dao Suan Jing (Sea Island Mathematical Manual)	Liu Hui	263	Extentions of problems in geometry and algebra from the *Nine Chapters*
Sun Zu Suan Jing (Master Sun's Mathematical Manual)	Sun Zu	400	A problem in indeterminate analysis; square root extraction; operations with rod numerals

continued

Figure 1.

and thought provoking for our students to solve non-trivial problems posed more than 3000 years ago.

Chapters 6 and 7 are on Chinese mathematics. It is very likely that most of our teachers are unaware of the long history of mathematical development that our neighbours had, and of the multiple transmissions between our cultures. Rather than list the many interesting topics, I have reproduced part of the chronology presented by Joseph (Table 6.1 of *Peacock*) in Figure 1.

Chapter 7 is devoted to a specific period, the late 13th and early 14th centuries, during the Song dynasty. Very fine mathematicians such as Qin Jiushao, Li Ye, Yang Hui, and Zhu Shijie lived in this period, and several schools of mathematics flourished. Joseph categorizes the essentially algebraic work of this era into three kinds: numerical equations of higher order, Pascal's triangle (note the period, for what was named after the 17th century French mathematician Blaise Pascal) and indeterminate analysis (solving a system of *n* equations with more than *n* unknowns).

We have all heard of the Chinese remainder theorem. Chapter 7 gives a very nice account of the historical development of ideas related to this (apart from geometry in Chinese mathematics). To trigger thought, consider the following problem from the 4th century mathematical text Sun Zu Suan Jing. *There are an unknown number of objects. When counted in threes, the remainder is* 2; *when counted in fives, the remainder is* 3; *and when counted in sevens, the remainder is* 2. *How many objects are there?* In modern notation,

$$N = 3x + 2, N = 5y + 3, N = 7z + 2,$$

or better,

$$N \equiv 2 \pmod 3),\ N \equiv 3 \pmod 5),\ N \equiv 2 \pmod 7),$$

and we seek the least integer value of N.
(The answer is 23.)

The chapter also has a brief discussion of mathematics in Japan, notably that of Seki Takakazu (1642–1709). Here was a mathematician who "discovered determinants ten years before Leibniz, . . ., discovered the conditions for the existence of positive and negative roots of polynomials, did innovative work on continued fractions, and discovered the Bernoulli numbers a year before Bernoulli."

George Joseph's centrepiece of the book is his account of mathematics in India, and it is laid out in Chapters 8, 9 and 10. The first of these talks of ancient India, ideas from the Vedic period, Indian numerals, and Jaina mathematics. The second is on the classical period, recording the contributions by Indian mathematicians to astronomy, algebra and trigonometry (Aryabhata I, 5th century CE; Brahmagupta, 6th century CE; Mahavira, 9th century CE; Bhaskara II, 12th century CE). The third is on what might justifiably termed the crest of the peacock, the Kerala school of mathematics, especially the results attributed to Madhava (14th century CE) and Nilakantha (15th century CE). Though this is perhaps the main section of the book, I will not discuss it in detail here since much of this mathematics was described in the review of Kim Plofker's book (*At Right Angles*, Volume 3, No. 3, November 2014, pp 82–87).

The final chapter is on mathematics from the Arab world which Joseph presents as a prelude to modern mathematics. While the development of algebra in the Islamic world and its impact on European mathematics is well known, much less is generally known of the work of Islamic mathematicians in number theory, geometry and trigonometry. This brief chapter has enough material to interest every mathematics teacher in India. The work of Ibn al-Haytham (965–1039), al-Biruni (973–1051), Omar Khayyam (1048–1126), and al-Kashi (1429) are important. Of these, Omar Khayyam is famous as a poet, but he was also a first-rate mathematician who propounded a geometric theory of cubic equations and tried (unsuccessfully) to derive the parallel postulate from other axioms. Al Kashi performed prodigious calculations, computing the value of π by circumscribing a circle by a polygon having 3×10^{28} sides!

As an appetizer, let me offer Joseph's illustration of the work of Omar Khayyam in Chapter 11. Suppose that we have a 'ratio problem' with a, b, c, d such that $\frac{b}{c} = \frac{c}{d} = \frac{d}{a}$. Then, $\left(\frac{b}{c}\right)^2 = \frac{c}{d} \cdot \frac{d}{a} = \frac{c}{a}$ and hence $c^3 = b^2a$. Letting $b = 1$, if there exist c and d such that

$$c^2 = d \text{ and } d^2 = ac,$$

then we can determine the cube root c of a.

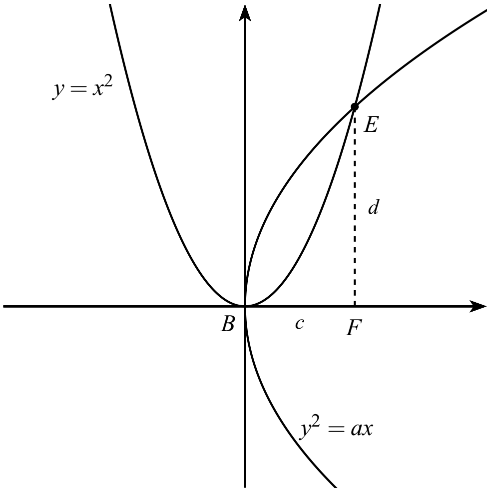


Figure 2.

Omar Khayyam perceived an underlying geometry in this problem. In the equation above, think of a as a constant and $c = x$ and $d = y$ as variables. Then we have two parabolas as in Figure 2, with equations $y = x^2$ and $y^2 = ax$; they have a common vertex B and mutually perpendicular axes, and they intersect a second time at E. It is easily checked that at E we have $x^3 = a$. Hence $BF = c$ is the cube root of a.

Khayyam extended his method to solve any third degree equation for positive roots. He solved equations for intersection of parabolas, of hyperbolas, of circle and parabola, and of parabola and hyperbola. Such application of geometric techniques to algebraic problems is of tremendous pedagogic value in the higher secondary stage in our schools, and I offer this only as a pointer to the rich lore available in mathematics from the Islamic world.

Pride and Practice

The *Peacock* is beautiful, and George Gheverghese Joseph has a pleasant style. To give you a flavour of his style, I quote from his concluding paragraph:

> *. . . [I]f there is a single universal object, one that transcends linguistic, national, and cultural barriers and is acceptable to all and denied by none, it is our present set of numerals. From its remote beginnings in India, its gradual spread in all directions remains the great romantic episode in the history of mathematics.*

Indeed it is, and the style makes for very pleasant reading. There are some natural criticisms of the book, and since the first edition appeared in 1991, historians have pointed to several flaws: the overuse of binary opposition of 'European' vs 'non-European' mathematics, when he himself is making the case for global transmissions; speculation where there is no documentary evidence; problems with his dating; insufficient demonstration that modern mathematics was indeed as strongly influenced by these 'eastern' contributions; and so on. Clemency Montelle's review of the third edition in Notices of the American Mathematical Society, December 2013, is a good place to not only read the critique but also get pointers to more recent and authoritative historical sources.

