Azim Premji University

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COMMUNITY MATH CENTRE · RISHI VALLEY SCHOOL (KFI)

At Right Angles

A RESOURCE FOR SCHOOL MATHEMATICS

Volume 3, No. 3
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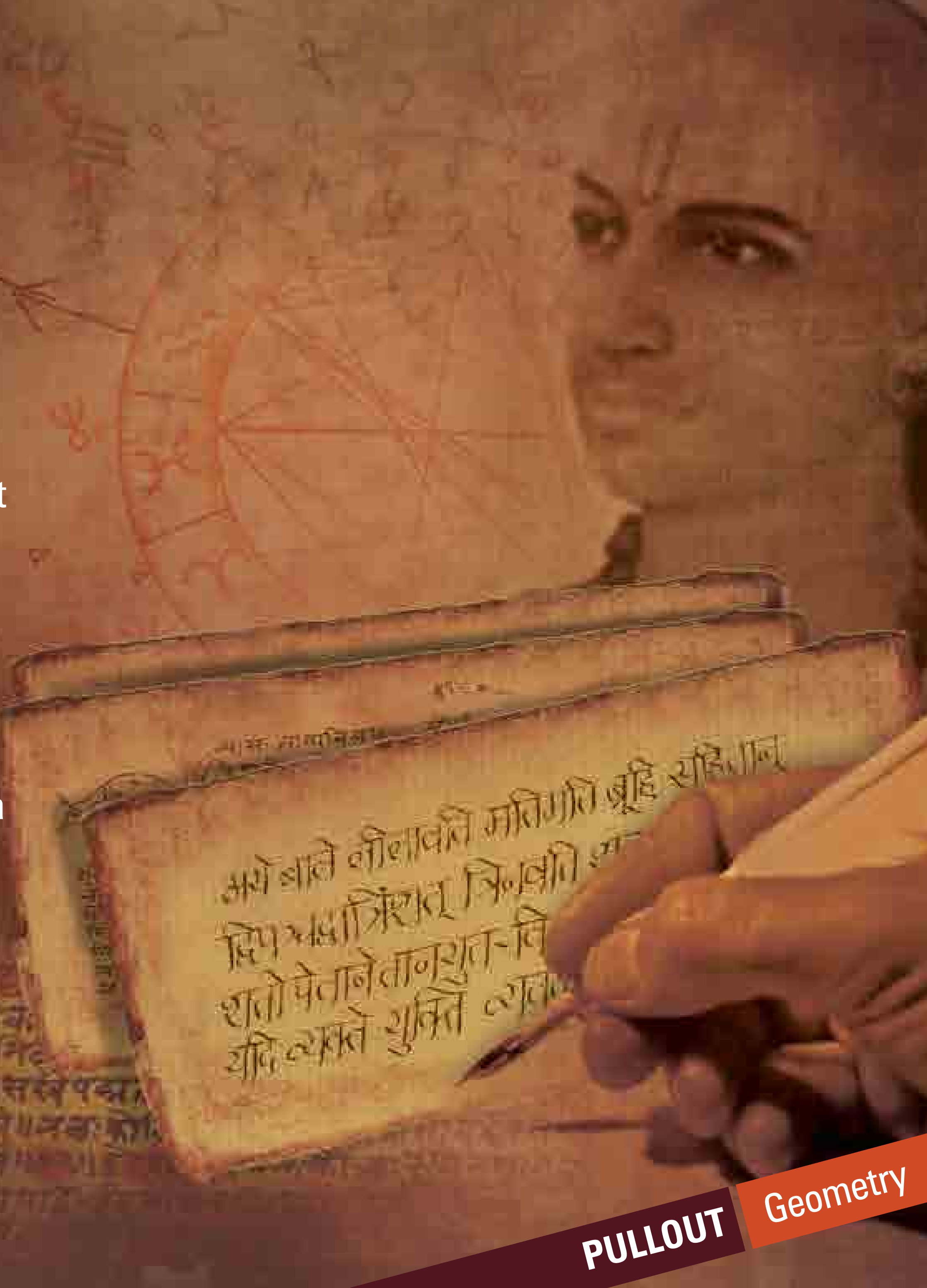

PULLOUT Geometry

ANCIENT INDIAN MATHEMATICS

India has had a rich mathematical tradition from the earliest of times, but we may never know the full extent of the knowledge the ancients had. For example, we do not know precisely when and how they stumbled on the notion of zero (not just the mathematical 'how' but also the cultural 'how'); or the decimal number system and the algorithms for long multiplication, long division, finding square roots, etc; or the notion of negative numbers. The picture below is of an inscription on the wall of a ninth-century temple in a fort in Gwalior (central India). One can see a zero at its centre (the number 270 clearly features in the document). This is the earliest usage of the symbol for zero for which we are able to assign a definite date.

For more information on this temple, please read the article "All for Nought" written by Bill Casselman, available at the following link:

http://www.ams.org/samplings/feature-column/fcarc-india-zero

Bhāskarācharya II, whose works we have featured in this issue of the magazine, wrote a work, Līlāvati, whose verses have even inspired dance performances in the classical Bharata-Natyam style. Shown below is one such verse. (It is written on palm leaf. The image is from

http://www.maa.org/publications/periodicals/convergence/mathematical-treasures-lilavati-of-bhaskara.)

The verse depicts the following problem: On a pillar 9 cubits high is perched a peacock. From a distance of 27 cubits, a snake is coming to its hole at the bottom of the pillar. Seeing the snake, the peacock pounces upon it. If their speeds are equal, tell me quickly, at what distance from the hole is the snake caught? We invite you to solve the problem!

We hope to feature more gems from ancient Indian mathematics in future issues. Stay tuned!

From The Editor's Desk . . .

The closing issue for 2014 features the second article by V G Tikekar on Morley's theorem, giving a trigonometric proof of the theorem, followed by a few articles that give a flavour of ancient Indian mathematics. This being the nine hundredth anniversary of the birth of the great Indian mathematician Bhāskarāchārya II (1114–1185), we have some pieces which give a sense of his work: an essay by Amartya Datta on the *Bījagaṇita*, an important work on algebra written by Bhāskarā; followed by an article by Shailesh Shirali on the Chakravāla algorithm to find solutions of the (wrongly named) Pell equation, which had been explored in great detail by many generations of early Indian mathematicians, with Brahmagupta, Jayadeva, Bhāskarā and Narayana Pandit all playing prominent roles; and followed in turn by an article featuring problems from *Bījagaṇita*. We hope to include more articles of this historical genre in succeeding issues of the magazine.

In 'Classroom', Bharat Karmarkar gives a pictorial proof of a theorem about a triangle connected with a quadrilateral. J Shashidhar dwells on a familiar topic: the joys of "completing the square" (for a quadratic expression). Shiv Gaur continues his series on Origamics, a rich genre in which explorations in geometry are made through the medium of paper folding. Sneha Titus and Sindhu Sreedevi dwell on Portfolio Assessment, an approach to assessment that is far more accurate and representative of students' work than assignments and tests; this is the closing piece of the series on the CCE. Following this we have 'Factor Fun', an article by A Ramachandran featuring problems about factors of numbers, and then we have an article on Angle Trisection — an evergreen topic as far as attractiveness to amateur mathematicians is concerned (not to mention cranks!). Shailesh Shirali continues his series on Proof.

In 'Tech Space', Jonaki Ghosh talks about the Hill Cipher. This cipher makes use of matrix algebra as well as modulo arithmetic and is thus highly suitable for demonstrating how simple mathematical ideas can combine to produce a hard-to-crack cipher. In the 'Review' section, R Ramanujam talks about a well written, critically acclaimed work on ancient Indian mathematics by Kim Plofker, *Mathematics in India*. And in the Pullout section, Padmapriya Shirali deals with the teaching of Geometry to pre-schoolers and children of class one. There's a lot packed into the issue. Enjoy!

— Shailesh Shirali

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At Right Angles is a publication of Azim Premji University together with Community Mathematics Centre, Rishi Valley School and Sahyadri School (KFI). It aims to reach out to teachers, teacher educators, students & those who are passionate about mathematics. It provides a platform for the expression of varied opinions & perspectives and encourages new and informed positions, thought-provoking points of view and stories of innovation. The approach is a balance between being an 'academic' and 'practitioner' oriented magazine.

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Contents

Features

This section has articles dealing with mathematical content, in pure and applied mathematics. The scope is wide: a look at a topic through history; the life-story of some mathematician; a fresh approach to some topic; application of a topic in some area of science, engineering or medicine; an unsuspected connection between topics; a new way of solving a known problem; and so on. Paper folding is a theme we will frequently feature, for its many mathematical, aesthetic and hands-on aspects. Written by practising mathematicians, the common thread is the joy of sharing discoveries and the investigative approaches leading to them.

In the Classroom

This section gives you a 'fly on the wall' classroom experience. With articles that deal with issues of pedagogy, teaching methodology and classroom teaching, it takes you to the hot seat of mathematics education. 'In The Classroom' is meant for practising teachers and teacher educators. Articles are sometimes anecdotal; or about how to teach a topic or concept in a different way. They often take a new look at assessment or at projects; discuss how to anchor a math club or math expo; offer insights into remedial teaching etc.

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Contents contd.

Tech Space

'Tech Space' is generally the habitat of students, and teachers tend to enter it with trepidation. This section has articles dealing with math software and its use in mathematics teaching: how such software may be used for mathematical exploration, visualization and analysis, and how it may be incorporated into classroom transactions. It features software for computer algebra, dynamic geometry, spreadsheets, and so on. It will also include short reviews of new and emerging software.

Problem Corner

Reviews

Pullout

feature

Lurking within any triangle…

Morley´s Miracle — Part II

…is an equilateral triangle

This article continues the series started in the last issue, wherein we study one of the most celebrated and beautiful theorems of Euclidean geometry: Morley's Miracle. In this segment we examine some approaches based on trigonometry.

V G Tikekar

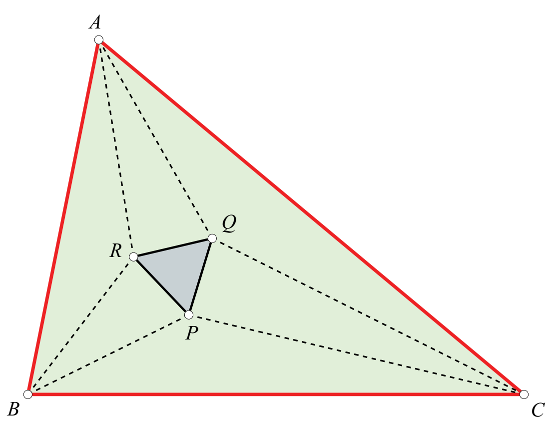

Figure 1. Morley's theorem: The angle trisectors closest to each side intersect in points which are the vertices of an equilateral triangle

In Part I of this article we had narrated the history of this theorem and discussed a beautiful 'pure geometry' proof found by M. T. Naraniengar over a century back. Readers will recall the curious logic used: the proof *starts* with an equilateral triangle and then constructs a configuration similar to the original one, and reaches the desired conclusion this way. (We remarked at that point that many of the pure geometry proofs known today proceed in this way. In Part III of this article, we will show another such proof.)

Keywords: *Angle trisector, equilateral triangle, supplementary angle, addition formula, triple angle formula, sine rule, cosine rule*

In contrast, the trigonometric proof is straightforward: it establishes that the triangle in question is equilateral simply by computing the lengths of its sides and checking that they are equal. In that sense it is very direct, and far from subtle. However, the algebraic steps are challenging! (Not, you might say, for the faint of heart.)

We start by summarizing (without proof) some facts we need from trigonometry. Proofs will be found in the textbooks used for classes 11–12. We use the usual notation: the sides of $\triangle ABC$ are a, b, c; the angles are A, B, C (with side a opposite $\angle A$ and so on); the radius of the circumcircle is R, and the radius of the incircle is r.

Supplementary angles identity: For any angle x,

$$\sin(180^\circ - x) = \sin x. \qquad (1)$$

Addition formula: For any two angles x and y,

$$\sin(x + y) = \sin x \cdot \cos y + \cos x \cdot \sin y. \qquad (2)$$

Triple angle formula: For any angle x,

$$\sin 3x = 3\sin x - 4\sin^3 x. \qquad (3)$$

Triple angle formula (product form): For any angle x,

$$\sin 3x = 4\sin x \cdot \sin(60^\circ - x) \cdot \sin(60^\circ + x). \qquad (4)$$

Sine rule: In $\triangle ABC$, the following identity holds:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R. \qquad (5)$$

Cosine rule: In $\triangle ABC$, the following identity holds:

$$a^2 = b^2 + c^2 - 2bc\,\cos A, \qquad (6)$$

with similar relations for sides b and c.

Of these, perhaps the only one which may look unfamiliar is (4): the product form of the triple angle formula. It is a nice exercise to prove it on one's own.

Now for the details of the trigonometric proof. Let $\angle A = 3x$, $\angle B = 3y$ and $\angle C = 3z$. Then the three angles created by the trisectors at vertex A are x each, the three angles created by the trisectors at vertex B are y each, and the three angles created by the trisectors at vertex C are z each. (See Figure 2.) Our strategy from this point on is straightforward:

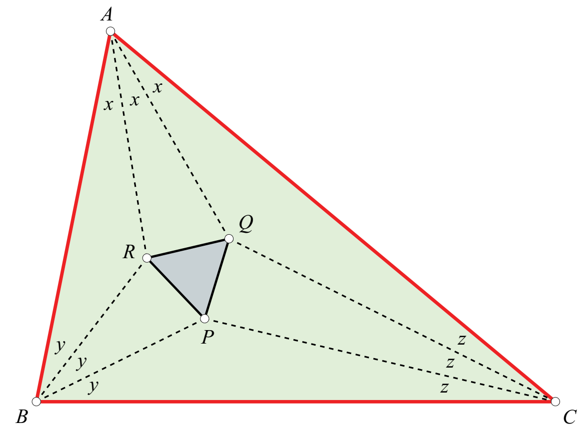

Figure 2.

(i) We use the sine rule in $\triangle PBC$ and $\triangle ARB$ to compute the lengths of BP and BR.

(ii) We transform the two expressions so obtained (for the lengths of BP and BR) using the triple angle formula (quoted above).

(iii) Armed with these expressions for the lengths of BP and BR, we use the cosine rule in $\triangle BPR$ and thus find the length of PR.

Steps (i) and (ii) are easy, but (iii) involves a substantial amount of manipulative algebra.

Now let us get started. Without any loss of generality we take the circumcircle of $\triangle ABC$ to have unit radius (i.e., $R = 1$). The sine rule applied in $\triangle PBC$ yields:

$$\frac{BP}{\sin z} = \frac{BC}{\sin \angle BPC} = \frac{2R\ \sin 3x}{\sin(180^\circ - y - z)} = \frac{2R\ \sin 3x}{\sin(y+z)} = \frac{2\sin 3x}{\sin(60^\circ - x)},$$

since $R = 1$ and $x + y + z = 60^\circ$. Hence:

$$BP = \frac{2\sin z \cdot \sin 3x}{\sin(60^\circ - x)}.$$

In the same way we get:

$$BR = \frac{2\sin x \cdot \sin 3z}{\sin(60^\circ - z)}.$$

Using the triple angle formula (product form), we transform these to the following:

$$BP = \frac{2\sin z \cdot 4 \cdot \sin(60^\circ - x) \cdot \sin x \cdot \sin(60^\circ + x)}{\sin(60^\circ - x)}$$
$$= 8\sin z \cdot \sin x \cdot \sin(60^\circ + x).$$

Similarly, $BR = 8\sin x \cdot \sin z \cdot \sin(60^\circ + z)$.

Now we apply the cosine rule to $\triangle BPR$, using the above expressions:

$$\begin{aligned} PR^2 &= BP^2 + BR^2 - 2BP \cdot BR \cdot \cos y \\ &= 64 \sin^2 z \cdot \sin^2 x \cdot \sin^2(60^\circ + x) \\ &\quad + 64 \sin^2 x \cdot \sin^2 z \cdot \sin^2(60^\circ + z) \\ &\quad - 128 \sin^2 z \cdot \sin^2 x \cdot \sin(60^\circ + x) \\ &\quad \cdot \sin(60^\circ + z) \cdot \cos y \\ &= 64 \sin^2 x \cdot \sin^2 z \cdot \left[\sin^2(60^\circ + x) \right. \\ &\quad + \sin^2(60^\circ + z) - 2 \sin(60^\circ + x) \\ &\quad \left. \cdot \sin(60^\circ + z) \cdot \cos y \right]. \end{aligned}$$

In the last line, note the angles occurring in the expression within square brackets: $60^\circ + x$, $60^\circ + z$ and y. Their sum is $120^\circ + (x + y + z) = 180^\circ$. Hence there exists a triangle with angles $60^\circ + x$, $60^\circ + z$ and y. The form of the expression within the square brackets now invites the next step.

Consider such a triangle UVW (see Figure 3) and apply the sine rule to it. We get:

$$\frac{UW}{\sin y} = \frac{UV}{\sin(60^\circ + z)} = \frac{VW}{\sin(60^\circ + x)} = 2k,$$

where k is the radius of the circumcircle of $\triangle UVW$. Now we apply the cosine rule to the same triangle. We get:

$$\begin{aligned} UW^2 &= UV^2 + VW^2 - 2\,UV \cdot VW \cdot \cos y, \\ \therefore\ 4k^2 \sin^2 y &= 4k^2 \sin^2(60^\circ + z) \\ &\quad + 4k^2 \sin^2(60^\circ + x) \\ &\quad - 8k^2 \sin(60^\circ + z) \\ &\quad \cdot \sin(60^\circ + x) \cos y, \\ \therefore\ \sin^2 y &= \sin^2(60^\circ + z) + \sin^2(60^\circ + x) \\ &\quad - 2 \sin(60^\circ + z) \\ &\quad \cdot \sin(60^\circ + x) \cos y. \end{aligned}$$

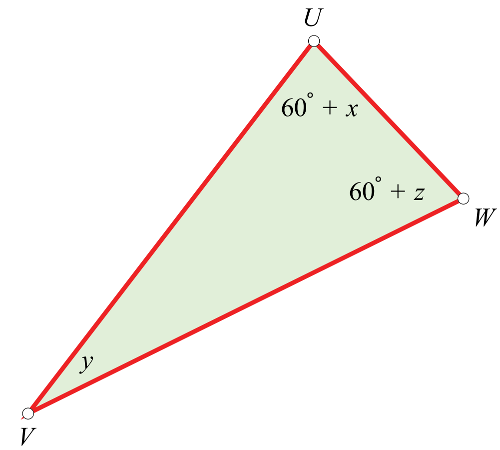

Figure 3. Triangle UVW with angles $60^\circ + x$, $60^\circ + z$ and y

This is an identity connecting any three angles x, y, z whose sum is 60°.

Going back to the expression we had found for PR^2 we find an amazing simplification:

$$PR^2 = 64 \sin^2 x \cdot \sin^2 y \cdot \sin^2 z,$$

and therefore:

$$PR = 8 \sin x \cdot \sin y \cdot \sin z.$$

What a lovely formula!

It is immediately obvious from the form of the above expression that we do not need to do any further computations. For, the expression obtained is completely symmetric in x, y, z (it does not 'prefer' any of x, y, z to the other two quantities), and this tells us that we will get exactly the same expression for PQ as well as QR. Hence $PQ = QR = PR$, and it follows that $\triangle PQR$ is equilateral.

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PROF. V.G. TIKEKAR retired as the Chairman of the Department of Mathematics, Indian Institute of Science, Bangalore, in 1994. He has been actively engaged in the field of mathematics research and education and has taught, served on textbook writing committees, lectured and published numerous articles and papers on the same. Prof. Tikekar may be contacted at vgtikekar@gmail.com.

QUOTES FROM MANJUL BHARGAVA

The Canadian-American number theorist Manjul Bhargava was awarded the Fields Medal at the ICM held in July 2014, in Seoul. Here are some quotes from an interview he gave recently to Mr. ChidanandaRajaghatta of Times of India.

[On his being awarded the Fields Medal.]
I am honored to be a recipient of the Fields Medal; beyond that, it is a source of encouragement and inspiration, and I hope that it is so also for my students and collaborators and colleagues who work with me.

[On whether math genius is a product of meticulous hard work and practice]
While a good memory and a copious supply of talent [are] very helpful, there is no substitute for hard work. Of course, this hard work has to be done in a way where one is always making progress, and where one is approaching this work with realistic short and long term goals with a global vision for what on is trying to achieve.

[On whether Ramanujan's genius was of the same kind as that of the character portrayed by Robin Williams in the movie Good Will Hunting]
Good Will Hunting was great as a movie, but as you might imagine, in reality most mathematics is not done the way it was portrayed in the movie. It requires years of hard work put in to get something out. Ramanujan was a talent of a level that has never been seen, but he certainly put in the hours as well to get the results he was interested in.

Bījagaṇita of Bhāskarācārya

What would have been Fermat's astonishment if some missionary, just back from India, had told him that his problem had been successfully tackled there by native mathematicians almost six centuries earlier!

feature

Amartya Kumar Dutta

The above sentence occurs in the book "Number Theory: An approach through history" (p 81–82) by André Weil (1906–98), one of the giants of 20th century mathematics. The mathematician Pierre de Fermat (1601–65) is regarded as the father of modern number theory. The "problem" referred to by Weil has a grand history. It was posed by Fermat in 1657 as part of his efforts to kindle the interest of contemporary mathematicians in the abstract science of numbers. The problem was to find all *integers* (or whole numbers) x, y which satisfy the equation $Dx^2 + 1 = y^2$, where D is a fixed positive integer which is not a perfect square. The unexpected intricacy of the problem can be felt from the case $D = 61$: the smallest solution (in positive integers) of the equation $61x^2 + 1 = y^2$ is $x = 226153980$, $y = 1766319049$. In his challenge, Fermat had specifically highlighted this case ($D = 61$).

The above problem turned out to be of paramount importance in algebra and number theory. It fascinated some of the greatest mathematicians of modern Europe like Euler (1707–83) and Lagrange (1736–1813). Powerful theories and techniques emerged out of the researches centred around the equation.

Keywords: *History of mathematics, ancient India, Fermat, Brahmagupta, Chakravala, Bhaskaracharya, infinity, mathematical verse, metre, anustup*

This article is a reprint of an article by Prof Amartya Dutta published in *Prabuddha Bharata*, Sept 2007, pages 545-546. We thank the editor Swami Narasimhananda for kindly permitting us to reprint the article. For details of the magazine, please refer to the website www.advaitaashrama.org.

A thousand years before Fermat, the ancient Indian mathematician-astronomer Brahmagupta (628 CE) had investigated the same problem and came up with a brilliant composition law *bhāvanā* on the solution space of the more general equation $Dx^2 + m = y^2$. Brahmagupta's work anticipates several basic principles of modern number theory and abstract algebra. Using Brahmagupta's rule, subsequent Indian algebraists developed an astonishing algorithm called *cakravāla* which gives a complete solution to the problem. The algorithm was discovered by the 11th century — a work of 1073 CE quotes the algebraist Jayadeva's solution to the problem!

The algebra text **Bījagaṇita** (1150 CE) of **Bhāskarācārya** (b. 1114 CE) gives a brief but lucid description, in Sanskrit verses, of the *bhāvanā* law followed by the *cakravāla* algorithm for solving Fermat's equation $Dx^2 + 1 = y^2$. The method is illustrated by two difficult examples including the peculiar example $D = 61$. No wonder that glowing tributes were paid by European scholars after a translation of the *Bījagaṇita* was published. The German mathematician H. Hankel (1874) wrote about the *cakravāla* method:

> It is beyond all praise: it is certainly the finest thing achieved in the theory of numbers before Lagrange.

Bhāskarācārya's *Bījagaṇita* also discusses integer solutions to the linear equation $ax - by = c$, a problem with applications in astronomy and calendar-making. The problem had been solved by Āryabhaṭa (499 CE) by a method called *kuṭṭaka* (pulverisation) which involves a subtle idea resembling Fermat's celebrated principle of descent. There are various other interesting examples of problems involving integer solutions in *Bījagaṇita*. From the verses, one can get a glimpse of the thrill and delight that the ancient Indian algebraists felt in handling such difficult number-theoretic problems. Even today, not many high-school students (or even college students) in India are familiar with this important branch of mathematics.

Bījagaṇita also covers topics in basic algebra that are now familiar to high-school students: negative numbers and zero, variables (unknowns), surds, and the fundamental operations with them; solutions of simultaneous equations in several unknowns; and the solution of the quadratic equation by the method of "elimination of the middle term" (or "completing the square") — an idea with far-reaching consequences in mathematics. As in modern school-texts, interesting concrete examples are given to illustrate applications of the principles.

Bhāskarācārya took the bold step of introducing infinity in mathematics and defining rules of interactions with usual numbers: $\infty + x = \infty$ and $\infty - x = \infty$. The idea of adjunction of infinity has now been put on a firm footing in several branches of higher mathematics like analysis or the valuation theory in commutative algebra and number theory.

The verses in *Bījagaṇita* are in the *anuṣṭup* metre. Ancient Indians had the perception that the metrical form has greater durability, power, intensity and force than the unmetrical and invariably recorded all important knowledge in verse form. It could be exciting for a modern reader to watch how Bhāskarācārya moulds the Sanskrit language to present technical terms and hard results of mathematics in the verse format!

Touches of mythological allegories enhance the charm of Bhāskarācārya's *Bījagaṇita*. While discussing properties of the mathematical infinity, Bhāskarācārya draws a parallel with Lord Viṣṇu who is referred to as *Ananta* (endless, boundless, eternal, infinite) and *Acyuta* (firm, solid, imperishable, permanent):

> During *pralay* (Cosmic Dissolution), beings merge in the Lord and during *sṛṣti* (Creation), beings emerge out of Him; but the Lord Himself — the *Ananta*, the *Acyuta* — remains unaffected. Likewise, nothing happens to the number infinity when any (other) number enters (i.e., is added to) or leaves (i.e., is subtracted from) the infinity; it remains unchanged.

The use of a mystic metaphor to explain the mathematical principle $\infty \pm x = \infty$ reflects the vibrant culture of the bygone era. Perhaps the spiritual culture had prepared the Indian mind for, and probably suggested to it, the concept of the mathematical infinity (or zero!) with its curious properties.

Again, in order to emphasise the importance, power and profundity of algebra, Bhāskarācārya begins the treatise with an Invocation involving an interesting "pun" on the words *Sāṁkhyāḥ* (the Sāṁkhya philosophers as well as the experts in *saṁkhyā*, the science of numbers), *Satpuruṣa* (the Self-Existent Being as well as the wise mathematician), *bīja* (root/cause as well as algebra) and *vyakta* (the manifested universe as well as the revelation of an unknown quantity).Thus, through the opening verse, Bhāskarācārya venerates the Unmanifested — the Self-Existent Being of the Sāṁkhya philosophy — who is the originator of intelligence and the primal Cause of the known or manifested universe; and, through the very same words, Bhāskarācārya pays tribute to the wise mathematician who, using algebra, solves a problem (i.e., reveals or manifests an unknown quantity)!

The importance of algebra is reiterated at the end of *Bījagaṇita*. Bhāskarācārya remarks that algebra is the essence of all mathematics, is full of virtues and free from defects; and that cultivation of algebra will sharpen the intellect of children. He concludes with the exhortation "*paṭha paṭha*" (Learn it, learn it) for the development of intelligence.

In this connection, I may mention here that one of our greatest contemporary mathematicians Shreeram S. Abhyankar (b. 1930) acknowledges the influence of Bhāskarācārya during his formative years. Abhyankar fondly recalls how his father (S.K. Abhyankar) used to teach him mathematics by reciting to him lines from Bhāskarācārya's text *Līlāvatī* and how he used to memorise them when he was around ten years of age.

Bhāskarācārya's *Bījagaṇita* not only makes us aware of the great advancements made by ancient Indian algebraists, it also gives us a feel for the charming atmosphere in which mathematical research and discourse — at both basic and advanced levels — used to take place in ancient times. While a study of *Bījagaṇita* will be enriching and inspiring for all cultured students of mathematics, a careful analysis of the treatise will also provide valuable insights to historians and scholars in general.

End-notes

- The quote (1874) by Hankel appeared in *Zur Geschichte der Mathematik in Alterthum und Mittelalter* (Leipzig, 1874), p 202.
- The remark by S. S. Abhyankar occurs on page 135 of his survey article "Resolution of Singularities and Modular Galois Theory", Bulletin of the American Mathematical Society, Vol. 38(2) (2001). The article is reproduced in the volume "Connected at Infinity: A Selection of Mathematics by Indians" ed R. Bhatia, pub Hindustan Book Agency (TRIM 25); the remarks occur in p 177.

End Notes for 'Bījagaṇita of Bhāskarācārya'

(1) The above article by Prof Amartya Kumar Dutta was published in the magazine *Prabuddha Bharata*, in the Sept 2007 issue (pages 545–546). It is reproduced here by kind permission of its editor, Swami Narasimhananda. For information on this publication, kindly refer to the website www.advaitaashrama.org.

(2) In connection with the sentence "In this connection, I may mention here that one of our greatest contemporary mathematicians Shreeram S. Abhyankar (b. 1930) ..." which appears in Prof Dutta's article, please note that this article was written in 2007. Unfortunately, Prof. Abhyankar passed away in November 2012.

(3) In the article there is a reference to the '*anuṣṭup* metre'. As the author has noted, it was the practice of ancient Indians to record all important knowledge in verse form. One of the metres they made use of frequently is *anuṣṭup*. For more information on this, please refer to https://en.wikipedia.org/wiki/Anustubh.

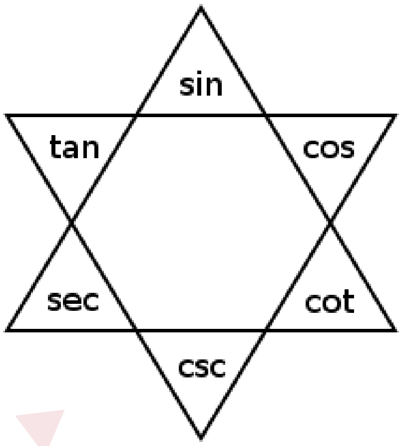
(4) In the same issue of *Prabuddha Bharata* (Sept 2007), there is an article on Indian mathematics by Prof. Kumar Murty (see http://www.advaitaashrama.org/Content/pb/2007/092007.pdf), which has a prescient paragraph on Prof Manjul Bhargava, the Indian-origin mathematician who was honoured with a Fields Medal at the International Congress of Mathematicians held in Seoul, South Korea, in August 2014: "The work of Bhargava, who is currently Professor of Mathematics at Princeton University, is deep, beautiful, and largely unexpected. It has many important ramifications and will likely form a theme of mathematical study at least for the coming decades."

AMARTYA KUMAR DUTTA is a Professor of Mathematics at the Indian Statistical Insitute, Kolkata. His research interest is in commutative algebra. He did his Ph.D. in Mathematics from TIFR, Mumbai. Prof. Dutta has contributed many papers on ancient Indian mathematics and is a teacher in the Indology course at the Ramakrishna Mission Institute of Culture, Kolkata. He may be contacted at amartya@isical.ac.in.

ALISON'S TRIANGLE

Here is a simple diagram which helps in remembering some trig identities.
It has been taken from the webpage
http://www.futilitycloset.com/2014/04/20/alisons-triangle/
It allows us to reconstruct all relations of the form a÷b=c or a×b=c, where a,b,c are the basic trigonometric functions of the same angle θ.

Here is the diagram :



Here's how it is used. Take any three functions arranged next to each other (e.g., sin, cos, cot). Then the product of the ends equals the middle.

For example, from "sin cos cot" we get: $\sin\theta \times \cot\theta = \cos\theta$.

Another way of stating this: the middle function divided by a function at the end equals the function at the other end. For example, from "sec tan sin" we get : $\tan\theta \div \sin\theta = \sec\theta$.

If we commit this diagram to memory, we can reconstruct all possible such relations. On the webpage mentioned, the writer notes that he found the diagram in Michael Stueben's book Twenty Years Before the Blackboard (Spectrum, 1998).

Solving a Famous Problem

The Chakravāla Method

Zeroing in on a Solution

feature

Let **n** *be a non-square positive integer. The problem of finding positive integers* $\mathbf{x}, \mathbf{y}$ *such that* $\mathbf{x^2 - ny^2 = 1}$ *is one of enormous interest to number theorists. This equation is commonly known today as 'Pell's equation', so-named after the English scholar John Pell (1611–1685). The name was given by Leonhard Euler (1707–1783), but we know now that this was done on an erroneous supposition. Before Pell and Euler, the equation had been explored by Pierre de Fermat. And much before Fermat — a whole millennium earlier — the same equation had been studied in great detail by the Indian mathematician Brahmagupta (598–670). The equation was referred to by Brahmagupta as the* ***Varga Prakriti****, or the "equation of the multiplied square". Some centuries later came Jayadeva (950–1000) who delved deeper into the problem and offered a more general procedure for its solution. Bhāskarachārya II (1114–1185) refined this work and gave a full account of the algorithm in his important work, Bījaganitam. He gave the name* ***Chakravāla*** *to the algorithm; the name reflects the cyclic or iterative nature of the procedure, for 'Chakra' means 'wheel'. Narayana Pandit (1340–1400) added further to this work. The above equation could therefore be called the Brahmagupta-Jayadeva-Bhāskarā equation, but we shall call it simply the Brahmagupta equation. In this note we explain the working of the Chakravāla algorithm to solve this equation. However, we do not give a proof that the algorithm works. For that we shall only refer you to published sources.*

SHAILESH SHIRALI

Introduction. Perhaps you have come across the 'house-number puzzle':

> *On a street with houses numbered* 1, 2, 3, …, A *where* A *is a three-digit number, I live in a house, numbered* B*, such that the sums of the house numbers on the two sides of my house are equal. Find* A *and* B*.*

We can ask, more generally: "Find possible values of A and B" (without the condition that A is a three-digit number). The condition tells us that the sums $1 + 2 + \cdots + (B - 1)$ and

$(B+1)+(B+2)+\cdots+A$ are equal. Hence:

$$\frac{(B-1)B}{2}=\frac{A(A+1)}{2}-\frac{B(B+1)}{2}, \quad \therefore B^2=\frac{A(A+1)}{2}.$$

From the last relation we get: $8B^2=4A^2+4A$, hence $8B^2+1=(2A+1)^2$. Let $x=2A+1$ and $y=2B$. Then we have:

$$x^2-2y^2=1.$$

Thus the house-number puzzle has given rise to a Brahmagupta equation with $n=2$. This is one of several different ways in which we arrive at such equations.

As we have a single equation in two variables, there could be more than one solution. (There could also be no solutions.) Is there a systematic way of finding the solutions? This question was considered by Brahmagupta. He solved the problem for some individual values of n, e.g., $n=83$ and $n=92$ and went on to say: "A person who is able to solve these problems within a year is truly a mathematician!" A year? Brahmagupta clearly had a high regard for the problem!

Three identities

We start by some identities which were used with great effectiveness by the main actors in this story.

Diophantus-Brahmagupta-Fibonacci identity. The following two identities go back all the way to Diophantus of Alexandria (they are found in his book *Arithmetica*):

$$\begin{cases}\left(a^2+b^2\right)\left(c^2+d^2\right) &= (ad-bc)^2+(ac+bd)^2,\\ \left(a^2+b^2\right)\left(c^2+d^2\right) &= (ad+bc)^2+(ac-bd)^2.\end{cases} \tag{1}$$

The identities may be verified by expanding all the terms. They tell us that *the product of two integers, each a sum of two squares, is itself a sum of two squares, and in two different ways*. For example, consider the two integers $10=3^2+1^2$ and $41=5^2+4^2$; both are sums of two squares. Their product $10\times 41=410$ may be written in two such ways:

$$410=(15+4)^2+(12-5)^2=19^2+7^2,$$
$$410=(15-4)^2+(12+5)^2=11^2+17^2.$$

Today we describe this as a *closure result*. We say: *The set of integers expressible as a sum of two squares is closed under multiplication*.