However, it is undeniable that George Joseph is pointing us to a serious lacuna in our education, and in our teaching and learning practices. When we appreciate the cultural rootedness of mathematics and the history of mathematical thought across diverse cultures, it expands our horizons in multiple ways: rather than mere pride, we obtain a nuanced appreciation of our own past and culture, and its deep connections with other cultures; rather than accepting definitions and concepts as given (by an alien culture), we engage with them, question them and conceive of how alternate trajectories may have altered them. The vast range of examples from across the world presented by Joseph encourages us to look closer at people's practices and unearth heuristics and algorithms that, on exploration, may pose interesting questions for mathematics.

There is a convenient (albeit oversimplified) classification of mathematics encountered in learning the subject: at school we learn mathematics from the 18th century and earlier; as undergraduates, we learn largely mathematics from the 19th century; and as graduates and researchers, we approach 20th century mathematics. Even as a thumb rule, this observation yields an important lesson. If we wish to question the components that constitute school mathematics by considering alternate definitions, methods and trajectories, it's a good idea to look at the past and across cultures. *The Crest of the Peacock* offers a panoramic view of what we are sure to find.

R Ramanujam is a researcher in mathematical logic and theoretical computer science at the Institute of Mathematical Sciences, Chennai. He has an active interest in science and mathematics popularization and education, through his association with the Tamil Nadu Science Forum. He may be contacted at jam@imsc.res.in.

ALL RIGHT THEN !

We were pleasantly surprised to receive several responses to the question retrieved from http://www.futilitycloset.com/2014/11/04/all-right-then-2/ and posed on page 15 of At Right Angles Volume 4 No 1 (March 2015).

AMONG THEM WERE THE FOLLOWING:

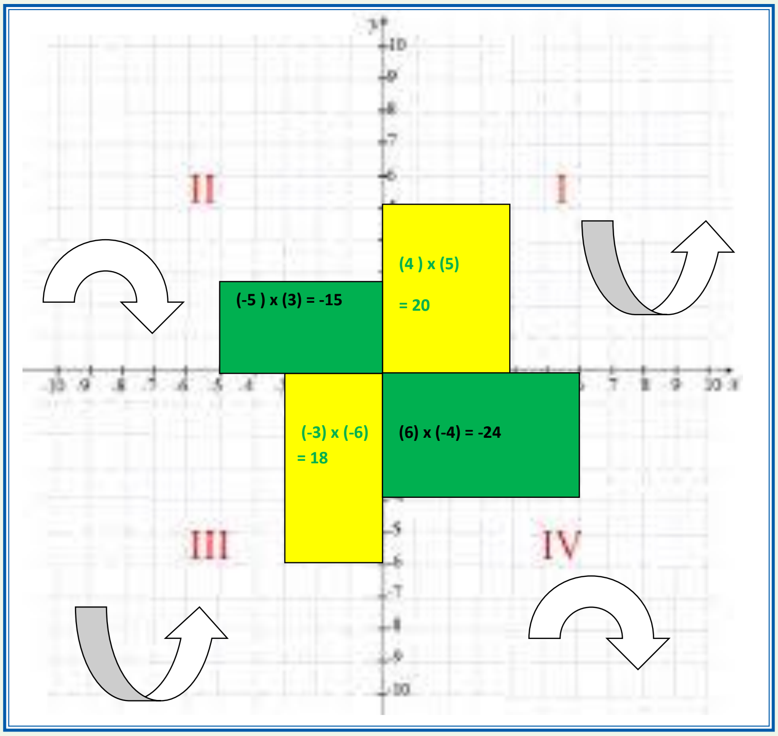
1

VAIDEHI MADHAVAN teaches at P.S. Senior Secondary School, Chennai and she writes, *"Many a time, I have heard high school students ask the question 'Why plus times plus is plus and minus times minus is also plus?' and I am amused at the way these questions have been coined or worded. According to me, where is the question of multiplying a plus sign and another plus sign? I generally correct these students to use, instead, the terms 'positive numbers' and 'negative numbers'.*

To illustrate the rule, she uses an ingenious visual representation which can be appreciated by any student who can understand the meaning of the co-ordinate system of mutually perpendicular axes. She starts with a few standard conventions.

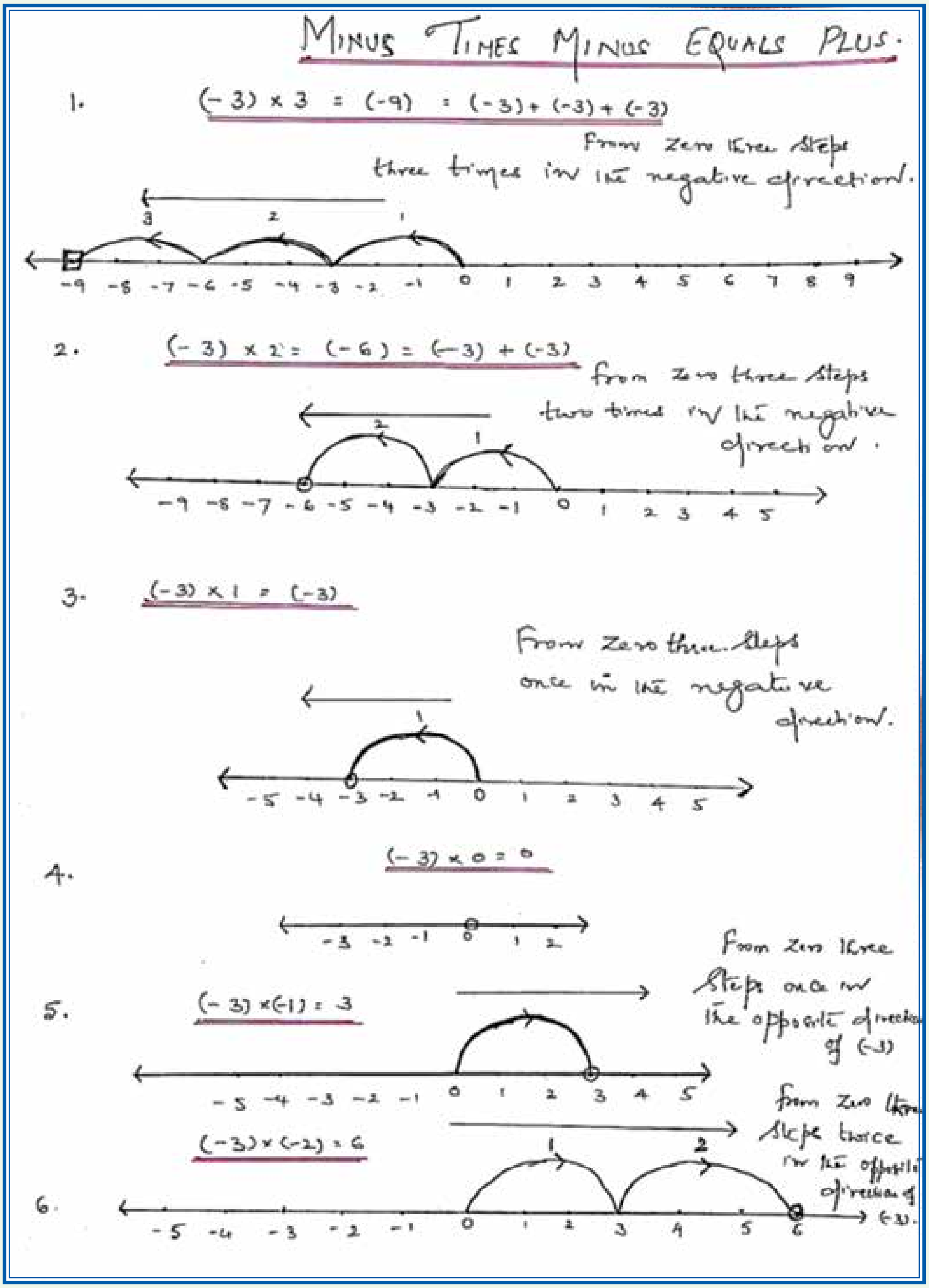
a. *In a number line, moving in the right direction of '0" is positive and moving in the left direction of "0" is negative.*
b. *Also, in a co-ordinate system of axes moving above or right of origin is positive, while moving below or left of the origin is negative.*
c. *In a similar way, when we move around a closed bounded region in the* ***anticlockwise*** *direction, the area covered is considered to be positive. If we move in the* ***clockwise*** *direction, the area is considered to be negative.*
d. *The product of two numbers* ***a*** *and* ***b*** *can be visualized as the area of a rectangle with sides of* ***a*** *units and* ***b*** *units.*

The following graph shows how the different products can be shown in the four quadrants.



Using these conventions, and a co-ordinate system of axes, she is able to provide a rationalization as to why the product of two negative numbers is positive and the product of two numbers of unlike signs is negative. However, the rationalization is based on the conventions used including the convention (which was not mentioned) that the order of multiplication is $x \times y$ and not $y \times x$.

2 **P.V. SATYA RAMESH** of Shanti Asiatic School, Ahmedabad uses the patterns that emerge when multiplication is viewed as repeated addition to justify the rule. The visual explains the logic.



MINUS TIMES MINUS EQUALS PLUS.

1. $(-3) \times 3 = (-9) = (-3) + (-3) + (-3)$

From zero three steps three times in the negative direction.

2. $(-3) \times 2 = (-6) = (-3) + (-3)$

from zero three steps two times in the negative direction.

3. $(-3) \times 1 = (-3)$

From zero three steps once in the negative direction.

4. $(-3) \times 0 = 0$

5. $(-3) \times (-1) = 3$

From zero three steps once in the opposite directio of (−3)

6. $(-3) \times (-2) = 6$

from zero thre steps twice in the opposit direction of (−3).

3

MEESUM NAQUI from KDAV Reliance, Jamnagar prefers unpacking the meaning of the arithmetic operations in order to understand the rule. Her rationale is as follows.

Addition of integers means to combine two sets of opposite events. For example:

1. *Go up 3 steps and go up 4 steps can be written in mathematical form as* (+3)+(+4)
2. *Go down 3 steps and go down 2 steps can be written in mathematical form as* (-3)+(-2)

Thus addition of (-2) *and* (+2) *will look like this:* (-2)+(+2)=0
Here zero means no change in final position. '-' and '+'sign inside the bracket shows the negative and positive integers and the '+' sign between brackets shows the operation of addition. Children often get confused in this and due care should be taken.
Multiplication of integers poses a problem as it refers to repeated operations. It makes little sense to say "move up 4 steps (-5) times". To solve this problem let's learn two basic terms, 'doing' and 'undoing'.

Doing something twice means doing the same thing again and again two times. Undoing something means to do just the opposite, so undo thrice means to do opposite of something three times. For obvious reasons we will take doing as +ve and undoing as –ve.

Now look at the following examples.

1. *Do twice (go up three stairs) can be written in mathematical form as* (+2)×(+3)=(+6)
2. *Undo thrice (go down two steps) can be written in mathematical as* (-3)×(-2)=(+6)
3. *Do twice (go down 4 steps) can be written in mathematical as* (+2)×(-4)=(-8)
4. *Undo thrice (go up four steps) can be written in mathematical as* (-3)×(+4)=(-12)

Above examples clearly show why the product of two negative integers is positive. This approach is based on real life situations and does not allow any scope for rote learning.

Editor's Note:

We are delighted that among our readers are teachers who clearly attempt to avoid rote learning and who use the tools of visualization, pattern recognition and logic to improve understanding!

The Closing Bracket . . .

I've lived a life that's full. I've traveled each and every highway;
And more, much more than this, I did it my way.

I planned each charted course; Each careful step along the byway,
And more, much more than this, I did it my way.

Yes, there were times, I'm sure you knew, when I bit off more than I could chew.
But through it all, when there was doubt, I ate it up and spit it out.
I faced it all and I stood tall; And did it my way.

For what is a man, what has he got? If not himself, then he has naught.
To say the things he truly feels; And not the words of one who kneels.
The record shows I took the blows -And did it my way!

Yes, it was my way.

Readers may recognize these lines from selected verses of **"My Way,"** a song popularized by Frank Sinatra. Its lyrics were written by Paul Anka and set to music based on the French song "'Comme d'habitude." We downloaded it from http://en.wikipedia.org/wiki/My_Way

So why is this song the Closing Bracket for the July 2015 issue? In this age when mathematics is packaged and re-packaged by teachers, parents and tuition classes, chewed up and broken into digestible pellets by online support sites, shredded into formulae lists and exam guides, what chance does a student have to explore, experiment, make mistakes, learn from them and grow in the process?

So here is the first question: Can one afford to be adventurous in mathematics? Can a teacher of mathematics allow her students to explore a variety of paths, knowing full well that not all of them will lead to a neat, satisfying solution or proof? When time is at such a premium in the school year can we really afford to do this?