How the early mathematicians hit upon this double identity is not clear, but it is easy for us today to reconstruct it and also to remember it by invoking the following result about complex numbers: *The magnitude of the product of two complex numbers is equal to the product of their magnitudes*. For, consider the complex numbers $a+bi$ and $c+di$. Here a,b,c,d are real numbers, and $i=\sqrt{-1}$. Now, observing that

$$|a+bi|^2=a^2+b^2=|a-bi|^2,$$
$$|c+di|^2=c^2+d^2=|c-di|^2,$$
$$(a+bi)(c+di)=(ac-bd)+(ad+bc)i,$$
$$(a-bi)(c+di)=(ac+bd)+(ad-bc)i,$$

we see how the Diophantus-Brahmagupta-Fibonacci identity is equivalent to the statement "magnitude of product equals product of magnitudes" for complex numbers.

Brahmagupta's identity. Somewhat more general are the following identities which were first discovered by Brahmagupta and used extensively by him:

$$\begin{cases} \left(a^2-nb^2\right)\left(c^2-nd^2\right) &= (ac+nbd)^2-n(ad+bc)^2, \\ \left(a^2-nb^2\right)\left(c^2-nd^2\right) &= (ac-nbd)^2-n(ad-bc)^2. \end{cases} \tag{2}$$

Thus: *For each fixed integer n, the set of integers expressible as* x^2-ny^2 *is closed under multiplication*. Example: Let $n=2$. The integers $3^2-2\times 1^2=7$ and $5^2-2\times 2^2=17$ are both of the form x^2-2y^2, and so is their product $7\times 17=119=19^2-2\times 11^2$.

The case $n=-1$ corresponds to the Diophantus-Brahmagupta-Fibonacci identity.

As earlier, it is easy to verify these relations by expanding all the terms. And it is easy to reconstruct them by studying numbers of the form $a+b\sqrt{n}$, as can be seen by multiplying together two numbers of this form. For we have:

$$\left(a+b\sqrt{n}\right)\cdot\left(c+d\sqrt{n}\right)=(ac+nbd)+(ad+bc)\sqrt{n}. \tag{3}$$

Brahmagupta used (2) in his approach to the equation $x^2-ny^2=1$. Indeed, he used it as a sort of 'composition law' which has a very modern look about it, for it looks like it is taken straight out of a book on modern algebra! In using such an approach, Brahmagupta was ahead of his time by a whole millennium.

There is another way of expressing the identity which may bring home its significance more strongly. *Suppose that* $(x,y)=(a,b)$ *is a solution of the equation* $x^2-ny^2=k_1$, *and* $(x,y)=(c,d)$ *is a solution of the equation* $x^2-ny^2=k_2$. *Then* $(x,y)=(ac+nbd,ad+bc)$ *is a solution of the equation* $x^2-ny^2=k_1k_2$.

Remark. The appearance of the identical identity in the works of Fibonacci (1170–1250) and Brahmagupta may seem an amazing coincidence but it is not a mystery. Brahmagupta wrote an extremely important work, the *Brahma-Sphuta-Siddhantha*, which reached the centre of the Arab world (Baghdad) in the eighth century and was translated into Arabic, at the instance of the Caliph. Three centuries later this work was translated into Latin in Spain, and it is this doubly translated work that came into the hands of Fibonacci, who was then a trader in Italy.

Bhāskarā's lemma. The third identity in our list is the following, which is an auxiliary result needed by the Chakravāla algorithm:

$$\text{If } a^2-nb^2=k, \text{ then } \left(\frac{ma+nb}{k}\right)^2-n\left(\frac{a+bm}{k}\right)^2=\frac{m^2-n}{k}. \tag{4}$$

The verification is straightforward:

$$\begin{aligned} \left(\frac{ma+nb}{k}\right)^2-n\left(\frac{a+bm}{k}\right)^2 &= \frac{m^2a^2+n^2b^2+2mnab-na^2-nm^2b^2-2mnab}{k^2} \\ &= \frac{m^2a^2+n^2b^2-na^2-nm^2b^2}{k^2} \\ &= \frac{(m^2-n)\cdot(a^2-nb^2)}{k^2}=\frac{m^2-n}{k}. \end{aligned}$$

The Chakravāla

In the description given below, n is a fixed non-square positive integer. For convenience we use the symbol or triple $[x,y,k]$ to indicate that the numbers x,y,k satisfy the relation

$$x^2-ny^2=k. \tag{5}$$

Observe that n does not appear in the triple. This is so because n is treated as a fixed integer in these relations.

The Brahmagupta identity (2) shows that from two triples $[a, b, k_1]$ and $[c, d, k_2]$ we can produce a third triple $[u, v, k_1 k_2]$, with

$$u = ac + nbd, \qquad v = ad + bc. \tag{6}$$

Example: Take $n = 10$. We have the triples $[7, 2, 9]$ and $[11, 3, 31]$ which correspond to the relations

$$7^2 - 10 \times 2^2 = 9, \qquad 11^2 - 10 \times 3^2 = 31.$$

We seek a triple of the form $[u, v, 9 \times 31] = [u, v, 279]$. We get: $u = 77 + 10 \times 6 = 137$ and $v = 21 + 22 = 43$. Check:

$$137^2 - 10 \times 43^2 = 18769 - 10 \times 1849 = 279.$$

The composition law

$$[a, b, k_1] \star [c, d, k_2] := [ac + nbd, ad + bc, k_1 k_2] \tag{7}$$

was given the name *Samasa bhāvanā* by Brahmagupta (or just *Bhavana* for short).

Deriving Bhāskarā's lemma from the composition law. Bhāskarā's lemma stated above may look mysterious and unmotivated; but it looks more meaningful when we invoke the composition law. Say we have a triple $[a, b, k]$; that is, we have $a^2 - nb^2 = k$. For any integer m, we also have the 'trivial triple' $[m, 1, m^2 - n]$ whose second coordinate is 1; it corresponds to the trivial statement $(m)^2 - n \cdot 1^2 = m^2 - n$. Now let us compose these two triples using the Bhavana. We get:

$$[a, b, k] \star [m, 1, m^2 - n] = [ma + nb, a + bm, k(m^2 - n)]. \tag{8}$$

In other words we have: if $a^2 - nb^2 = k$, then

$$(ma + nb)^2 - n(a + bm)^2 = k(m^2 - n). \tag{9}$$

Dividing through by k^2 we get Bhāskarā's lemma:

$$\left(\frac{ma + nb}{k}\right)^2 - n\left(\frac{a + bm}{k}\right)^2 = \frac{m^2 - n}{k}. \tag{10}$$

Heuristic motivation for the Chakravāla. We seek to solve the equation $x^2 - ny^2 = 1$; that is, we seek a triple $[x, y, 1]$ whose third coordinate is 1, which is the smallest it can possibly be (it can never be 0, because n is non-square, implying that $\sqrt{n}$ is irrational). Now when we repeatedly apply the composition law, the number in the third coordinate steadily increases, for it is subject to integer multiplication, and this continues unless we effect a division at some stage. For this it must happen that $ma + nb$, $a + bm$ and $m^2 - n$ are all divisible by k. This is how Bhāskarā's lemma steps in, and this simple-minded reasoning is the heuristic logic behind the Chakravāla. Here, then, is how the algorithm due to Jayadeva, Bhāskarachārya II and Narayana Pandit works.

Steps of the Chakravāla algorithm.
Step 0: Start with a triple $[a, b, k]$ in which a and b are coprime.

Step 1: Look for values of m such that $a + bm$ is divisible by k. These values will form an arithmetic progression (AP) with common difference $|k|$. From this AP, select that value of m which makes $|m^2 - n|$ the least.

Step 2: Let a', b', k' be computed thus:

$$a' = \frac{ma + nb}{|k|}, \qquad b' = \frac{a + bm}{|k|}, \qquad k' = \frac{m^2 - n}{k}. \tag{11}$$

Replace $[a, b, k]$ by $[a', b', k']$. (That is, $[a, b, k] \longleftarrow [a', b', k']$, to use "computer grammar".) If the third coordinate of the new triple is 1, then this is the triple we seek; end of story. Else, go back to Step 1 and start a fresh cycle of computations.

The 'cycle' tells us why Bhāskarā called this the Chakravāla. We show the working of the algorithm using a few examples.

Example 1. Let $n = 10$. Let us start with the trivial triple $[4, 1, 6]$, that is, with the relation $4^2 - 10 \times 1^2 = 6$, and see where it leads us.

Step 0: $[a, b, k] = [4, 1, 6]$.

Step 1: We want $a + bm$ to be divisible by k, i.e., $4 + m$ to be divisible by 6. Hence $m \in \{2, 8, 14, 20, \ldots\}$. The value of m for which $m^2 - 10$ is least is $m = 2$.

Step 2: $a' = \dfrac{8+10}{|6|} = 3,\ b' = \dfrac{4+2}{|6|} = 1$ and $k' = \dfrac{4-10}{6} = -1$.

We have obtained the triple $[a, b, k] = [3, 1, -1]$. Since the third coordinate is not 1, we go back to Step 1.

Step 1: We want $3 + m$ to be divisible by -1. Any integer value will do. We also want $|m^2 - 10|$ to be as small as possible. So we choose $m = 3$.

Step 2: $a' = \dfrac{9+10}{|-1|} = 19,\ b' = \dfrac{3+3}{|-1|} = 6$ and $k' = \dfrac{9-10}{-1} = 1$.

We have obtained the triple $[a, b, k] = [19, 6, 1]$. Now the third coordinate is 1, so the computations are over, and we have obtained a solution to the problem:

$$19^2 - 10 \times 6^2 = 1.$$

Example 2. Let $n = 13$. We start with the trivial triple $[4, 1, 3]$, that is, with the relation $4^2 - 13 \times 1^2 = 3$.

Step 0: $[a, b, k] = [4, 1, 3]$.

Step 1: We want $a + bm$ to be divisible by k, i.e., $4 + m$ to be divisible by 3. Hence $m \in \{2, 5, 8, 11, \ldots\}$. The value of m for which $m^2 - 13$ is least is $m = 2$.

Step 2: $a' = \dfrac{8+13}{|3|} = 7,\ b' = \dfrac{4+2}{|3|} = 2$ and $k' = \dfrac{4-13}{3} = -3$.

We have obtained the triple $[a, b, k] = [7, 2, -3]$. As the third coordinate is not 1, we go back to Step 1.

Step 1: We want $7 + 2m$ to be divisible by -3. Hence $m \in \{1, 4, 7, 10, \ldots\}$. The value of m for which $|m^2 - 13|$ is least is $m = 4$.

Step 2: $a' = \dfrac{28+26}{|-3|} = 18,\ b' = \dfrac{7+8}{|-3|} = 5$ and $k' = \dfrac{16-13}{-3} = -1$.

We have obtained the triple $[a, b, k] = [18, 5, -1]$. We go back to Step 1.

Step 1: We want $18 + 5m$ to be divisible by -1. Any integer value will do. We want m such that $|m^2 - 13|$ is least. So we choose $m = 4$.

Step 2: $a' = \dfrac{72+65}{|-1|} = 137,\ b' = \dfrac{18+20}{|-1|} = 38$ and $k' = \dfrac{16-13}{-1} = -3$.

We have obtained the triple $[a, b, k] = [137, 38, -3]$. Back to Step 1.

Step 1: We want $137 + 38m$ to be divisible by -3. Hence $m \in \{2, 5, 8, 11, \ldots\}$. The value of m for which $|m^2 - 13|$ is least is $m = 2$.

Step 2: $a' = \dfrac{274+494}{|-3|} = 256,\ b' = \dfrac{137+76}{|-3|} = 71$ and $k' = \dfrac{4-13}{-3} = 3$.

We have obtained the triple $[a, b, k] = [256, 71, 3]$. Back to Step 1.

Step 1: We want $256 + 71m$ to be divisible by 3. Hence $m \in \{1, 4, 7, 10, \ldots\}$. The value of m for which $|m^2 - 13|$ is least is $m = 4$.

Step 2: $a' = \dfrac{1024 + 923}{|3|} = 649$, $b' = \dfrac{256 + 284}{|3|} = 180$ and $k' = \dfrac{16 - 13}{3} = 1$.

We have obtained the triple $[649, 180, 1]$. Now the third coordinate is 1 so the computations are over. Here is our solution:

$$649^2 - 13 \times 180^2 = 1.$$

Remark. We have described the working of the Chakravāla, but we have not attempted to show that it will always yield an answer. For this, we must show that we will reach a triple $[u, v, 1]$ no matter with which triple $[a, b, k]$ we start the computation; of course, we do need GCD $(a, b) = 1$. Interested readers may refer to [1] or [3] for proof that the Chakravāla algorithm works.

Had the ancients considered such a question? Did Jayadeva, Bhāskarā and Narayana Pandit *know* that the Chakravāla algorithm would invariably give a solution? We do not know.

In closing we note that there are some steps that serve as shortcuts. Brahmagupta found that if we ever reach a triple $[a, b, k]$ with $k \in \{\pm 4, \pm 2, -1\}$, then we can reach the desired end (with third coordinate equal to 1) in a single step, thus shortening the computations greatly. We do not study these shortcuts here. Please refer to [3] for details.

Exercises. Using the Chakravāla, find some positive integer solutions to the following equations. Use any convenient starting triple of your own choice in each case.

(1) $x^2 - 8y^2 = 1$

(2) $x^2 - 11y^2 = 1$

(3) $x^2 - 15y^2 = 1$

(4) $x^2 - 17y^2 = 1$

(5) $x^2 - 19y^2 = 1$

(6) $x^2 - 41y^2 = 1$

Remark. Using the Chakravāla, Bhāskarā found the following positive integer solution to the equation $x^2 - 61y^2 = 1$ (it is the *least* such!): $x = 1766319049$, $y = 226153980$.

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Ingenuity in Algebra

Gems from Bhāskarāchārya II

Number Problems from Ancient India

feature

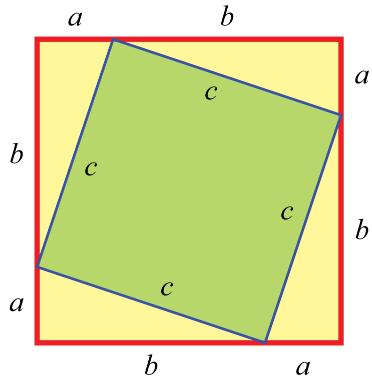
C⊗MαC

Bhāskarā II, pre-eminent mathematician-astronomer of ancient India, was born in 1114 AD in Bijapur (present-day Karnataka). Thus, the present year marks his nine hundredth birth anniversary. Events have been organized across the country in honour of this event.

Bhāskarā served as head of an astronomical observatory located at Ujjain (in present-day Madhya Pradesh). He is best known for his work *Līlāvatī* ('The Beautiful'). This book occurs as part of a larger work, *Siddhānta Siromani* ('Crown of Treatises'), which also has an important section on advanced algebra, *Bījagaṇita* ('Seed Arithmetic'), and two works on astronomy, *Grahagaṇita*, on the motions of planets, and *Golādhyāya* ('The Sphere'). This work was written when he was 36 years old.

Līlāvatī is a work on basic mathematics, covering arithmetic, simple algebra, geometry and simple mensuration. The famous 'Behold!' proof-without-words of the Pythagorean theorem (see Figure 1) is from this book.

There is a lovely legend behind *Līlāvatī* which is worth quoting. The legend goes that Bhāskarā had a daughter named Līlāvatī. She came of age, her wedding was arranged, and the long-awaited day finally came. The young Līlāvatī, ever curious about things around her, peeped into the water clock



$$(a+b)^2 = c^2 + \left(4 \times \tfrac{1}{2}ab\right)$$
$$\therefore\ a^2 + b^2 = c^2$$

Figure 1. Proof of the Pythagorean theorem. No words needed …

being used to ensure that events took place at the auspicious hour. As she did so, a pearl from her necklace fell into the water, blocked one of the channels, and so interrupted the flow. This upset the calculations, and the wedding took place at the wrong time, with the result that Līlāvatī was widowed at an early age. To console her, Bhāskarā composed his masterpiece, *Līlāvatī*, ... Did it really happen this way? Of course, we do not know, and we have no way of knowing. We do not even know if Bhāskarā had a daughter by that name!

Bījagaṇita is mainly about the extraction of roots of equations of different kinds; its level of exposition is higher than that of *Līlāvatī*. Interestingly, some of the material of *Līlāvatī* is dealt with again in *Bījagaṇita*, but the analysis is deeper, and algorithms are not presented merely in cook-book fashion. It covers topics such as:

- The rules of operations for working with zero and infinity (Bhāskarā II is the first mathematician to state explicitly and correctly the rules for working with infinity);
- Solutions of several Diophantine equations, including instances of the Brahmagupta equation $Nx^2 + 1 = y^2$ (this is also known, wrongly, as the 'Pell equation'). For example, he solves the equation $61x^2 + 1 = y^2$, and finds its *least* positive integral solution: $x = 226153980,\ y = 1766319049$.

Isolated cases of higher degree equations occur (a cubic and a biquadratic, respectively), but the equations considered are not of a general nature.

Some Diophantine Problems from Bījagaṇita

A *Diophantine equation* or *Diophantine problem* is one for which we seek solutions in integers or in rational numbers; sometimes in only the positive integers. For example, we have the equation associated with the Pythagorean theorem (generally remembered as $a^2 + b^2 = c^2$), whose solutions yield the Pythagorean triples.

Among the many Diophantine problems posed and solved in *Bījagaṇita*, mention may be made of the following.

Problem 1: *Find four numbers whose sum is equal to the sum of their squares.* (This is Example 56 of *Bījagaṇita*.)

Bhāskarā offers the following solution. Take any four integers, say 1, 2, 3, 4. Their sum is 10, and the sum of their squares is 30. As $10 : 30 = 1 : 3$, we multiply all the numbers by the ratio $1 : 3$, and so arrive at the nice equality

$$\frac{1}{3} + \frac{2}{3} + \frac{3}{3} + \frac{4}{3} = \left(\frac{1}{3}\right)^2 + \left(\frac{2}{3}\right)^2 + \left(\frac{3}{3}\right)^2 + \left(\frac{4}{3}\right)^2.$$

Simple and neat! Note that the method yields solutions in rational numbers, not integers.