And here is the next question: Can we make mistakes in mathematics? Isn't this a subject that is all about the 'right answer'?

Pressed for time, with the right answer and the right method up her sleeve, can one fault a teacher who hands these over to a student with the best of intentions? But at what cost? We don't claim to have answers for all these questions but enough has been said about 'math-phobia' and the number of students who can't wait to drop the subject. What of the student who does well in it? What of the teacher who is reputed to teach it well? Are marks the sole criteria for these accomplishments? Has the teacher produced a mathematician or a mark-machine? Teachers in institutions which conduct difficult entrance examinations have noted that getting into and doing well in and appreciating a course seem to require two entirely different skill sets and student types. What about the work space? Do we have graduates who have honed their thinking skills, developed their own algorithms and learnt to be problem-solvers?

Is there virtue in letting students experience the subject in all its beauty in a slow and measured manner?

As lovers of the subject, let's think long-term and create opportunities for students to say 'yes, it was my way.'

— Sneha Titus
Associate Editor

Specific Guidelines for Authors

Prospective authors are asked to observe the following guidelines.

1. Use a readable and inviting style of writing which attempts to capture the reader's attention at the start. The first paragraph of the article should convey clearly what the article is about. For example, the opening paragraph could be a surprising conclusion, a challenge, figure with an interesting question or a relevant anecdote. Importantly, it should carry an invitation to continue reading.
2. Title the article with an appropriate and catchy phrase that captures the spirit and substance of the article.
3. Avoid a 'theorem-proof' format. Instead, integrate proofs into the article in an informal way.
4. Refrain from displaying long calculations. Strike a balance between providing too many details and making sudden jumps which depend on hidden calculations.
5. Avoid specialized jargon and notation — terms that will be familiar only to specialists. If technical terms are needed, please define them.
6. Where possible, provide a diagram or a photograph that captures the essence of a mathematical idea. Never omit a diagram if it can help clarify a concept.
7. Provide a compact list of references, with short recommendations.
8. Make available a few exercises, and some questions to ponder either in the beginning or at the end of the article.
9. Cite sources and references in their order of occurrence, at the end of the article. Avoid footnotes. If footnotes are needed, number and place them separately.
10. Explain all abbreviations and acronyms the first time they occur in an article. Make a glossary of all such terms and place it at the end of the article.
11. Number all diagrams, photos and figures included in the article. Attach them separately with the e-mail, with clear directions. (Please note, the minimum resolution for photos or scanned images should be 300dpi).
12. Refer to diagrams, photos, and figures by their numbers and avoid using references like 'here' or 'there' or 'above' or 'below'.
13. Include a high resolution photograph (author photo) and a brief bio (not more than 50 words) that gives readers an idea of your experience and areas of expertise.
14. Adhere to British spellings – organise, not organize; colour not color, neighbour not neighbor, etc.
15. Submit articles in MS Word format or in LaTeX.

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together with Community Mathematics Centre,
Rishi Valley

TEACHING
Measurement

PADMAPRIYA SHIRALI

A PRACTICAL
APPROACH

At Right Angles
A Resource for School Mathematics

Measurement

Measurement occupies a unique position in the curriculum, for various reasons. As it is an essential everyday activity in human life, children are naturally exposed to measurement in various situations at home and elsewhere. Also, measurement overlaps both with numbers and geometry. It involves spatial dimensions as well as counting. In measurement, one is measuring one attribute in terms of another attribute. Also one is expressing a non-discrete quantity in terms of discrete numbers. Since there are different ways of measuring the given length or mass, the choice of the measure is dependent on the purpose that needs to be served. Children need to understand that some contexts require precision to a fine degree while some require approximate figures. A strong foundation in measurement concepts leads to a better understanding of decimal numbers in particular.

The focus of the teaching of measurement needs to be more on developing a proper concept of measuring rather than on practising measuring activities. As the topic is developed, children need to begin to appreciate under what situations measurement can be used, and the suitability of different measures in different situations. It is also important to create activities to the extent possible which have an inbuilt task in them, a task which is sufficiently interesting and challenging for the children to take up with enthusiasm.

I have focussed only on length, weight and capacity in this article. Measurement includes many other measures as well: measurement of time, of temperature, of speed and so on. I shall go into these in a subsequent piece.

Measurement activities often require tools which may be limited in supply. Also typically the activities require children to cooperate (like holding a rope tight or bringing two things together). It is best to organise children in groups of four and set each group a task.

Young children require both experience and maturity to understand measurement concepts. The awareness of quantity and appropriate language associated with that happens simultaneously. Words such as big, small, more and less are learnt at a very early age. However, they may take time to understand the principle of conservation of quantity or number. It is dependent on the experiential understanding and maturity of the child and develops slowly and varies from child to child. Also, they may hold false assumptions about weight or volume. For instance, they may believe that between two objects, the one that looks larger also weighs more. Or they may think that a taller container can hold more than a shorter container. However, a teacher can quicken the pace of development by exposing the children to meaningful activities and guiding them along by getting them to articulate their observations and ideas. Through conversation, a teacher can clear some of the misconceptions and help the child to acquire right understanding. The principle of conservation is a prerequisite in developing an understanding of the principles of measurement.

Keywords: *Measurement, length, height, weight, capacity, size, comparison, estimation*

ACTIVITIES

Preliminary activities for young children (3 to 5 year olds):

To understand the attribute of LENGTH:

Materials: Straws, used sketch pens, ice-cream sticks or toothpicks, coloured paper strips or wooden rods of different lengths, coloured ropes or shoe laces of different lengths, pencils, beads and rope for threading.

Selection of materials is to be done carefully so that children can focus on one attribute. Large 2-D shapes and 3-D objects can be introduced at a later point as they have more than one measure (length, width and height).

Language to be introduced: long, short, tall, longest, shortest, thick, thin, wide, narrow, distance - related words (far, near).

Comparison of two objects of the same type: Let children compare two standing objects like two trees or two children standing on level ground to identify the taller and the shorter. Since the objects stand at the same level, there is no scope for confusion. Let them now do the same for two sticks or two paper strips to identify the longer and the shorter. An advantage of measuring strips is that they can be placed one over the other. To compare the strips, the children will need to bring them close to each other. At this point, the teacher must check to see that the child has placed the strips so that the lower ends have the same starting point. If not, the teacher will need to help the child understand that in order to compare lengths, the objects must be aligned at the starting point.

Comparison of two objects of different types: Let them compare the lengths of a pen and scale, chalk and duster, a pencil and scissors.