This method clearly generalizes to any number of numbers. For example, one can solve problems of this kind: *Find two positive numbers for which the sum of their squares is equal to the sum of their cubes.*

Problem 2: *Find two numbers the sum of whose squares is a cube, and the sum of whose cubes is a square.* (This is Example 59 of *Bījagaṇita*.)

The way this is expressed in *Bījagaṇita* is: "The sum of the cubes of two numbers, and the sum of their squares is a cube. If you know such numbers, I shall consider you great among algebraists."

Let the numbers be x and y. Then $x^2 + y^2$ is a cube, and $x^3 + y^3$ is a square. To find such pairs of numbers, Bhāskarā makes use of the fact that 1 and 2 are two numbers the sum of whose cubes is a square (namely: $1^3 + 2^3 = 9 = 3^2$). So he puts $x = z^2$ and $y = 2z^2$. Then $x^3 + y^3 = z^6 + 8z^6 = 9z^6 = \left(3z^3\right)^2$ which is a square. Thus the second condition is automatically satisfied. This allows him to focus on just the first condition.

We now have: $x^2 + y^2 = z^4 + 4z^4 = 5z^4 = 5z \times z^3$. This is a cube if $5z$ is a cube, which it will be if z is of the form $25r^3$ for any integer r, giving $x = 625r^6$ and $y = 1250r^6$. So we can generate pairs of numbers with the required property. Thus, $r = 1$ yields the pair $\{625, 1250\}$, while $r = 2$ yields $\{40000, 80000\}$.

Remark. We can use other pairs of numbers (instead of $\{1, 2\}$) the sum of whose cubes is a square. For example: $\{11, 37\}$, with $11^3 + 37^3 = 228^2$, and $\{56, 65\}$, with $56^3 + 65^3 = 671^2$. The question can also be asked whether this problem has solution pairs which are not generated by this kind of logic.

Problem 3: *Find pairs of integers A, B such that $(A + B)^2 + (A + B)^3 = 2(A^3 + B^3)$.* (This is Example 89 of *Bījagaṇita*.)

In *Bījagaṇita* we read: "The square and cube of the sum of two numbers is equal to twice the sum of their cubes. O Mathematician, please state the numbers."

Bhāskarā takes the integers A and B to be given by $A = x + y$, $B = x - y$. Substituting these into the given equation, he gets $A + B = 2x$, $A - B = 2y$, and so:

$$A^3 + B^3 = 2(x^3 + 3xy^2).$$

Hence we have:

$$4x^2 + 8x^3 = 4x^3 + 12xy^2,$$
$$\therefore\ 4x + 4x^2 = 12y^2.$$

The quantity on the left is $(2x + 1)^2 - 1$. Let $X = 2x + 1$ and $Y = 2y$; then we get an instance of a Brahmagupta equation,

$$X^2 - 3Y^2 = 1,$$

and Bhāskarā is quite at home in this territory! Working backwards, he obtains solutions of the original equation.

(X, Y)	$(7, 4)$	$(97, 56)$	$(1351, 780)$	...
(x, y)	$(3, 2)$	$(48, 28)$	$(675, 390)$	...
(A, B)	$(5, 1)$	$(76, 20)$	$(1065, 285)$	...

The solution $(A, B) = (5, 1)$ corresponds to the equality

$$6^2 + 6^3 = 2(5^3 + 1^3),$$

while $(A, B) = (76, 20)$ corresponds to

$$96^2 + 96^3 = 2(76^3 + 20^3).$$

Problem 4: *Find all positive integers x such that $5x^4 - 100x^2$ is a square.* (This is Example 90 of *Bījagaṇita*.)

The original version: "Five times the fourth power of the unknown reduced by hundred times its square gives a square. O Mathematician, give that number quickly."

Let x be an integer with the stated property. Then $5x^2(x^2 - 20)$ is a square, hence $5(x^2 - 20)$ is a square. Let $5(x^2 - 20) = y^2$. Since y^2 is a multiple of 5, it must be that y itself is a multiple of 5. Let $y = 5u$. Then we have $x^2 - 20 = 5u^2$. From this we see that x too is a multiple of 5. Let $x = 5v$; then $5v^2 - 4 = u^2$. So we must solve the equation $u^2 - 5v^2 = -4$.

Yet again we get a Brahmagupta equation, which we solve by the usual means. Its solutions, and the corresponding values for x and y, are given below.

(v, u)	$(1, 1)$	$(4, 2)$	$(11, 5)$	$(29, 13)$	$(76, 34)$	$(199, 89)$	...
(y, x)	$(5, 5)$	$(20, 10)$	$(55, 25)$	$(145, 65)$	$(380, 170)$	$(995, 445)$	...

Hence, x-values that fit the original equation are: 5, 10, 25, 65, 170, 445, ...

Remark. If we list the u-values alone, we get the following list:

$$1,\ 2,\ 5,\ 13,\ 34,\ 89,\ 233,\ 610,\ \dots .$$

These are alternate members of the Fibonacci sequence! Note the nice feature of the v-values too: 1, 4, 11, 76, 199, ..., with:

$$4 = 5 - 1,\ \ 11 = 13 - 2,\ \ 76 = 89 - 13,\ \ 199 = 233 - 34,\ \ \dots .$$

So the v-numbers are the differences between successive pairs of alternate members of the Fibonacci sequence.

Problem 5: *Find all positive integers x for which both $3x + 1$ and $5x + 1$ are squares.* (This is Example 99 of *Bījagaṇita*.)

Let x be a positive integer such that $3x + 1 = u^2$, $5x + 1 = v^2$ where u, v are integers. Bhāskarā now supposes that u is of the form $3y + 1$. This yields

$$3x + 1 = (3y + 1)^2, \quad \therefore 3x = 9y^2 + 6y, \quad \therefore x = 3y^2 + 2y.$$

Substituting this into the relation $5x + 1 = v^2$ we get:

$$v^2 = 5\left(3y^2 + 2y\right) + 1 = 15y^2 + 10y + 1.$$

Multiplication by 15 on both sides yields:

$$15v^2 = 225y^2 + 150y + 15 = (15y + 5)^2 - 10.$$

Let $z = 15y + 5$. Then we have $z^2 = 15v^2 + 10$, and we get a Brahmagupta equation once again. Here are its solutions and the corresponding x-values:

(z, v)	$(5, 1)$	$(35, 9)$	$(275, 71)$	$(2165, 559)$	$(17045, 4401)$	...
x	0	16	1008	62496	3873760	...

If we list the z-values along we get the sequence

$$1,\ 9,\ 71,\ 559,\ 4401,\ \dots,$$

with a pleasing underlying pattern which allows us to guess the succeeding terms.

Remark. What happens if at the initial stage Bhāskarā supposes that u is of the form $3y+2$? Then he would get:

$$3x+1=(3y+2)^2, \quad \therefore 3x=9y^2+12y+3, \quad \therefore x=3y^2+4y+1.$$

Substitute this into the relation $5x+1=v^2$:

$$v^2=5\left(3y^2+4y+1\right)+1=15y^2+20y+6.$$

Multiplication by 15 on both sides yields:

$$15v^2=225y^2+300y+90=(15y+10)^2-10.$$

Let $z=15y+10$. Then we have $z^2=15v^2+10$, and we get the very same Brahmagupta equation as earlier! So they yield the same x-values.

We remark here that in general, Bhāskarā seems to be interested in listing an infinity of solutions, rather than a single isolated solution.

Bhāskarā II and the seeds of the Calculus

Most remarkably, Bhāskarā seems to have been practically on the doorstep of calculus — several centuries ahead of Newton and Leibnitz! The evidence for this claim is the following.

- He determines the surface area and volume of a sphere by 'calculus-like' methods which are nearly the same as those used by Archimedes. Thus, area is determined by dividing the surface into a large number of small parts, and likewise for volume.
- In constructing his table of sines, Bhāskarā states (as always, in verse form) that if x and x' are close to one another, then

  $$\sin x'-\sin x=(x'-x)\cos x.$$

  This is equivalent to the statement that "the derivative of the sine function is the cosine function", i.e.,

  $$\frac{d}{dx}(\sin x)=\cos x.$$

  The statement is found in his book on astronomy.
- Lastly, the following statements are found: "Where the motion of the planet is an extremum, there the fruit of its motion is absent" (that is, the motion is stationary), and: "At the commencement and end of retrograde motion, the apparent motion of the planet vanishes."

  These statements together with the examples given earlier strongly suggest the presence of the seeds of the differential and integral calculus in Bhāskarā's work.

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The **COMMUNITY MATHEMATICS CENTRE** (CoMaC) is an outreach arm of Rishi Valley Education Centre (AP) and Sahyadri School (KFI). It holds workshops in the teaching of mathematics and undertakes preparation of teaching materials for State Governments and NGOs. CoMaC may be contacted at shailesh.shirali@gmail.com.

FOLDING A 45°, 60°, 75° TRIANGLE FROM A SQUARE SHEET OF PAPER IN 6 EASY STEPS !

1. Start with a square sheet

2. Fold it in half, make a crease, then unfold.

3. Fold a corner so that it falls on the crease.

4. Fold another corner to the same crease.

5. Fold the remaining portion as shown.

6. This triangle has the stated property!

Tale of a Quadrilateral and a Triangle

in the classroom

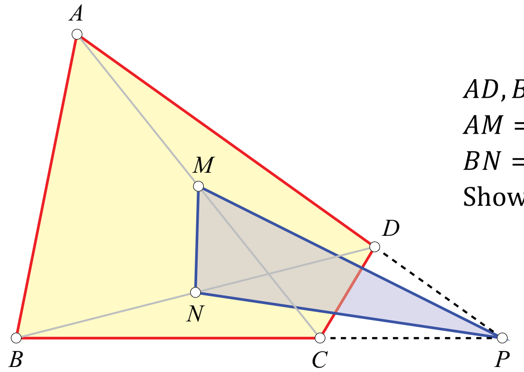
Bharat Karmarkar

This note is devoted to a proof of the following geometrical statement:

> *Let $ABCD$ be a convex quadrilateral in which AD is not parallel to BC. Let AD and BC meet, when extended, at P. Let M and N be the midpoints of diagonals AC and BD, respectively. Then the area of triangle PMN is one-quarter the area of quadrilateral $ABCD$.*

We present the proof in the form of pictures for which we give a light justification in each case. We use the following notation: if X denotes any plane geometric figure, then [X] denotes the area of X. So the square brackets stand for "area of ...".

Proposition

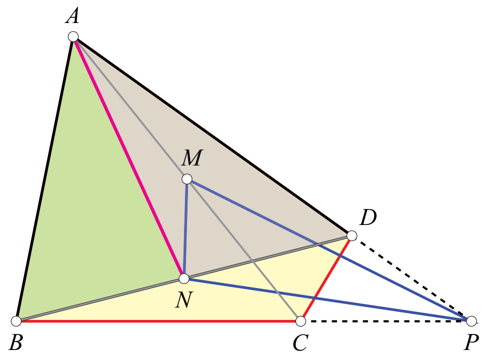


AD, BC meet at P
$AM = MC$
$BN = ND$
Show: $[PMN] = \frac{1}{4}[ABCD]$

Try to find your own proof before reading on!

Proof in Seven Movements

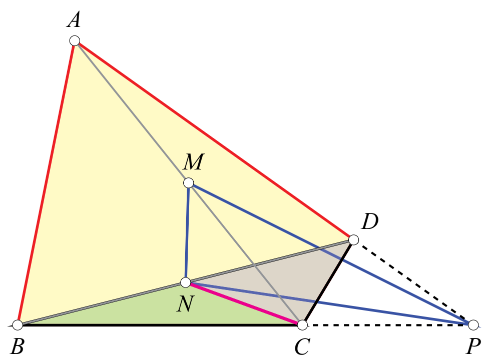
Step 1.



$[ABN] = [AND] = \frac{1}{2}[ABD]$.

Reason: AN is a median of $\triangle ABD$.

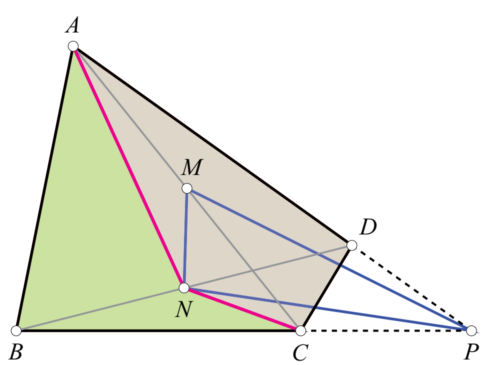
Step 2.



$[CBN] = [CND] = \frac{1}{2}[CBD]$.

Reason: CN is a median of $\triangle CBD$.

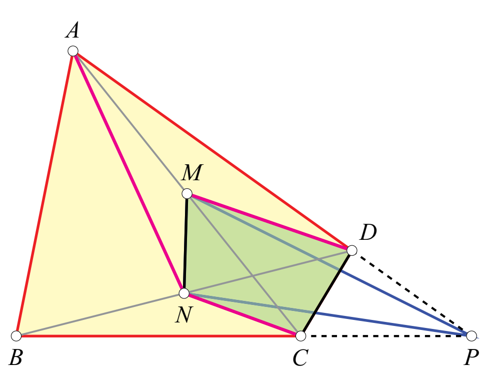
Step 3.



$[ANCD] = \frac{1}{2}[ABCD]$.

Proof: Follows by addition of the equalities in Steps 1 & 2.

Step 4.



$[CMN] = \frac{1}{2}[CAN]$

$[DMC] = \frac{1}{2}[DAC]$

Hence $[MNCD] = \frac{1}{2}[ANCD]$.

But $[ANCD] = \frac{1}{2}[ABCD]$.

Hence $[MNCD] = \frac{1}{4}[ABCD]$.

Step 5.

$$[PNB] = \tfrac{1}{2}[PDB],$$
$$[CNB] = \tfrac{1}{2}[CDB].$$

Now subtract:

$$[PNB] - [CNB] = [PNC],$$
$$[PDB] - [CDB] = [PDC].$$

Hence:

$$[PNC] = \tfrac{1}{2}[PDC].$$

Step 6.

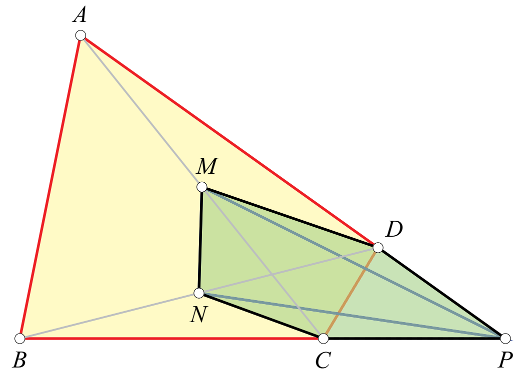
$$[PAM] = \tfrac{1}{2}[PAC],$$
$$[DAM] = \tfrac{1}{2}[DAC].$$

Now subtract:

$$[PAM] - [DAM] = [PDM],$$
$$[PAC] - [DAC] = [PDC].$$

Hence:

$$[PDM] = \tfrac{1}{2}[PDC].$$



Step 7.

Consider the polygon $PDMNC$. We have:

$$\begin{aligned}[PDMNC] &= [MNCD] + [PDC]\\ &= \tfrac{1}{4}[ABCD] + [PDC]. \qquad (1)\end{aligned}$$

We also have:

$$\begin{aligned}[PDMNC] &= [PMN] + [PDM] + [PNC]\\ &= [PMN] + \tfrac{1}{2}[PDC] + \tfrac{1}{2}[PDC]\\ &= [PMN] + [PDC]. \qquad (2)\end{aligned}$$

Comparing equalities (1) and (2), we get:

$$[PMN] = \tfrac{1}{4}[ABCD],$$

as required.

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in the classroom

Quadrilateral and Triangle: A Further Look

C⊗MαC

The following proposition was proved in the preceding pages: *Let $ABCD$ be a convex quadrilateral in which AD is not parallel to BC. Let AD and BC meet, when extended, at P. Let M, N be the midpoints of diagonals AC, BD, respectively. Then* $[PMN] = \frac{1}{4}[ABCD]$. (Here square brackets denote area. See Figure 1.)

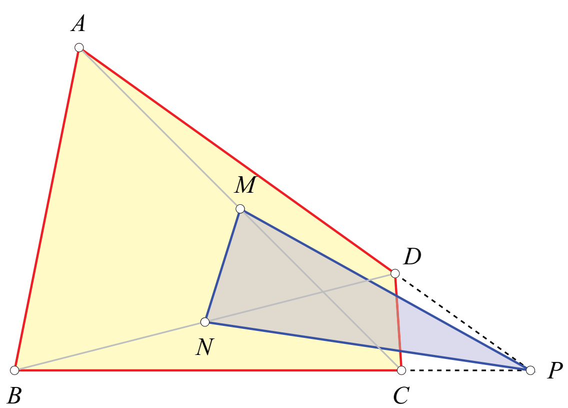

Figure 1.

It is of interest to look at this proposition through the lens given to us by George Pólya: that of tweaking a problem and seeing what we get. A very useful tweak is that of looking at *extreme situations*. In our context we identify the following extreme configurations when the quadrilateral $ABCD$ becomes 'degenerate' in some way:

(1) Quadrilateral $ABCD$ collapses into a triangle because two of its vertices coincide.
(2) Quadrilateral $ABCD$ collapses into a triangle because three of its vertices are collinear.

There are other possibilities, but we will mention them later.

Cases (1) and (2) can be considered as part of a continuum. We imagine that vertex D lies somewhere along segment AC. If D coincides with either A or C, we have case (1), and if D lies in the interior of segment AC, we have case (2).

The first possibility, of D coinciding with A, does not yield anything of interest, as line AD is undefined and hence point P is undefined as well. So we discard this.

If D coincides with C, we get a result which is well known; see Figure 2. For now, point P too coincides with C, which means that M is the midpoint of side AC and N is the midpoint of side BC. The statement that $[PMN] = \frac{1}{4}[ABCD]$ now simply reads: $[CMN] = \frac{1}{4}[CAB]$. This is easily seen to be true via the midpoint theorem.

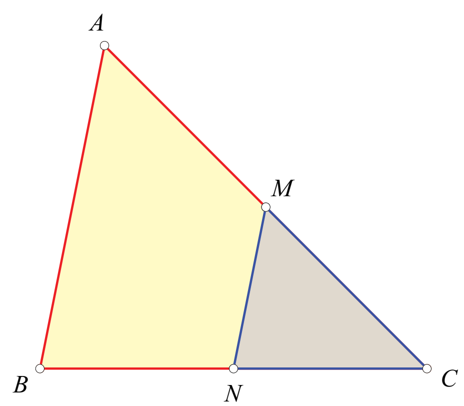

Figure 2.

In this figure, both D and P coincide with C. So M is the midpoint of AC, and N is the midpoint of BC. It is easy to see that $[CMN] = \frac{1}{4}[CAB]$.

It is always reassuring to find that a result being explored yields something well known as a special case. It means that the result under study cannot be completely wrong!

Of greater interest is the case when D lies in the interior of segment AC (Figure 3). Once again, P coincides with C. Constructing points M and N as earlier (M is the midpoint of AC and N is the midpoint of BD), the claim is: $[CMN] = \frac{1}{4}[CAB]$.

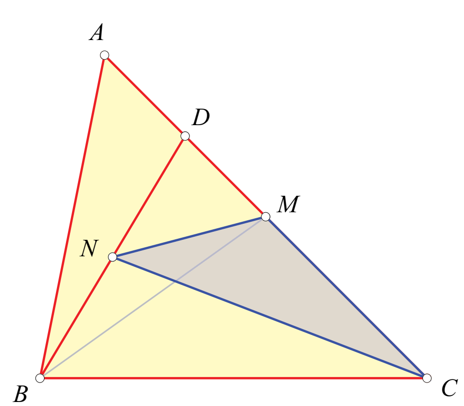

Figure 3.

In this figure, D is any point on AC; N is the midpoint of BD; M is the midpoint of AC. The claim is now: $[CMN] = \frac{1}{4}[CAB]$.

The claim is easy to prove:

$$
\begin{aligned}
[CMN] &= [CDN] - [MDN] \\
&= \frac{1}{2}[CDB] - \frac{1}{2}[MDB] \\
&= \frac{1}{2}[CMB] \ = \ \frac{1}{4}[CAB].
\end{aligned}
$$

Remark. There are two other ways in which the configuration under study can become special or degenerate:

(3) Quadrilateral $ABCD$ becomes a trapezium in which the sides AD and BC are parallel to each other (so they do not meet when extended).
(4) Quadrilateral $ABCD$ becomes a parallelogram.

But these cases are clearly rather troublesome. In case (3), the extended sides AD and BC fail to meet each other at all, so the point P does not exist. Or one may say that "P lies at an infinite distance along line BC (or line AD)". In case (4), the points M, N coincide; at the same time P lies at an infinite distance along line BC. (So (4) is in a way even "worse" than (3).)

A vector proof of the main proposition

We conclude with a vector proof of the proposition quoted at the start. Let position vectors of the various points in the diagram be with reference to P as the origin, and let the position vectors be denoted by lower case letters in boldface (Figure 4). Then:

$$
\begin{aligned}
2[PMN] &= \mathbf{m} \times \mathbf{n} \\
&= \frac{1}{2}(\mathbf{a}+\mathbf{c}) \times \frac{1}{2}(\mathbf{b}+\mathbf{d}), \\
\therefore\ 8[PMN] &= (\mathbf{a}+\mathbf{c}) \times (\mathbf{b}+\mathbf{d}) \\
&= \mathbf{a}\times\mathbf{b} + \mathbf{a}\times\mathbf{d} + \mathbf{c}\times\mathbf{b} + \mathbf{c}\times\mathbf{d} \\
&= \mathbf{a}\times\mathbf{b} + \mathbf{c}\times\mathbf{d}, \qquad \text{since } \{A,D,P\} \text{ and } \{B,C,P\} \text{ are collinear.} \\
\therefore\ 4[PMN] &= \frac{1}{2}(\mathbf{a}\times\mathbf{b}) - \frac{1}{2}(\mathbf{d}\times\mathbf{c}) \\
&= [PAB] - [PDC] \\
&= [ABCD].
\end{aligned}
$$

The smooth elegance of this proof is a testimony to the power of the vector approach.

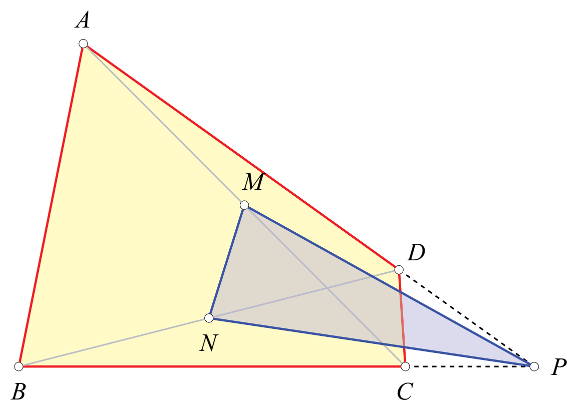

Figure 4.

in the classroom

Completing the Square...

A powerful technique, not a feared enemy!

Shashidhar Jagadeeshan

1. Introduction

The technique of completing the square is often introduced to high school students in the context of deriving the quadratic formula. Many students find the technique rather irritating, tedious and hard to remember and use appropriately. On the other hand, as a teacher I have grown to admire this technique for the range of problems it helps us solve and understand! I would like to share my perspective in the hope that it may save the idea of completing the square from being a dry technique to being a powerful tool with a rich history that solves many problems.

Completing the square helps us understand entirely the shapes of quadratic equations, locate the vertex of a parabola, derive the quadratic formula, find the range of a second-degree equation, find the inverse of a quadratic (with appropriate domain) and locate the centre and radius of a circle. What more can one ask for from a humble technique? Plenty more it seems! In the article "A Tale of Two Formulas" (*At Right Angles*, Vol. 2, No. 3, November 2013) many other applications are discussed.

This article repeats some of the ideas covered in our November 2013 issue, but since completing the square is such a rich idea, we have decided to explore it in greater detail in this issue.

2. The technique

Let us begin by looking at the technique. In its simplest form, the idea is to change an expression like $x^2 + bx + c$ into an expression of the form $(x - \alpha)^2 + \beta$. That is, a squared expression plus a pure number.

To do so we use a geometric illustration (Figure 1) similar to the one in the November 2013 issue, assuming for the moment $c = 0$. Here clearly $\alpha = -\frac{b}{2}$ and $\beta = \left(\frac{b}{2}\right)^2$. What is wonderful about this illustration is that it clearly explains why we often tell students: "to complete the square, add and subtract the square of half the coefficient of x"! Moreover, it now becomes clear why the technique is referred to as 'completing the square'.

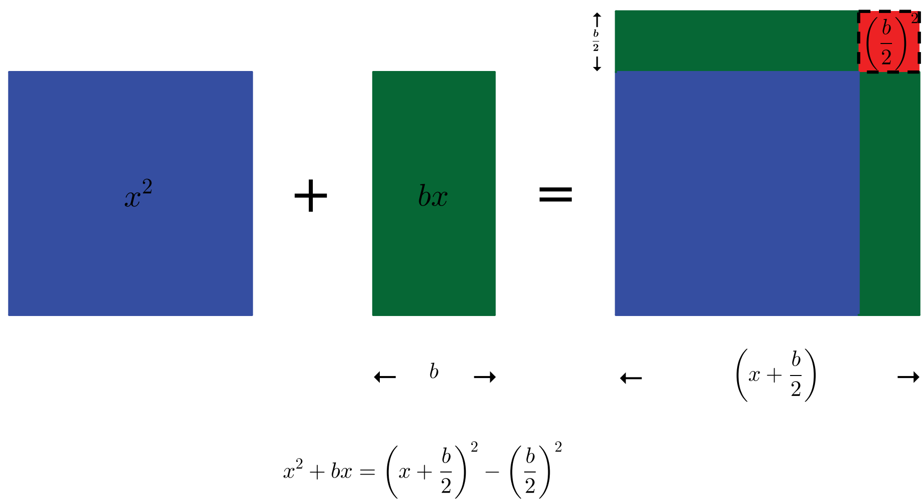

Figure 1.

If $c \neq 0$ then we get

$$x^2 + bx + c = \left(x + \frac{b}{2}\right)^2 + \frac{4c - b^2}{4} \tag{2.1}$$

Notice so far we have assumed that the coefficient of x^2 is 1. What if it is not? Perhaps as most of you would have guessed by now, that is not a big deal, we can always factor out the coefficient of x^2 and make it 1. So, if we have $ax^2 + bx + c$, we can rewrite it as $a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right)$ and all we need to do is replace the coefficient of x (that is, our old b) in equation 2.1 by $\frac{b}{a}$ and the constant term (our old c) by $\frac{c}{a}$ to get

$$ax^2 + bx + c = a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right) = a\left[\left(x + \frac{b}{2a}\right)^2 + \frac{4\left(\frac{c}{a}\right) - \left(\frac{b}{a}\right)^2}{4}\right].$$

Simplifying we get

$$ax^2 + bx + c = a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a} \tag{2.2}$$

So once again we can write $ax^2 + bx + c$ as $a(x - \alpha)^2 + \beta$, where $\alpha = -\frac{b}{2}$ and $\beta = \frac{4ac - b^2}{4a}$.

We will now see how this amazing idea of representing a second degree equation in the above manner yields many applications.

3. All quadratics have the shape of a parabola

In this section we will show by the means of applying many transformations that the graphs of all quadratic equations of the form $y = ax^2 + bx + c$ have the same shape as that of $y = x^2$.

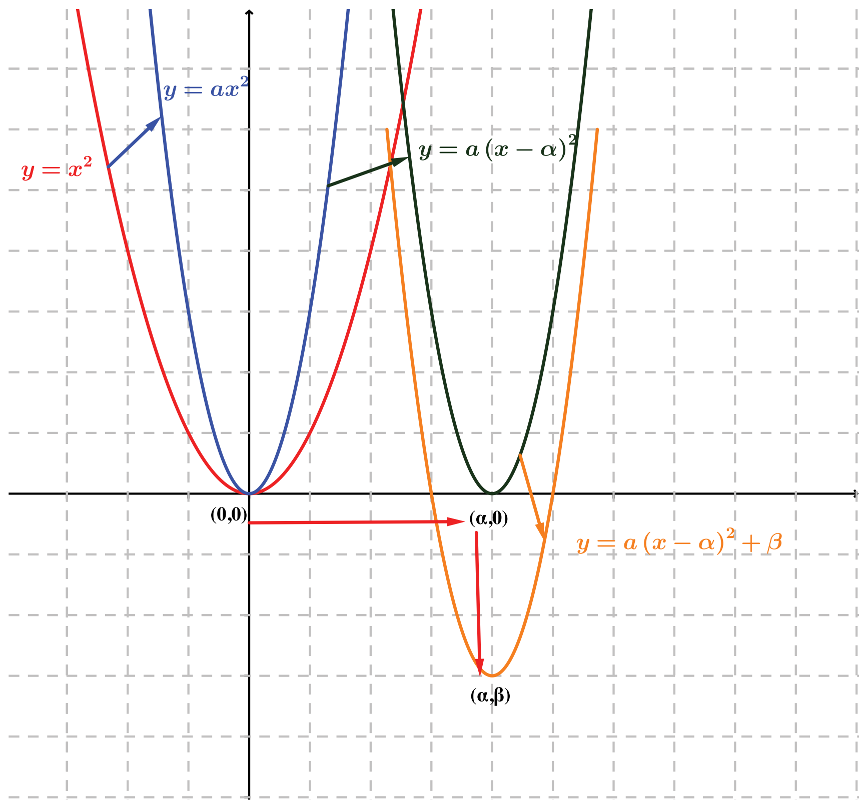

Figure 2.

The red graph (see Figure 2) is that of the function $y = x^2$, when we change the coefficient of x^2 from 1 to a, then the graph becomes narrower or broader depending on whether a is bigger than 1 or smaller than 1. Moreover, if a is positive then the graph is concave up (shaped like ∪) and if a is negative then the graph is concave down (shaped like ∩). The blue graph is that of $y = ax^2$. Notice the vertex of both the curves $y = x^2$ and $y = ax^2$ is $(0, 0)$. Now if we shift $y = ax^2$ to either the left or the right we get the green graph $y = a(x - \alpha)^2$. In this case the vertex has moved from $(0, 0)$ to $(\alpha, 0)$. If we now move the graph up and down then we get the orange graph whose equation is of the form $y = a(x - \alpha)^2 + \beta$. The vertex has now moved to (α, β). Readers familiar with computer software like GeoGebra can use sliders to change the values of a, α and β to see how the graphs move around. Can you explain why when a graph $y = f(x)$ is moved to the left or the right by α it becomes $y = f(x - \alpha)$, and when it is moved up or down it becomes $y = f(x) + \beta$?

Basically, what we have managed to do is to show that the graph of any function of the form $y = a(x - \alpha)^2 + \beta$ has the same shape as that of $y = x^2$. But in Section 2, using the technique of completing the square we saw that all quadratic expressions of the form $ax^2 + bx + c$ can be rewritten in the form $a(x - \alpha)^2 + \beta$! Which means that the graphs of all quadratic equations have the same shape as that of $y = x^2$, in other words that of a parabola.

What is very satisfying here is that we completely understand the graphs of all quadratic equations, without the use of calculus.

4. The quadratic formula for free!

Before we go on to other applications, let us quickly derive the quadratic formula! So if $ax^2 + bx + c = 0$ from equation 2.2 we have $a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a} = 0$. Rearranging we get $\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$ and from there we get

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

5. The vertex of a parabola and other applications

Let us come back to our equation $ax^2 + bx + c = a\left(x - \alpha\right)^2 + \beta$, where $\alpha = -\frac{b}{2}$ and $\beta = \frac{4ac - b^2}{4a}$. It is clear from Figure 2 that the vertex is (α, β). So if we are just given the function $f(x) = ax^2 + bx + c$, can we find the minimum or maximum value of $f(x)$? Obviously, before we have learned calculus!

Well! All we need to do is examine the expression $a\left(x - \alpha\right)^2 + \beta$. Let us assume first that $a > 0$. Then it is clear that no matter what value we plug in for x, since $a\left(x - \alpha\right)^2 > 0$, $f(x) > \beta$. Therefore the *minimum* value for $f(x)$ is β and this least value occurs when $x = \alpha$. Since we know that the shape of a quadratic is a parabola, the vertex then has to be (α, β). If we now assume $a < 0$, one can easily see that the *maximum* value of $f(x) = \beta$ and occurs once again when $x = \alpha$.

For example, consider the quadratic function $f(x) = -2x^2 - 4x + 1$. You can do the math and find that we can rewrite it as $f(x) = -2\left(x + 1\right)^2 + 3$. Therefore the vertex of the parabola is $(-1, 3)$ and the function has a maximum value of 3 at $x = -1$.

By completing the square it is now easy to see that the range of the above function is all real numbers less than or equal to 3. And if we are asked to find the inverse of $f(x) = -2x^2 - 4x + 1$, when $x \leq -1$ (making the function one-to-one), we can rearrange $y - 2\left(x - 1\right)^2 + 3$ to get $f^{-1}(x) = -1 - \sqrt{\frac{3-x}{2}}$. All this becomes very clear when we graph the functions concerned (see Figure 3).

One last application before we end this article with a historical note on completing the square. We all know that an equation in two variables like $x^2 + 2x + y^2 - 4y = 20$ is the equation of a circle. But, how do we find its centre and radius? Of course, by completing the square! I leave it to the reader to show that the circle given above has radius 5 and centre $(-1, 2)$.

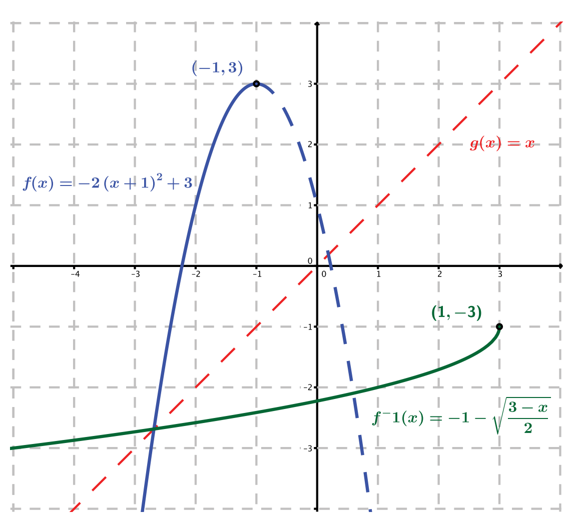

Figure 3.

6. Historical note

It appears that almost all ancient cultures, the Egyptian, Babylonian, Greek, Chinese and Indian, seem to have been aware of quadratic equations. Many of them obtained their solutions through the technique of completing the square. In this article I will touch upon some highlights that I found interesting.

We begin by describing Euclid's (300 BCE) method of completing the square (*The Elements*, Proposition 6, Book II). He solved the quadratic equation $x^2 + bx = c$, by completing the square as in Figure 4.

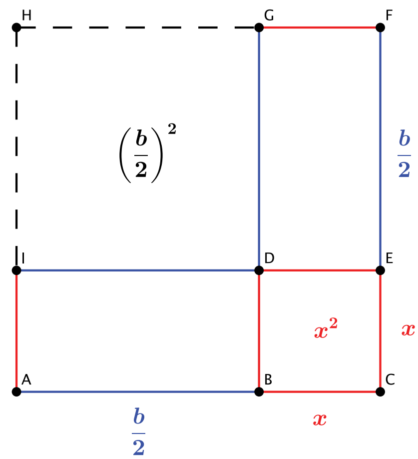

Figure 4.