Task 1: Give each child a straw and ask the children to get an object as long as the straw. They may find a pencil, a book, a leaf or a broom stick. In the process of locating the object, they would have tested it against many objects and acquired practice in comparing lengths of objects.

Comparison of more than two objects: Given a set of objects (pencils, chalks, paper strips, sticks and so on) with different lengths, they can arrange the objects in order of their length. Children can make bead chains and hang them in length order. Note that bead chains provide an opportunity to make numerical comparisons too: “The blue bead chain is two beads longer than the red bead chain.”

Task 2: Ask children to form groups of four and have each group stand in height order.

Task 3: Sorting activity: Materials: Coloured straws or crayons of four different lengths as shown in the picture shown in Figure 1, with a few of each size. Ask the children to sort the straws or crayons into groups of equal length.

Introduce the words 'wide' and 'narrow' in various contexts: Opening and closing the door, pointing to the gap (wide and narrow); or wide/narrow paper strip. Young children enjoy physical activities and they can demonstrate the meanings of these words by opening their eyes ('wide open', 'tightly shut'), parting their fingers, etc.

Provide opportunities for checking other forms of lengths like *width* and *thickness*. Again, ensure that they

Figure 1

have aligned the objects correctly. Let them compare the widths of two note books, widths of two pencil boxes, etc. Encourage them to use appropriate words: "The English notebook is wider than the Math note book" or "His pencil box is narrower than her pencil box", etc. Let them compare the width of a chalk box and a pencil box, width of a duster and ruler.

Similarly introduce words like *thick* and *thin* by pointing to a thick book and a thin book, and a thick line and a thin line. They can observe the different fingers on their hand and describe them in terms of these words.

Task 4: Let each group of students collect four sticks and use the words thick, thin, thicker than, thinner than, as thick as, in between, etc to describe them. Let them arrange the sticks in order of thickness.

Comparisons of objects with more than one attribute: In the case of 2-D shapes and 2-D objects, comparisons can be made based on different attributes. "The text book is longer and wider than the note book." "The door is taller than the window, but the window is wider than the door." "The bench is wider than the table, but the table is taller than the bench." And so on.

The teacher may pose a problem: "Is there a box in which I can keep the duster?" Let the children experiment with various boxes in the class and give reasons for why it will fit or why it will not. "Can it fit into the chalk box? No, the chalk box is not long enough. Can the duster fit into the pencil box? Yes, it can as the box is longer than the duster and wider than the duster." Many more questions of this kind can be posed. Can the book go into this bag? Will the map fit in this space?

To understand the attribute of CAPACITY/ VOLUME:

Materials: Large tub or a decent sized sand pit, cups of different sizes, narrow tall containers, wide short containers, discarded transparent plastic bottles, transparent plastic bowls, cardboard boxes of different sizes, sand, beads, cubes and boxes, bricks or wooden blocks. Bucket, mugs, funnel, sieve and water.

Children need to experience activities involving filling, pouring, packing, fitting and emptying to understand the principles involving capacity. With small children we do not use the words *volume* or *capacity*. Instead we pose questions such as: "Does this hold more than that?"

Children take great pleasure in filling slightly damp sand into containers and inverting them to get sand moulds. This is potentially an excellent opportunity to bring out concepts related to shape and capacity. Once the children have made several sand moulds with different objects on the ground, an interesting question to pose is: "Identify the container used to make each mould." Secondly, moulds come in interesting shapes based on the containers used: cylinders, cones, truncated cones, cuboids and cubes. Children can describe these shapes in their own way. Thirdly, a discussion can ensue on finding the bigger mould.

Let children compare capacities of two containers by filling one with sand and then pouring out the same sand into the other container. Let them repeat this activity with different containers. Initiate a conversation with the child to check whether the child has realised that if the sand does not fill the second container, then the second one has greater capacity. And if it overflows, then the first one has greater capacity.

It is quite common for young children to think that a tall container has greater capacity than a short container. It is only through experiences of filling and testing that they begin to realise that this may not always be true. Usage of transparent containers makes it easier for children to see the capacities of different containers. Give children two plastic containers of equal height, one with a narrow base and one with a broad base. Let them fill each one with a cupful of water. Ask: "Why is the height of the water in one more than in the other?" Let them understand that the size of the base is more in one than in the other. Devise more such experiments to remove misconceptions.

Figure 2

To understand the attribute of WEIGHT:

Materials: Large tub, few cups or mugs of same size, small cardboard boxes, sand, water, beads, used pens, leaves, chalk pieces, different objects of varying weights.

Let children pick up one object in one hand and one in the other to compare their weights.

Let children play around with filling a cup with various materials such as sand, chalk pieces, stones etc, and ask: "Which material is the heaviest?" "Which is the lightest?"

Children may have notions about length and weight or volume and weight which are incorrect. Give simultaneously opposite examples as given below to prevent and clear any misconceptions that may arise with regard to weight.

A larger object does not always weigh more.

Give children a large balloon and a tennis ball. Let them compare their weights and see that even though the balloon is larger in size, it weighs less. Also, give two other objects where the larger object does weigh more, say a thick book and a thin one.

A larger quantity does not always weigh more.

Fill a large packet either with cotton or saw dust. Fill a smaller packet with sand so that the weight of the sand packet is more than the sawdust packet. Let children lift both objects and find out which has greater weight.

Objects of the same length may have different weights.

Give children different objects of the same length, say a paper strip and a wooden or metal rod. Let them pick up the objects one in each hand and compare the weights. Do the same with more pairs of objects of the same length.

Weight of a larger number of one type of object is not always more than the weight of a smaller number of some other type of object.

Give children five balloons or table-tennis ('ping pong') balls and a cricket ball. Let them compare the weights of both and see. Let them realise that a greater number does not necessarily imply more weight.

Tie a rope from one end of the class to another with a basket hanging from it as shown in figures 3 and 4. Pull it taut. Let children experiment with placing different objects in the basket to find the heaviest and the lightest.

Measuring activities for 5 to 7 year olds:

Measurement of LENGTH with 5 to 7 year olds:

Comparisons of objects which need another object to be used as a measure:

Task 5: Is the cupboard in the class wider than the door?

This requires comparison of objects which are fixed and cannot be placed next to each other.

Questions of this kind require the use of another object to be used as a measure, say a string. The cupboard is first measured with a string and a blue mark is made on the string marking the endpoint. Then the same string is used to measure the door and a red mark is made on it at the new endpoint. From the two marks we can determine which one is the wider object. Or two strings can be used and laid next to each other for comparison.

Children can be encouraged to use their feet or arms to make comparisons of fixed objects. Example: The width of the classroom steps is longer than my foot, and the width of the steps on the slide is shorter than my foot.