The figure shows that $x^2 + bx + (\frac{b}{2})^2 = \left(x + \frac{b}{2}\right)^2$. Hence $c + \left(\frac{b}{2}\right)^2 = \left(x + \frac{b}{2}\right)^2$, and from here we can find x. Exactly the same method can be found in a Mesopotamian tablet (BM13901, between 1800 BCE to 2000 BCE) with $b = \frac{3}{4}$ and $c = 1$. Euclid of course had generalized the construction and had provided a proof. Euclid also had an elaborate geometric construction for the solution of certain quadratic equations, but we will not get into them here.

We now move to the Indian contribution. Various books mention that quadratic equations show up in the *Sulba-sutras* and in the *Bakshali Manuscript*, but we will mention the work of Brahmagupta and Sridhara.

Brahmagupta (7th century CE) is credited with having derived the quadratic formula, perhaps using the technique of completing the square, and he may well have been the first mathematician to recognize that quadratic equations had two roots, and he permitted negative roots.

Sridhara (8th or 9th century CE) came up with the following ingenious trick, often referred to as 'Sridhara's rule'. This method can be used to derive the quadratic formula 'from scratch' without the use of fractions. His trick was to first multiply $ax^2 + bx + c = 0$ throughout by $4a$, yielding $4a^2x^2 + 4abx + 4ac = 0$, then adding b^2 to both sides, giving us $(2ax)^2 + 4abx + b^2 = b^2 - 4ac$. This in turn gives

$$(2ax + b)^2 = b^2 - 4ac$$

and from here the quadratic formula is a cinch! In fact, for the fun of it, why don't you illustrate this rule geometrically?

In the November 2013 issue al-Khwarizmi's (9th century CE) geometric solution of the equation $x^2 + 10x = 39$ was given, in fact he had solutions to six different types of quadratic equations (he did not permit negative solutions). I am sure many of you are aware of this, but it bears repeating, that the words 'Algebra' and 'Algorithm' come from al-Khwarizmi's name and his famous book *Hisab al-jabr w'al-muqabala*. Another Arab mathematician Thabit ibn Qurra (9th century CE) using Euclid's theorems showed how to solve the general quadratic equation geometrically.

The quadratic formula as it is currently used was published by the French mathematician Rene Descartes in *La Geometrie* in 1637.

We leave the readers with a wonderful problem from Bhaskaracharya's *Lilavati* (source: *The Crest of the Peacock* by George Gheverghese Joseph, Example 9.2, page 274) .

From a swarm of bees, a number equal to the square root of half the total number of bees flew out to the lotus flowers. Soon after $\frac{8}{9}$ *of the total swarm went to the same place. A male bee enticed by the fragrance of the lotus flew into it. But when it was inside the night fell, the lotus closed and the bee was caught inside. To its buzz, its consort responded anxiously from outside. O my beloved! How many bees are there?*

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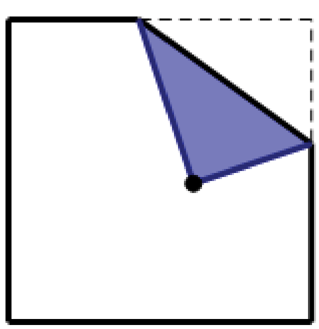
Folding and Mapping Turned-Up Folds (TUFs)

in the classroom

In this issue we take up another 'origamics' exploration by Dr. Kazuo Haga from the chapter INTRASQUARES AND EXTRASQUARES *of his book.*

Shiv Gaur

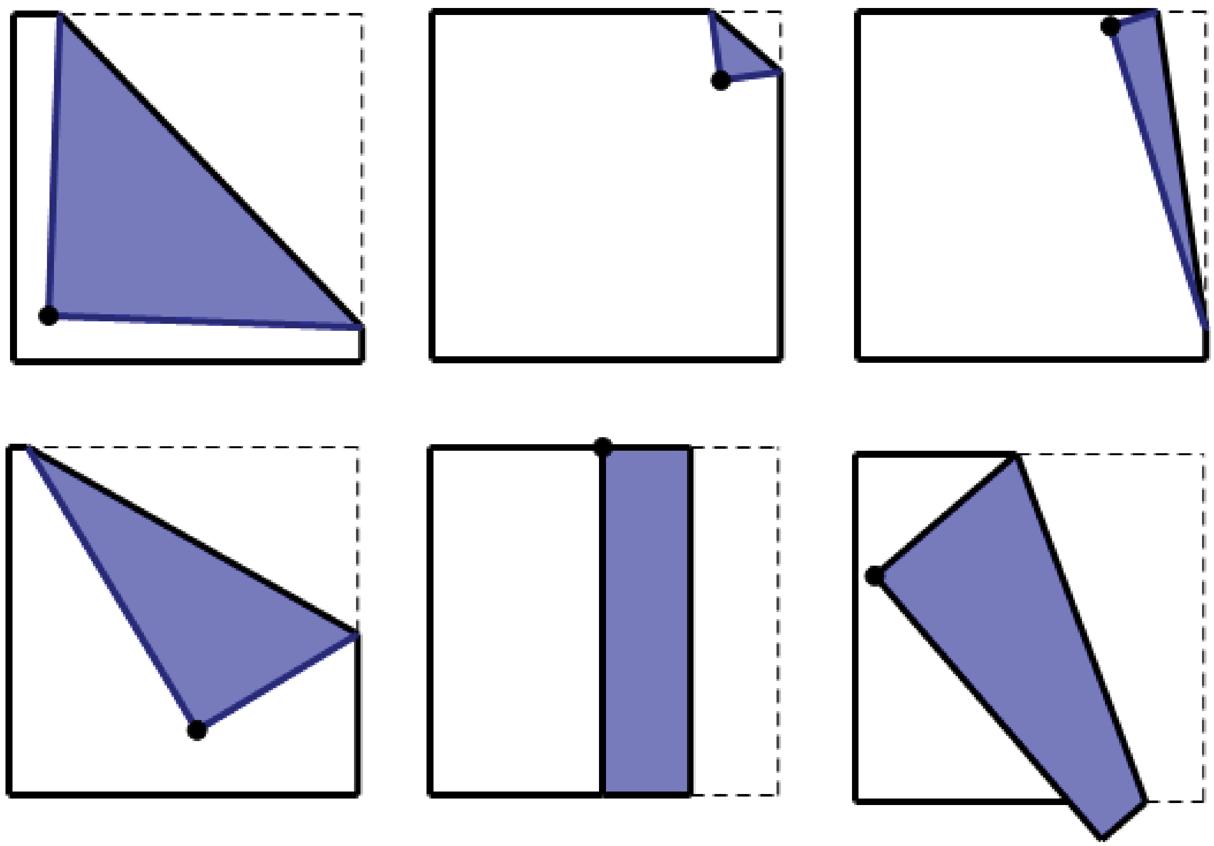
The Task: Take a square piece of paper (colour side down) and choose the upper right corner as your reference point. Pick a point at random anywhere on the paper, and fold the corner to that point. This creates a flap, and we will call it the Turned-Up Fold ('TUF').



Experiment with many TUFs. How many sides does your TUF have? Three? Four? Five? In other words, what types of polygons do you get?

Try to find an answer to the question, "How can we tell how many sides a TUF will have?"

Here are some possible TUFs:



Some possible conjectures relating to the shape of the TUF

Student A: Open the folded paper and examine the line of the fold. If the line connects two adjacent sides of the original origami square, then the flap is a triangle; if the line connects two opposite sides, then the flap is a quadrilateral.

Student B: If during folding only one vertex of the original square moves, then the resulting flap is a triangle; if two vertices move, then the flap is a quadrilateral.

Student C: If the colored portion is contained wholly in the origami square, then it is a triangle; if a part of it is outside the square then the flap is a quadrilateral.

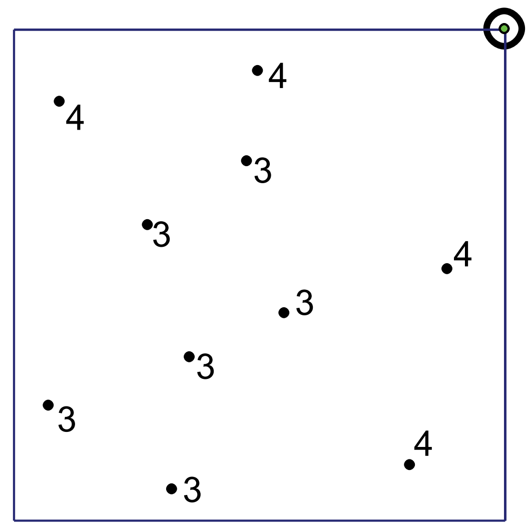
Student D: The shape of the flap depends on the position of the moved vertex on the origami paper.

Are these conjectures valid? Please do some explorations and make your own conjectures!

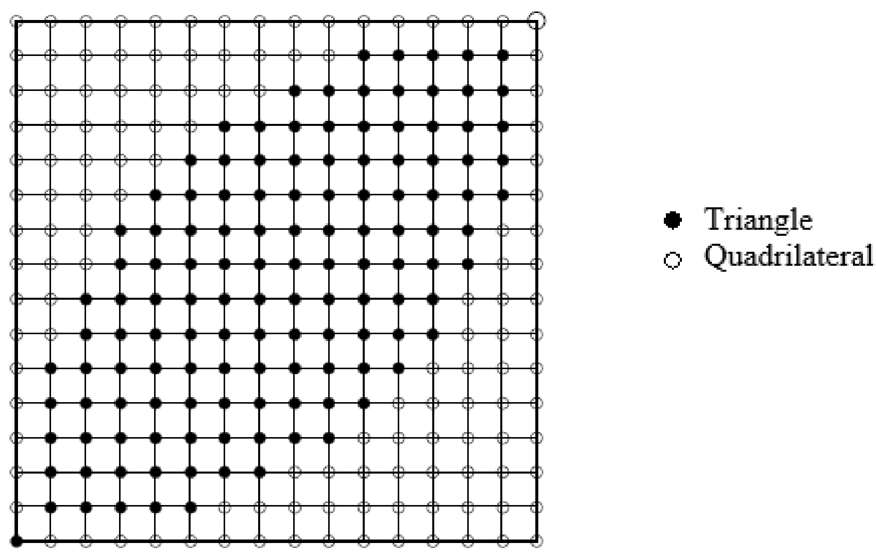
Making a Map

We continue looking for conditions that lead to the formation of a triangular or a quadrilateral flap. Use a new sheet of square paper.

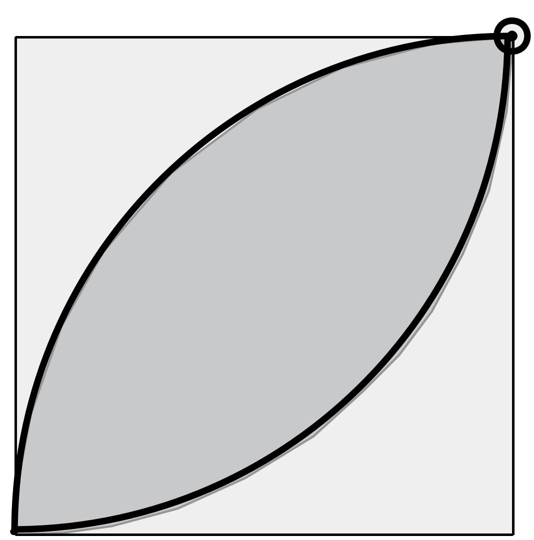
Mark at random ten points on the sheet. Select the upper right corner of the sheet to be the reference vertex. Carefully move it to a marked point and determine the shape of the flap formed. Write "3" or "4" beside the point or color it, to indicate the number of the vertices of the polygonal flap.



Next we get more organized and make a grid and the distribution area gradually appears.



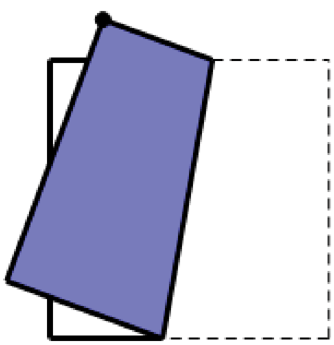
The eye-shaped region inside the square is the “triangle region” whereas the outsides form the “quadrilaterals region”.

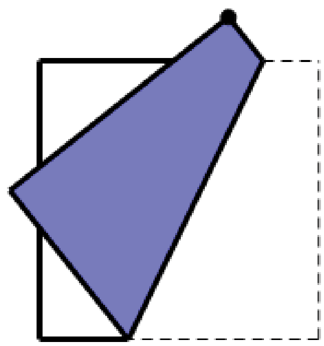


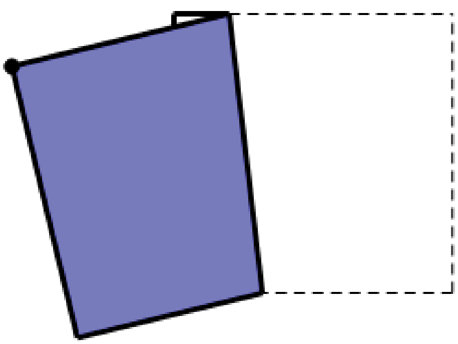
The boundary between the Triangle and Quadrilateral regions is the path traced by the reference vertex as it moves, with one end of the line of the fold fixed at one of the vertices adjacent to the reference vertex. It would be a nice challenge to explain this curious shape.

Extended Task: What if we allow the point to be outside the square? Then what are the possibilities? As an exercise try extending the above map to include all possible shapes which emerge when the reference vertex moves outside the square.

Here a few possible TUFs when the vertex moves out of the square:







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Solution to the 'Origamics' Problem

in the classroom

Shiv Gaur
Swati Sircar
Shailesh Shirali

In this note, we offer an explanation to the observations made in the 'Origamics' article (November 2013 issue of *At Right Angles*). The following observations had been made: (a) The points of intersection of the X-creases fall on the vertical midline of the square. (b) The points of intersection of the X-creases vary along a short distance from below the centre of the square.
(c) The three lines connecting the point of intersection to the starting point on the edge and the two lower vertices are equal in length. Of these, (a) and (c) are easy to explain, while (b) offers greater challenge.

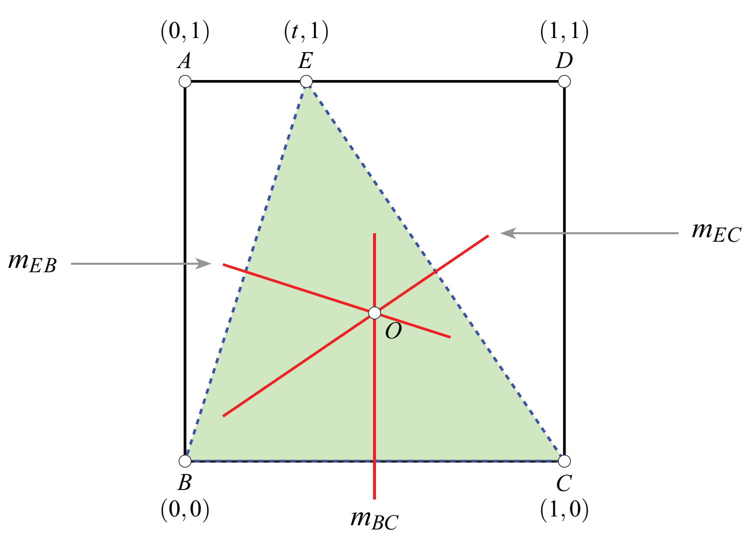


Figure 1.

In Figure 1, $ABCD$ is the square sheet, and E is the point on the upper side of the square which is folded successively to B and C. The fold lines (creases) are shown in red color. Segments EB and EC do not appear on the original sheet but we have drawn them (in dashed blue colour), this creating $\triangle EBC$. It should be clear that the X-creases are perpendicular bisectors of the sides EB and EC of $\triangle EBC$. These two lines necessarily meet on the perpendicular bisector of the third side BC of the triangle. This explains observations (a) and (c): why the X-lines meet on the midline of the square, and also why the point of concurrence of the lines is equidistant from E, B, C, for this point is the circumcentre of $\triangle EBC$.

To explain (b), we assign coordinates: $B = (0,0)$, $C = (1,0)$, $D = (1,1)$, $A = (0,1)$, $E = (t,1)$. Here $0 \le t \le 1$. The slope of EB is $\frac{1}{t}$, hence the slope of the perpendicular bisector of EB is $-t$. The coordinates of the midpoint of EB are $\left(\frac{t}{2}, \frac{1}{2}\right)$. Therefore the equation of the perpendicular bisector of EB (line m_{EB} in Figure 1) is

$$y - \frac{1}{2} = -t\left(x - \frac{t}{2}\right).$$

The equation of the perpendicular bisector of BC (line m_{BC} in Figure 1) is simply $x = \frac{1}{2}$. Let m_{EB} and m_{BC} meet at $O = (u, v)$; this point is, of course, the circumcentre of $\triangle EBC$. Since $u = \frac{1}{2}$, we have:

$$v = \frac{1}{2} - t\left(\frac{1}{2} - \frac{t}{2}\right) = \frac{1}{2} - \frac{t(1-t)}{2}.$$

Now $0 \le t \le 1$. The maximum and minimum values taken by $t(1-t)$ over $0 \le t \le 1$ are $\frac{1}{4}$ (taken when $t = \frac{1}{2}$) and 0 (taken when $t = 0$ and $t = 1$), respectively. To see why the maximum value is $\frac{1}{4}$, observe that

$$t(1-t) = t - t^2 = \frac{1}{4} - \left(t - \frac{1}{2}\right)^2 \le \frac{1}{4},$$

with equality just when $t = \frac{1}{2}$.

Hence v lies between $\frac{1}{2} - \frac{1}{2} \cdot \frac{1}{4}$ and $\frac{1}{2}$, i.e.,

$$\frac{3}{8} \le v \le \frac{1}{2}.$$

Thus O travels on a line segment with endpoints $\left(\frac{1}{2}, \frac{3}{8}\right)$ and $\left(\frac{1}{2}, \frac{1}{2}\right)$. This short segment is the locus of O.