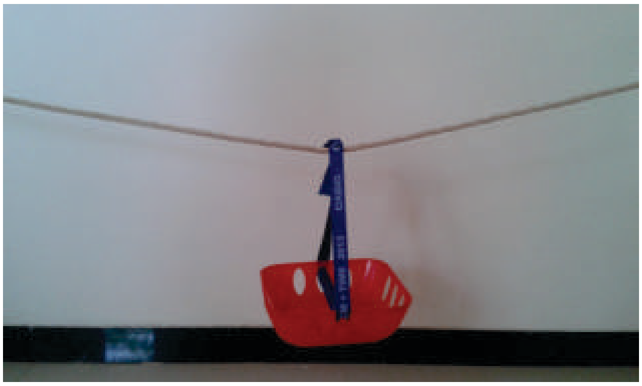

Figure 3

Figure 4

Task 6: Who has the biggest head? Let children work in pairs and use a string to measure the length around the forehead. They can mark the strings and lay them out straight for comparison.

The level of accuracy needed in measurement depends very much on the context. In many of our day to day activities, we use non-standard units (NSUs) routinely as we need only approximate values. By approaching measurement through the usage of NSUs, the measurement concept becomes clear to young children. A scale with numbers and fine divisions can be better appreciated by the child when he or she constructs the scale and marks the numbers on it.

Pose a task: Is the length of the class more than the width? If so, by how much?

Children select a unit available in the class like a straw to measure the length of the room and the width of the room in terms of number of straws and determine which is bigger. Selection of the unit should be attempted by the child on his/her own. He/she needs to develop the skill of choosing the right unit. It is only through trial and error that the child begins to realise that a large unit will not give a useful answer, while a very small unit will make the number large and unwieldy. Later, guide the children to use straws or wooden blocks for measuring longer lengths and ice-cream sticks or paper clips for measuring smaller lengths. Beads can be used for measuring very small lengths but they tend to roll.

While they measure with NSUs the teacher needs to watch out for the following:

- Do the children know where to begin and end? Have they left any gap at the start, or is it aligned correctly?
- Have the children placed the units one after another without gaps or overlaps?
- Have they managed to form a reasonably straight line or has the line become curved?

If the teacher notices that any of the above conditions have not been met satisfactorily, he/she should initiate a discussion or demonstrate to them why it would give an incorrect result. Also, while using straws and sticks an uncovered space might result at the end. Teacher can help them to do rounding.

I once asked a group of preschool children, varying in ages from 3 to 5, to make the longest possible train in the class using blocks. It was very interesting to see the engagement that ensued. One child began to form a line of blocks from the middle of the room. Soon they laid out a string of blocks and reached one end of the room. Just as they thought they had finished another child noticed that they had started from the middle and that they could extend it in the other direction too. Quickly more blocks were picked off the shelf and unloaded to complete the train to the other end as well. Since they had not lined the blocks against the wall, their train was not absolutely straight. When I asked 'how many bogies are there in your train?, with great enthusiasm they counted off in chorus and found it to be 114 bogies. The counting itself did not go smoothly as the younger ones were uncertain about some numbers. But the older ones corrected them, and they finally arrived at the correct number. I then asked them: “How many bogies long is the room?” They took some time to respond and this time it was the older ones who saw the correspondence between their train and the length of the room and answered correctly.

Measurement of curved lines: Let children use small units like ice-cream sticks or paper clips and measure the length of curved lines or length around a circular shape. “Whose shape has the biggest round?” “How much bigger is this flower pot than the other?”

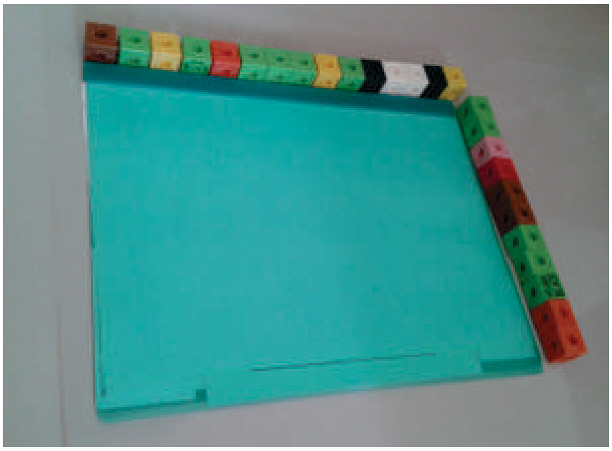

Figure 5

As children go through the experience of measuring lengths with NSUs they need to absorb certain essential principles involved. One principle is that they need to use the same unit repeatedly while measuring one object. Secondly to make comparisons between two lengths they need to use the same unit for measuring both. Thirdly they begin to see that measurement with a larger unit will result in a smaller count and measurement with a smaller unit will result in a larger count. They will later be able to appreciate that the choice of the measuring unit depends on the level of accuracy that is needed.

The teacher can devise activities which lead to an understanding of measurement processes and skills.

Materials: A long strip, three measuring units of different length (chalk, tooth pick, straw).

Let children measure the long strip with each unit. They will see that when they measure with straws they get a small number, when they measure with a chalk they get a larger number and when they measure with a toothpick they get a still larger number.

Body lengths: Let children work in pairs. One child lies on the floor, and the other traces the outline of his body with a chalk. Let each child measure the length of his body using straws or sketch pens.

Help them to record the information. Similar activities can be done by tracing the foot or palm of each child and selecting a suitable measurement unit.

Using body measures as NSUs: Children now begin to measure lengths using the foot as a unit of measurement. They also use hand span and paces, though this requires greater motor co-ordination to get it right. Sports and games activities provide many measurement opportunities.

Task 7: Mark a circle on the ground outside or in the veranda of the school. Let children take turns to stand in the circle and throw a stick, one by one. Let them measure the distance thrown using feet or paces.

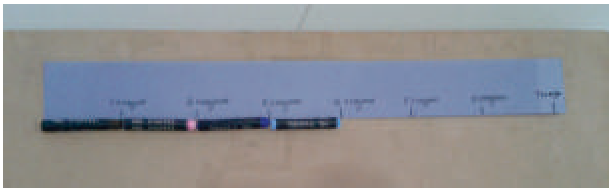

Figure 6

Constructing Rulers of different units: Let children build cardboard rulers using a straw or ice-cream stick as a unit, as shown in Figure 6. Let them number the markings as '1 straw' or '1 stick.' This will help them understand that '1' points to the place where one straw ends and the second one begins. Children often make errors in measuring by not aligning an object with zero on the scale as they do not understand that the number '1' is at the end point of the measuring unit.