Remark. The above derivation shows that the expression $t(1-t)$ is symmetric about $t = \frac{1}{2}$. Another way of seeing this is to note that the replacement $t \mapsto 1 - t$ leaves the expression $t(1-t)$ unchanged (the two factors simply swap places). The geometric expression of this symmetry is the following: if we reflect the entire configuration in the midline of the square (the line m_{BC}), triangle EBC is mapped to a triangle congruent to itself, so its circumcentre lies at the same height above BC as earlier.

Two more loci. It is of interest to inquire into two more loci: those traced out respectively by I, the incentre of $\triangle EBC$, and by H, the orthocentre

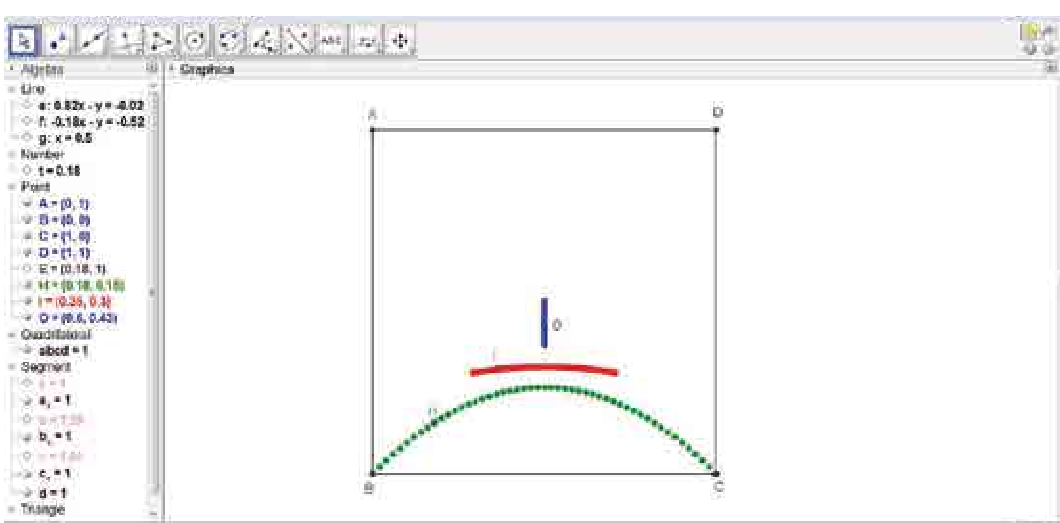


Figure 2. Three loci

of $\triangle EBC$. The latter locus is easily found. When $E = (t, 1)$, the equations of the altitudes through E and C are, respectively:

$$x = t, \quad y = -t(x-1).$$

Solving these, we get $H = \left(t, t - t^2\right)$. So H travels on a parabolic arc with end-points B and C, with its highest point on the midline of the square, at height $\frac{1}{4}$.

The locus of I is a much less familiar curve. Using the vector formula for the coordinates of the incentre in terms of the coordinates of the vertices, we find that when $E = (t, 1)$, the coordinates of I are $(x(t), y(t))$, where

$$x(t) = \frac{t + \sqrt{1+t^2}}{1 + \sqrt{1+t^2} + \sqrt{1+(1-t)^2}},$$

$$y(t) = \frac{1}{1 + \sqrt{1+t^2} + \sqrt{1+(1-t)^2}}.$$

The GeoGebra screen shot in Figure 2 shows all the three loci.

A suggested extension. This activity can be done using a rectangular sheet of paper as well. Try doing it with a rectangular sheet with point E on the shorter edge and then on the longer edge. Also try it with a long thin rectangle, e.g., an A4 sheet halved along its shorter edge. Check whether you get configurations with the same features as described above.

An interesting question: Find the dimensions which the rectangle must have for the point O to lie outside the paper. (In this case the folds will not meet on the sheet at all.) Modify the algebraic calculations (done above) for the case of a rectangle with sides a and b, and check whether the calculation gives back the above formulas for the case $a = b = 1$.

A SLICE OF HISTORY: THE BIRTH OF LOGARITHMS

The following extract is taken from the 'Futility Closet' (http://www.futilitycloset.com/about/) and its page http://www.futilitycloset.com/2014/09/11/likewise/?utm_source=rss&utm_medium=rss&utm_campaign=likewise.

A charming little scene from mathematical history — in 1615 Gresham College geometry professor Henry Briggs rode the 300 miles from London to Edinburgh to meet John Napier, the discoverer of logarithms. A contemporary witnessed their meeting: of logarithms. A contemporary witnessed their meeting :

He brings Mr. Briggs up into My Lord's chamber, where almost one quarter of an hour was spent, each beholding the other with admiration, before one word was spoke: at last Mr. Briggs began. 'My Lord, I have undertaken this long journey purposely to see your person, and to know by what engine of wit or ingenuity you came first to think of this most excellent help unto Astronomy, viz. the Logarithms: but my Lord, being by you found out, I wonder nobody else found it before, when now being known it appears so easy.'

Their friendship was fast but short-lived: The first tables were published in 1614, and Napier died in 1617, perhaps due to overwork. In his last writings he notes that "owing to our bodily weakness we leave the actual computation of the new canon to others skilled in this kind of work, more particularly to that very learned scholar, my dear friend, Henry Briggs, public Professor of Geometry in London."

To find out more about the early history of logarithms, please refer to any of the following:

- http://www.maa.org/publications/periodicals/convergence/logarithms-the-early-history-of-a-familiar-function-introduction
- Shailesh Shirali, A Primer on Logarithms (Universities Press)
- http://www.westcler.org/gh/outtda/pdf_files/History_of_Logarithms.pdf

in the classroom

Portfolio Assessment

Sneha Titus
Sindhu Sreedevi

As I looked back at the year's teaching learning process and the assessment activities done so far, I felt that it was time to collate the evidence of each student's learning in the form of a portfolio. By doing this I could get authentic representations of classroom performance over the assessment period. Portfolios also provide students with the context for assessing their own work and setting meaningful goals for learning. Basically it is a collection of samples of the students' work over the year and an important part of it is the component of self-reflection that explains the rationale for selecting each sample.

Portfolios score high over verbal and written reports because the reader can see the improvement that the child has made in learning each concept through the lens of the selected sample. Not only this, it would give me as well as the students themselves insight into their thoughts as they reviewed their learning. The next step was to compile these processes. This compilation would be a good source of evidence and it led me to design a portfolio assessment. I shared my idea with the children in the next class about the purpose and the process of creating a good portfolio.

The process of collecting evidences that was shared with the children was:

- Choosing important work that demonstrated their road map of learning the concept
- Compiling these documents and creating a file
- Writing their reflections on the journey of learning the concept

Keywords: CCE, mathematics, portfolio, anecdotal, evidence

The students showed interest in the fact that they could demonstrate their learning of the concept of Mensuration. Then we had a conversation on how a portfolio differs from a note book. This question led us to an interesting discussion on the different types of portfolios such as *collaboration portfolio*, *showcase portfolio* and *evaluation portfolio*. In a *collaboration portfolio* teacher and student collaboratively collate the work of each individual, while in a *showcase portfolio* the purpose is to view the journey of learning a particular concept and reflect upon it. An *evaluation portfolio* is for the determination of the student's readiness to move to the next concept. We needed to include the activities that we did in the classroom, formative assessments, the teacher's feedback and the corrections that the child has made based on this feedback. All these will provide information about the path that each individual has taken to master the concept. Including the self-reflection on their journey of learning will provide information to other stakeholders such as teacher, parents and school authorities.

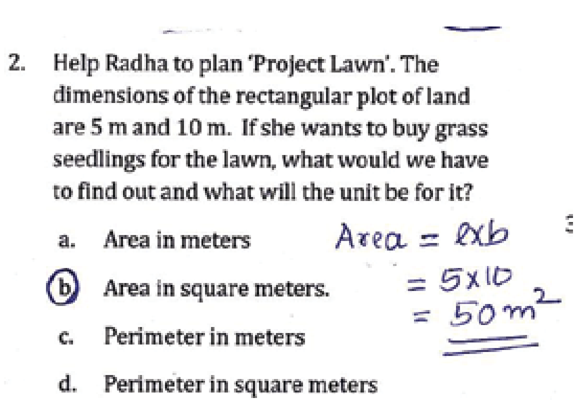
At the end of the discussion we decided to create a collaborative portfolio and arrived at a common list of what the contents of the portfolio would be:

- Entry level assessment paper that we did in the beginning of the chapter
- Worksheets that we did during the derivation of area of the trapezium with my feedback
- Successive drafts of an important work or problem that the child has created
- Records of informal assessments that were made
- Responses for open ended questions
- Same work that repeated over a period of time
- Favorite work
- The best work that was done in the whole process
- The individual project plan and contributions to the project
- Self-reflection
- Teacher's reflection
- Observation record of activities of the individual done by teacher

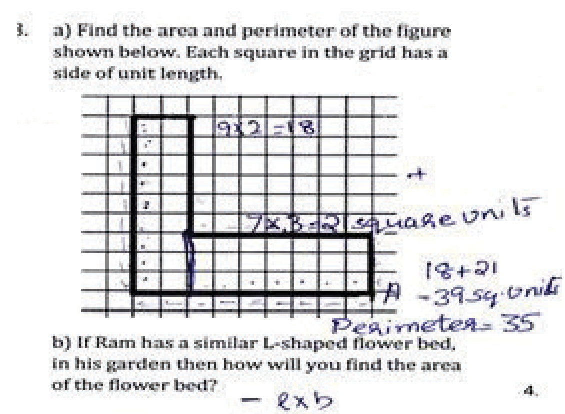
Following are some of the samples which were considered for the portfolio. I could draw many inferences from them about Ranjith's learning.

1. For which of the following given shapes can we find the area and perimeter? Justify your answer

Fig. 1


2. Help Radha to plan 'Project Lawn'. The dimensions of the rectangular plot of land are 5 m and 10 m. If she wants to buy grass seedlings for the lawn, what would we have to find out and what will the unit be for it?

a. Area in meters
b. Area in square meters.
c. Perimeter in meters
d. Perimeter in square meters


3. a) Find the area and perimeter of the figure shown below. Each square in the grid has a side of unit length.

b) If Ram has a similar L-shaped flower bed, in his garden then how will you find the area of the flower bed?

In the entry level test Ranjith has shown some basic idea of area and perimeter but he depended heavily on formulae.

Fig.2

The following samples are chosen from the collection of formative assessments carried out during the lesson on Mensuration.

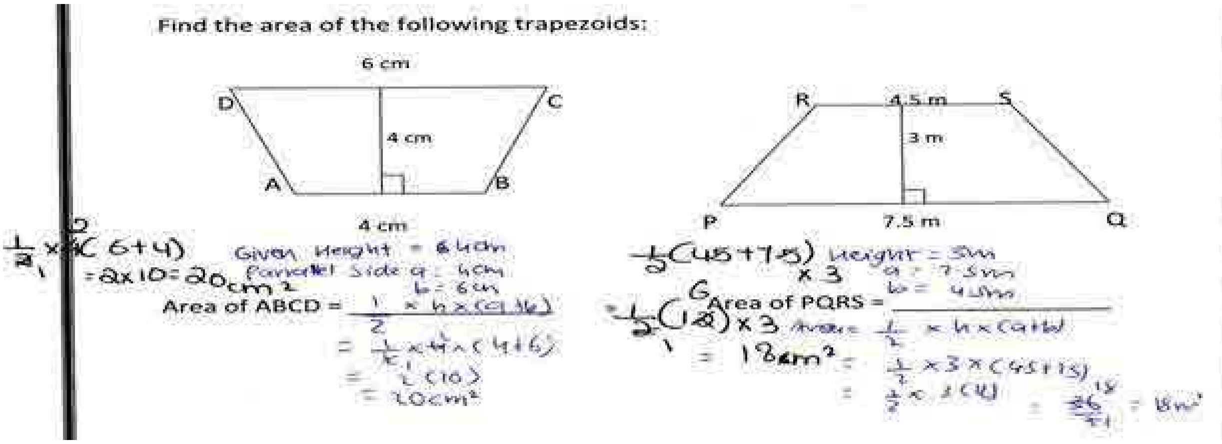
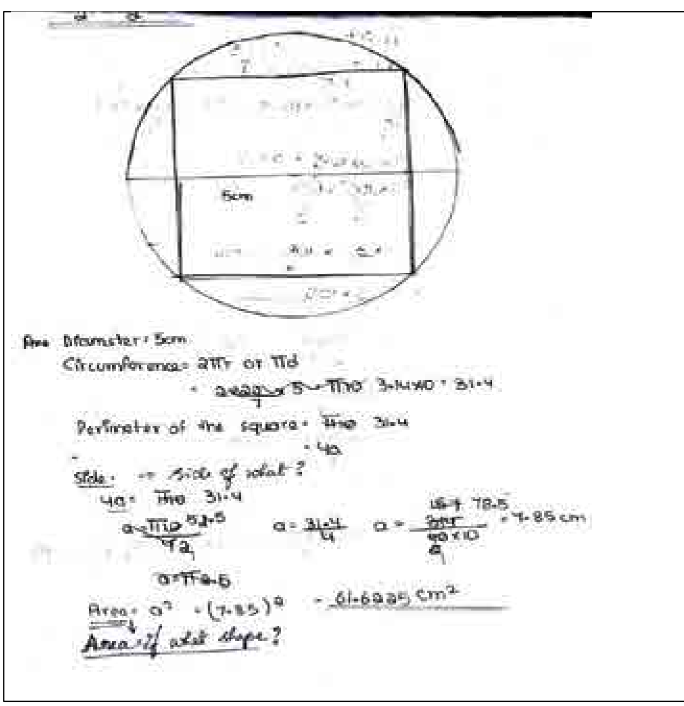


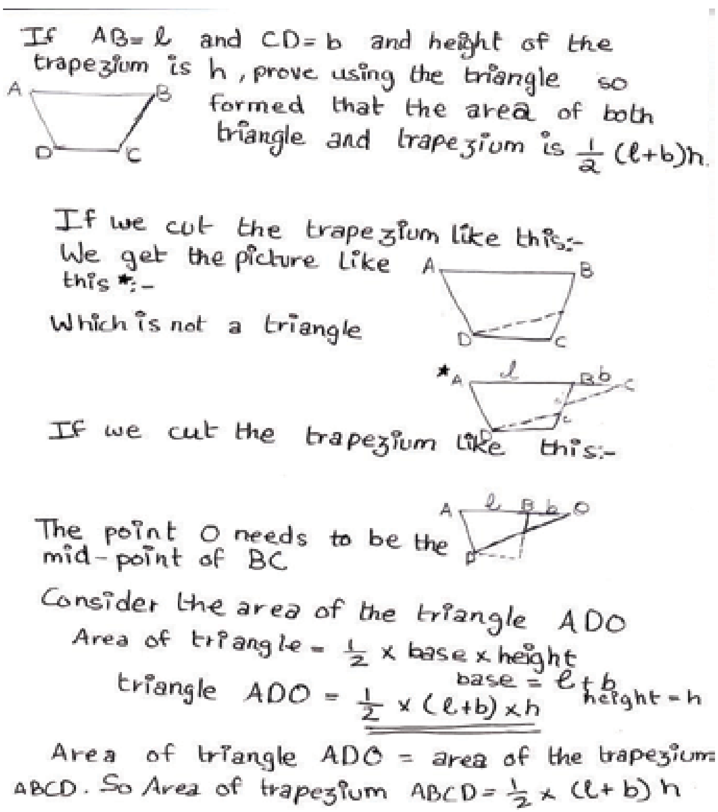
Fig.3

Draw a circle of a diameter 5cm using a compass. What is the area of a square with the same perimeter as the circumferences of the circle?



In the sample in Fig. 3, he was using the formula to find the area of the trapezium. After a series of conversations he tried to solve the problem by considering what is given, what needs to be found out and how to find it out. In the sample shown in Fig. 4 he tried to analyse the situation but he made some mistakes such as considering 5 cm to be the radius in spite of recording it as the diameter. Here also he was obsessed with formulae like circumference = $2\pi r$ or πd.

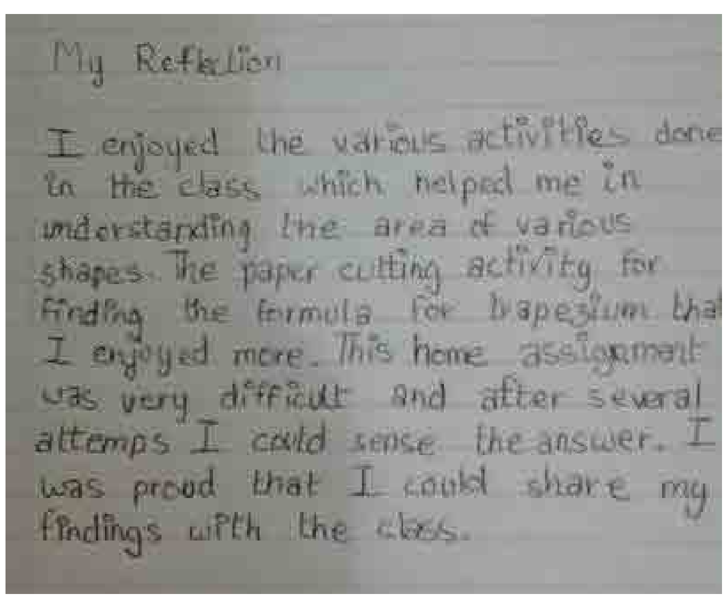
Fig.4



In the home assignment he analysed the problem and tried to give logical reason for each steps.

Fig 5: Attempting to cut a trapezium to form a triangle

He also included his reflection in the portfolio


My Reflection

I enjoyed the various activities done in the class which helped me in understanding the area of various shapes. The paper cutting activity for finding the formula for trapezium tha I enjoyed more. This home assignment was very difficult and after several attemps I could sense the answer. I was proud that I could share my findings with the class.

Fig. 6

My anecdotal records for activities during the teaching learning process helped me in analysis of the child's learning or in creating detailed reports. One of the anecdotal records for Ranjith was this.

Name: Ranjith Date :23/8	
Activity	Deducing the area of a trapezium
Observation	*Ranjith shows interest and logical reasoning for each step in the worksheet. He has back calculated the area of given trapezium from the formula and used it for finding the area of triangle DCB. He could communicate the findings and provide the rationale for the steps in the deduction.*

As I mentioned above these anecdotal records direct me in writing my reflection on his learning.