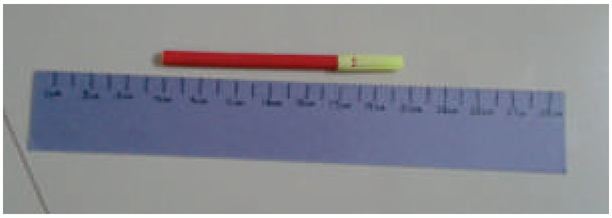

Figure 7

Show them why aligning the starting end of an object with zero facilitates reading of information. At the same time, expose them to measuring lengths of objects which are not aligned, as shown in Figure 7, so that they understand the need to subtract the first number from the second number in order to determine the length of the object.

Measurement of CAPACITY for 5 to 7 year olds:

Materials: tub, plastic bottles, plastic containers (some with small base and some with large base), cups or glasses, cardboard boxes (all these items should be of different sizes), plastic or wooden cubes for fitting into the boxes, sand and water.

Figure 8

Children can compare the capacities of two or more containers by filling the containers with sand measured with a cup and count the number of cupfuls each one holds. Ask questions which require them to apply their understanding: "Does this cup hold double of that cup?" "Will the container fill up if I pour one more cupful?"

Let each group of four children select one large container and three differently sized cups for measuring the capacity of the container. Ask questions which will require them to reason out their answers. "How many small cupfuls of sand did you use to fill the bowl?"

"How many big cupfuls of sand did you use to fill the bowl?" "Why did you get a smaller number?" Point to a size which is in-between and ask: "What will happen if you use this cup?" Point to a bowl which is smaller in size than the one measured and ask: "How many small cupfuls of sand will this hold?"

They can measure the capacities of their water bottles. Children's water bottles come in various shapes and sizes, and as they measure them they will see that objects which look different in size and shape may have the same capacity. They can also measure the capacities of their tiffin boxes by filling it with cubes.

Measurement of **WEIGHT** for 5 to 7 year olds:

Materials: Give children a collection of objects and ask them to pick up pairs of objects which are similar in weight.

Building a sense of weight: Give children some identical cardboard boxes each containing an object which they have handled before (examples: chalk, duster, crayon box, tennis ball, stapler, punch), of different weights. Mention the names of the objects used. Let children pick up each box in turn and guess the object in the box based on the weight.

Hanger balance: Build a hanger balance as shown in Figures 9 and 10 for children to compare weights of different small objects.

Figure 9

Figure 10

With older children in the primary school, the teacher should use the knowledge and experiences that children already possess with regard to usage of length, weight and capacity. Children would have already witnessed usage of length while buying cloth, when they visit a tailor, while buying shoes, height chart in a doctor's clinic, etc. They would have experienced weights while buying vegetables and sweets, on packaged products, in a doctor's clinic, in baking activities, etc. They would have experienced capacity while having cool drinks, buying milk or oil, drinking water cans, metre readings of petrol consumption, etc.

Measuring activities for 7 to 9 year olds

Measurement of LENGTH for 7 to 9 year olds:

Need for standard units: Select a long object in the classroom, say the blackboard or a wall, for measuring its length. Ask different children to measure it using hand spans. The teacher can also measure using his or her hand span. It will be noticed that slightly different answers are obtained. Discuss with the children the reasons for such differences. Point out the difficulties that can arise if we had to ask a carpenter to make a frame for the board and we specified the dimensions using NSUs. Bring in the need for standard units as measures.

Sense of centimetre: Now one can make the transition to the standard units of centimetre and metre. Show a centimetre ruler to the children and help them understand the markings. Typically, the zero mark is a little off the edge of the ruler. The teacher must explain this so that the children clearly see that the cm length begins at the zero mark.

Normal rulers come with millimetre markings as well. If one wishes to avoid teaching millimetres at this stage, it is good to prepare and keep centimetre rulers in the class, so that one can introduce millimetres at a later point. During the days when wooden rulers were available, I used the back of the ruler to create cm markings. However, if children are curious and ask about the smaller divisions within the centimetre markings, the teacher can explain about millimetres as well. Let the children count the divisions and see that 10 millimetres makes a centimetre, and that a millimetre is useful for very small lengths. However, while recording measurements they can write lengths as 1 cm and 5 mm, or 15 mm. Decimal notation can be taught later.

Construct a ruler with only centimetres marked: Even though children have rulers of their own, it is still worthwhile to get children to build rulers with cm markings and number the markings as 1 cm, 2 cm, etc. In the act of constructing and making markings and labelling them, their understanding of a measuring scale and its divisions becomes clearer.

Figure 11

Let them now observe a normal ruler and describe it in terms of the main divisions they see, how each division is further divided into subdivisions, the numbering on the ruler, etc. Verify whether they understand this completely and whether they are able to use it correctly.

Let children measure various objects in their bag, recording the information in a table form. Some questions are always big hits with children: "Who has the widest smile?" "Who has the longest nose?" "Who has the longest palm?" While measuring length, ensure that children do not confuse length with area measure. Length is a linear measure whereas area is a surface (2-D) measure.

Building estimation skills using body measures: Let children find and look for cm sized parts in their body. Ask: "Which finger nail is about 1 cm wide?" Once that is clearly established, they can learn to use their sense of a centimetre to estimate lengths of other small objects.

Practice activities: Children can be given coloured streamers to cut 1 cm pieces without measuring. They can later make a collage and hang it in the class. Toilet roll paper can also be used for cutting and estimation activities involving bigger lengths like 10, 15, 30 cm.

Let them go on a cm hunt and find natural objects which are about a cm long.

Shoe size: Let them measure their foot size and see how it corresponds with their shoe size.

Footprints: Make children into groups of four. Using old newspapers, each group can print their footprints using poster colour. Other groups have to match the footprint with the right person.

Let children collect some small objects like screws, pen caps, chalk pieces, bottle caps, etc. Let them make a table listing these objects and write their estimations before checking and writing the actual measurements next to them.

Construction activities: Measurement becomes meaningful when children are given construction tasks for which they need to use measurements. Create a box to hold a given object. Make a paper T-shirt for a friend, using a newspaper. Create a streamer decoration for the classroom. Make a *rangoli* design in the centre of the floor.

Sense of metre: Show them a metre ruler and demonstrate usage of it in measuring a few objects in the class. It is good to cut and keep a few 1 metre pieces of rope for measurement activities. Let the children use it for measuring arm length, height, etc. They can describe these measures in terms of "my arm is less than a metre," "my leg is less than a metre," "my height is more than a metre."

Building estimation skills using body measures:

Demonstrate that the distance from one shoulder to the tip of the opposite arm is roughly a metre (for an 'average' adult). They can now begin to use this as a rough measure in various estimation activities.

On a daily basis, pose questions about the lengths of new objects inside or outside the class which require them to use their sense of cm and metre. To reinforce the sense of how large is a metre, ask questions such as: "Guess the length of a cycle and a car. Guess the width of a road." They can later measure the width of a narrow road and a broad road, and the width and length of a car.

More on metres:

Materials: Measuring tapes, metre rulers.

By class 5, they can measure lengths of paths from the school gate to the school, or the length of the school veranda, or the road in front of the school. At this point one needs to discuss rounding to the nearest whole figure.

They should also be given tasks which require them to measure lengths in parts and add them like total length around a building or a garden bed. They can also be given broken up tasks so that pairs of children measure lengths from opposite directions and add them to arrive at the total figure.

Different routes: Ask the children to think of the various paths they can take from the school gate to their classroom. Let them guess the length of these routes. They can then measure these lengths and share their measurements in class.

Children of primary school find it difficult to develop a sense of kilometre. However, sports events which involve running on 50 m, 100 m and 200 m tracks will help children develop a sense of these measures.

Task 8: Given a simple local map depicting lengths in kilometres, children will be able to work out the lengths of the shortest routes connecting various pairs of places.

Foot and inches: Since many measurements are done using feet and inches as measures (rather than metre and cm), children can be taught these in a subsequent year once they thoroughly understand cm and metre.

Building estimation skills using body measures: Let children use the knowledge of their own height to estimate the heights of various objects in the class. Get them to articulate how they think about it. They may say: "this seems to be double my height" or "This is slightly more than half my height" or "This is close to my height," etc.

Let them estimate height of the door, height of the class room, height of the tube light, bench, chair, height of the flag pole, etc. This can be followed by actual measurement activities using a foot ruler or a metre scale.

Task 9: Match object and length.

Prepare a set of object cards and length cards. Object cards can have labels and pictures of familiar objects and animals with varying heights (dog, elephant, coconut tree, table, stool, flower pot, doorstep) and the length cards can have possible heights in metres, centimetres.

Children will have to use their sense of length to match these cards.

Measurement of CAPACITY for 7 to 9 year olds:

Materials: Litre bottles, half litre bottles, 100 ml, 50 ml, 250 ml measuring cups, measuring spoons, 5 ml and 10 ml measuring caps of medicine bottles, normal glass, standard cup, paper cups, spoon, bucket and bathing mug, containers with labels showing capacity, sand, water, cardboard boxes, cubes

Building sense of capacity of everyday objects (drinking glass, teacup, spoon, water bottle): Let children fill a tumbler using a 100 ml measure and check its capacity. Let them also fill a tea cup and a spoon with appropriate measuring units and check their capacity. As 1 litre bottles are widely used, children are quite familiar with them. They can also measure a bath mug and a small bucket. Once they are completely familiar with the capacities of these objects, they will be able to use this knowledge for estimating the capacities of other containers.

Create a calibrated bottle: Let children use a standard transparent bottle. They can fill it with 100 ml measure and mark on the bottle with a marker pen. They can record in multiples of 100 or 250, 500, 750, etc.

Let children use cubes to fill different cardboard boxes (toothpaste box, soap box) and compare capacities of these boxes.

Task 10: Building boxes with interlocking cubes or plain cubes: Give each group of children 36 cubes.

One challenge could be to build open boxes with them. What is the maximum capacity of these boxes? Another challenge can be to make all the

Figure 12

Figure 13

Figure 14

possible types of cuboids with them to see how many differently shaped cuboids can be made from the same number of cuboids.

Checking capacity through displacement: Place a small bucket filled completely with water in a tub. Lower a closed bottle into the bucket. Let the water which has spilled out be collected carefully and poured into the immersed bottle. What is noticed?

Measurement of WEIGHT for 7 to 9 year olds:

Use a rubber band and a clip along with a basket as shown in figures 15 and 16. Let children select three objects and guess the heaviest and lightest among them. Children can now hang the objects from the hooks and measure the lengths of the stretched rubber band. Do they see a relationship between the weight of the object and the length of the stretch?

Materials: Balance made by the children, real balance if possible, 50 gm weight, 100 gm weight, 250 gm weight, 500 gm weight, 1 kg weight (it is easy to put together some stones of equivalent weights packed into cloth bags).

Let children measure the weights of many everyday objects.

Let them build a sense of the weights of some objects they use every day, weights of their tiffin box, water bottle, notebook, biggest textbook, pencil box, pencil, etc. They should use actual weights for the measurements and record the results. Using knowledge of these weights, they should be able to estimate the weights of other objects in the class.

Game: Find a match! One child picks up any object, e.g. a rubber ball, and asks the others: "Find a match for the rubber ball." The other children try to find another object with a similar weight. They can later compare the actual weights using the balance. Whoever comes closest is the winner.

Figure 15

Figure 16

Home project: Let children look around in the house, particularly the kitchen, and ask their parents to make a list of items bought in gm, kg, litres, etc. Discuss the need for smaller measures like gm in medicines and cooking, and the need for larger measures like tonnes.

Building estimation skills: Estimation is a skill learnt through trial and error. One learns from the feedback one gets and the skill gets refined. Children use estimation frequently in daily life. They may not even be aware that they use estimation when they race or jump or leap from one place to another, when they estimate the length of paper needed to make a paper plane or the time they need to complete their homework.

Build the estimation skills by specifying the measure of one object to determine the measure of other objects. If the weight of an orange is known, weights of other fruits like apple, sweet lime, banana, lemon and coconut can be estimated by comparing them mentally and then multiplying or dividing by a suitable factor. Similarly, if the height of the classroom is known, then other heights like the height of the building, the flag pole, the gate, the door, the black board can be estimated. Another important approach used in estimation is to estimate the length or weight of a small portion of the actual object. Example: If one wants to estimate the height of a book shelf with 6 divisions, one will estimate the height of one division and then multiply by the appropriate factor.

Activity: Find the unit used.

Create a worksheet or a set of cards which give measures of various objects without specifying the unit used. Let children use their sense of measure to decide which unit has been used. Example: Rishi is 120 ___ tall; weight of this apple is 125 ___; the handkerchief is 25 ___ long; the shoe is 130 ___ long; this jug's capacity is 2000 ____.

Body awareness: Height: Class room can be a place where the heights of children are recorded on the wall (behind the door) with a pencil. Children can record their height in centimetres, or feet and inches, in their math notebook at the start of the year. They can check it again half way through the year and at the end of the year.

Weight: A weighing scale can be brought to the class at the start of the year and children can record their weight in their note book. They can repeat this activity at the end of the year.

The Teacher can discuss the results. Correlation between increase in height and weight can be observed in many cases.

Padmapriya Shirali

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