My reflection

In the entry level test Ranjith showed the basic understanding of area and perimeter of rectangle. After performing various activities I could see the improvement and evidences in analytical thinking. He has progressed far away from using the formula for finding the area to show higher order thinking skills. He also started enjoying the activities and participates in group discussion.

I could understand not only the strengths and weaknesses of Ranjith's learning but also about his efforts. At the end of the whole process some of the children shared that the whole process helped them to understand their strengths and weaknesses and motivated them to work on these. One of the students shared that "the process helped me to compete with myself not with others and I feel that I am responsible for my learning". The students were discussing about creating a similar kind of collection for other topics. I felt very happy to hear this discussion and their feeling towards this assessment. I felt there are so many reasons for this acceptance. It occurs in the child's natural environment and also it invites the child to be reflective about his/her work. Assessment through portfolio is a true learning experience and it gives authentic information about my teaching process. It also made a good platform to increase the interaction with children, with parents and school authorities about the child's progress as it is purely evidence based and focuses on the path the child has traversed to reach to a certain level.

Once I collected the evidences about learning through different assessment strategies, it was time to sum up all the findings. Summative assessment is always considered for evaluation and grading purpose. Thus it is carried out at the end of a lesson or month or term. I consider summative assessment to be for summing up of learning about a concept or topic at a particular point in time before moving on. It also provides scope for improvement so at such points, it is hard for me to distinguish between summative and formative assessment. I consider all types of assessment for both formative and summative – the purpose of assessment is to know about the learning and change the instruction as per the need.

After collecting the detailed evidences through different ways/strategies, I needed to analyse these and provide detailed feedback to the child and parent about the learning. These evidences have been compared with a predefined set of references for each topic/concepts. I had already decided these set of references for each concept which aligned to the goals and objectives of the topic. These frames of reference in each topic are called indicators and they helped me in analyzing the evidences collected. Indicators that I decided for the area of a trapezium are

- Identify the basic types of polygons – triangles and rectangles- after folding or cutting the trapezium.
- Use the mensuration formulae of rectangle and triangle.
- Derive the proposed formula and give justification for the steps taken
- Communicate the logical reasoning for each step of the deduction

In this manner I drew up indicators for each topic and these indicators are linked to the development of critical skills like problem solving, logical reasoning, mathematical communication, estimation, number sense, generalization, data collection etc. that a child need to develop in mathematics. Analysis of the evidences gives references for the development of these skills. Apart from these skills I also get evidences about the child's behavior, attitude and interest towards mathematics. These detailed evidences collected on daily basis help me in writing a detailed report on the child's learning and it benefits me in communicating with the child, parent and school authority in a structured way.

As I mentioned in the earlier entry, a cooperative head teacher and higher authority empowered me in choosing appropriate assessment strategies to collect the evidences about student learning. This also provided a sense of responsibility for the child's learning. The freedom to choose the appropriate assessment strategies to collect the evidences was an important thing in CCE that a teacher like me felt empowered by in the classroom. I am able to know about my students as a teacher rather than as an external observer. By carrying out CCE I could actively assist the learning process. Earlier I could use only written tests for evaluating the children. This was a barrier for me to collect valid evidences about them. CCE also provided an opportunity for in depth discussion with each of my students through these assessment strategies. This equipped me to understand each of my students and help him/her to learn the concepts through the way he/she is comfortable. The continuous aspect of CCE enabled me in strengthening the assessment procedure in terms of its reliability and validity. Through these effective assessments I could devise a self-evaluation strategy among children at the end which made them responsible for their learning.

I also think that by the recording of anecdotal evidence, teachers have a clear idea about his/her students and their learning. Earlier we never used to document the information anywhere. I look at CCE to provide me the platform to think, analyse, draw inferences and document about the child's learning. This enables me to have continuous conversation with students, parents and policy makers about the nature of learning and to avoid subjectivity. My experience over the last year with formative assessment convinced me that the CCE had indeed shown me the way to develop a finer sensitivity to the nuances of my teaching and the student's learning. But it was important not to get overwhelmed by the workings of the system. It was important that CCE works for me, not that I work for CCE.

The CCE column is the product of the Azim Premji University Resource Centre. The team members who worked on this article are Sindhu Sreedevi and Sneha Titus. **SINDHU SREEDEVI** was a Mathematics teacher for 5 years during which she had opportunities to learn the practical aspects of teaching, learning and assessment processes. Along with that she was also coordinating several external assessments in the school. **SNEHA TITUS** is Associate Editor of *At Right Angles* and Assistant Professor and a mathematics resource person at the University Resource Centre.

in the classroom

How To Prove It

This continues the 'Proof' column begun earlier. In this 'episode' we study some results from geometry related to the theme of concurrence.

SHAILESH A SHIRALI

Concurrence of lines. An extremely common theme in plane geometry is that of proving the concurrence of three or more lines. (The dual problem: proving the collinearity of three or more points.) It is of interest to study the different strategies used. We study some well known results in this area and contrast the approaches used to prove them.

Perpendicular bisectors

Perhaps the easiest of all results on concurrence is this: ***The perpendicular bisectors of the three sides of a triangle concur.***

This is best proved using the idea of a locus, namely: ***Given two distinct points* A *and* B*, the locus of points* P *such that* PA = PB *is the perpendicular bisector of segment* AB**; see Figure 1 (a). Here's how the proof uses this locus idea.

Let the perpendicular bisectors of sides AB and AC meet at O; see Figure 1 (b). Join OA, OB, OC. Then $OA = OB$, and also $OA = OC$, hence $OB = OC$. The last equality means that O is equidistant from B and C and hence lies on the perpendicular bisector of BC. Therefore the three perpendicular bisectors meet in a point.

How can we be sure that the perpendicular bisectors of sides AB and AC do meet? That's easy: they meet because they are not parallel to each other, and this is ensured by the fact that they are respectively perpendicular to AB and AC; and AB and AC are

Keywords: *Concurrence, locus, ratio, intersection, area, monotone, monotonic*

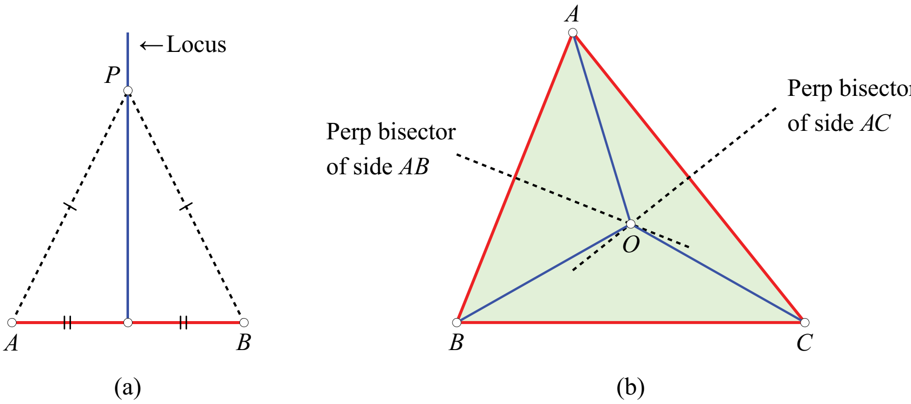


Figure 1. Concurrence of the perpendicular bisectors of the sides of a triangle

certainly not parallel to each other — after all, they meet at A!

Internal angle bisectors

Of the same nature, and proved the same way, is this: ***The internal bisectors of the three angles of a triangle concur.***

To show this, we use a different locus fact which just as basic: ***Given an angle* ABC*, the locus of points* P *equidistant from the arms* BA *and* BC *is the internal bisector of* ∡ABC.** See Figure 2 (a).

Let the internal bisectors of $\measuredangle ABC$ and $\measuredangle ACB$ meet at I. Draw perpendiculars ID, IE, IF from I to sides BC, CA, AB respectively; see Figure 2 (b). Using the locus fact stated above, we see that $ID = IF$ and $ID = IE$. Hence $IE = IF$, which implies that I lies on the internal bisector of $\measuredangle BAC$. So the three internal angle bisectors meet in a point.

How can we be sure that the internal bisectors of $\measuredangle ABC$ and $\measuredangle ACB$ do meet? The lines must meet because they are not parallel to each other, and we can be sure of this because $\measuredangle ABC + \measuredangle ACB < 180°$, which implies that $\measuredangle IBC + \measuredangle ICB < 90°$.

A small tweak in this argument yields a related but slightly less familiar result: ***Given any triangle, the external bisectors of any two of its angles and the internal bisector of the third angle concur***; see Figure 3.

A general remark. The above two proofs have a common theme; namely, to prove that two quantities u and v are equal, show that both of them are equal to a third quantity w. Viewed thus in generality, we see a theme used frequently in mathematics, at all levels.

Medians

The result generally encountered next is:

Theorem. The medians of a triangle concur.

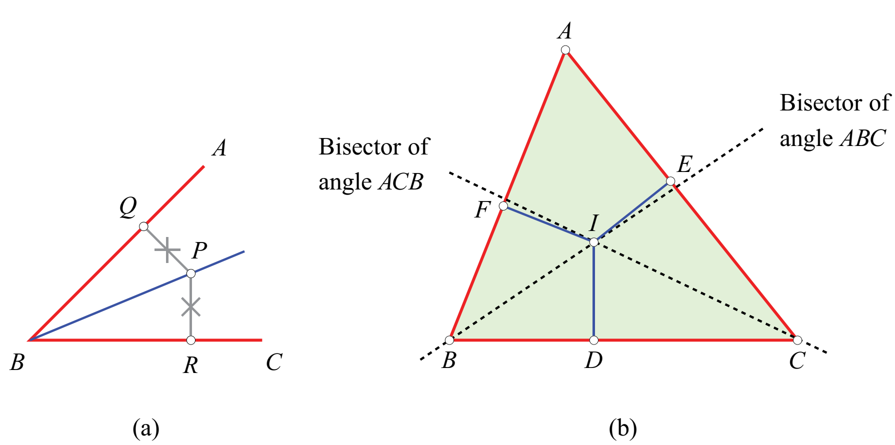


Figure 2. Concurrence of the internal bisectors of the angles of a triangle

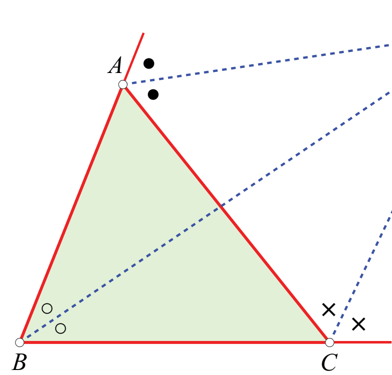


Figure 3. Concurrence of two external angle bisectors and one internal bisector

This turns out to be more challenging to prove, because the median is not so obviously a locus, unlike the perpendicular bisector of a line segment or the bisector of an angle. We offer three proofs of concurrence. The first is based on a well known theorem of elementary geometry: ***The segment joining the midpoints of two sides of a triangle is parallel to and half the third side.*** This is, of course, the **midpoint theorem**.

Proof based on the midpoint theorem. In Figure 4 (a) we have drawn two medians, BE and CF; they intersect at G_1. Consider $\triangle G_1EF$ and $\triangle G_1BC$. Since $EF \parallel BC$ (midpoint theorem), the two triangles are similar to each other, hence their sides are in proportion. Since $EF = \frac{1}{2}BC$ (midpoint theorem, again), it follows that $G_1E = \frac{1}{2}G_1B$. Hence G_1 divides segment BE in the ratio $2 : 1$.

In Figure 4 (b) we have drawn BE and the remaining median AD; they intersect at G_2. Considering $\triangle G_2AB$ and $\triangle G_2DE$ and invoking the midpoint theorem twice, like earlier, we deduce that G_2 divides segment BE in the ratio $2 : 1$.

So G_1 and G_2 divide BE in the identical ratio $(2 : 1)$. This means that they are the same point! In other words, the point where BE and CF meet is identical to the point where BE and AD meet. This obviously means that AD, BE, CF concur.

Remark. Note the reasoning involved. It is rather more subtle than the reasoning used in the proofs for concurrence of the perpendicular bisectors and the angle bisectors. The underlying principle by which we concluded that points G_1, G_2 are the same is illustrated in Figure 5. Let AB be a given segment, and let P be a variable point located strictly in its interior. (This means that P cannot coincide with either A or B.) Consider the ratio $t = AP/PB$. *Then, as P moves from A towards B, this ratio assumes every possible positive value exactly once.* No

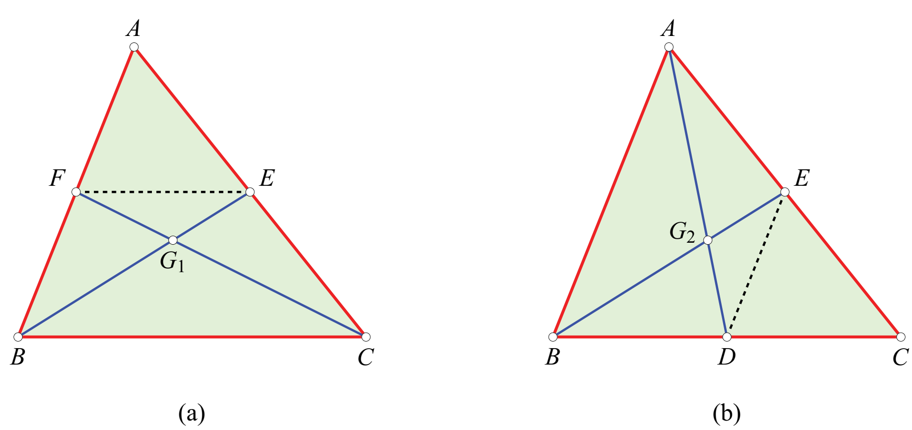


Figure 4. Concurrence of the medians of a triangle: proof using similar triangles

Figure 5. Variation of the ratio $t = AP/PB$ as P moves along AB

ratio is repeated. If P is very close to A, the ratio is close to 0, and it can get as close to 0 as we may want. Similarly, if P is very close to B, the ratio becomes very large, and it can get as large as we may want (there is no upper bound).

Or, to use terminology generally heard in a calculus class rather than a geometry class, we may say that the ratio $t = AP/PB$ is a strictly *monotonic function* of the position of P.

Proof based on area considerations. We offer a second proof which is of a very different nature. It uses one basic idea over and over again: *A median of a triangle divides it into two triangles with equal area*. (See Figure 6. Observe the notation used carefully: if X is any geometric figure, then $[X]$ denotes the area of X.)

Here's how we invoke this idea. In Figure 7 we have shown $\triangle ABC$ with two medians BE and CF which intersect at G. The line through A and G is then drawn; it intersects BC at D. Note that there is nothing being said about AD being a median and (therefore) D being the midpoint of BC. Rather, we have to *prove* that AD is a median. To make sure we do not fall into the trap of assuming implicitly that it is a median (and thus assuming the very thing we wish to prove), we have drawn it using a dashed line.

The various lines drawn within $\triangle ABC$ create six smaller triangles. Certain pairs of these are immediately seen to have equal area, by making use of the property mentioned above. (i) Since BE is a median, and G is a point on BE, $[GCE] = [GAE] = x$, say. (ii) Since CF is a median, and G is a point on CF, $[GAF] = [GBF] = y$, say. In Figure 7 we have written these symbols within the respective regions. Let $u = [GBD]$ and $v = [GCD]$, respectively. Since $[BCE] = [BAE]$, we have $u + v + x = y + y + x$; hence:

$$u + v = 2y.$$

Again, since $[CAF] = [CBF]$, we have $u + v + y = x + x + y$; hence:

$$u + v = 2x.$$

From the above equalities we deduce that

$$x = y.$$

Now we argue algebraically. Let $BD : DC = t$. Since $\triangle GBD$ and $\triangle GCD$ have equal altitude, and their bases are in the ratio $t : 1$, their areas bear this same ratio to one another. Hence:

$$u = tv.$$

Similarly, the areas of $\triangle ABD$ and $\triangle ACD$ bear this same ratio to one another. Hence:

$$2y + u = t(2x + v).$$

By subtraction the above two equalities yield:

$$2y = 2tx, \qquad \therefore\ y = tx.$$

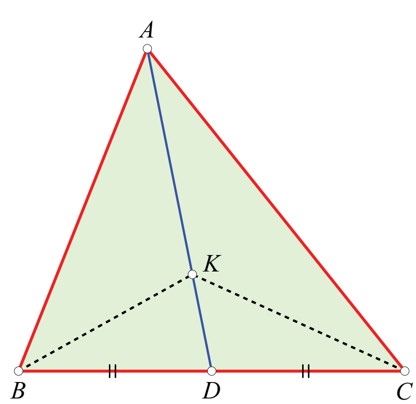


Figure 6. Property of a median of a triangle

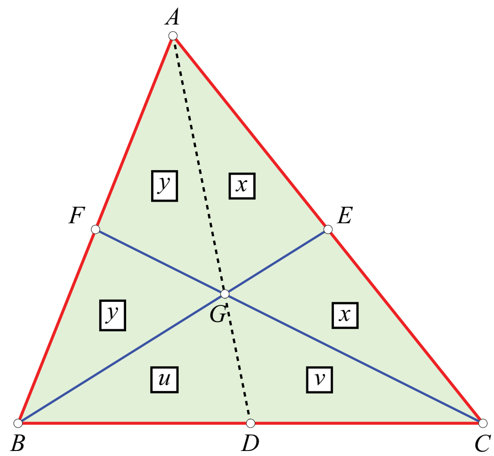


Figure 7. Making use of the area bisection property to prove concurrence

But we have already shown that $x = y$. Hence $t = 1$. But this means that D is the midpoint of BC. Hence AD is a median of $\triangle ABC$.

It follows that the third median passes through the point of intersection of BE and CF. That is, the medians of the triangle concur.

Remark. Note the use of algebraic manipulations in this proof. As such, this is not a "pure geometry" proof, and purists will frown at it. But the central idea is, surely, a pleasing one. We hope you enjoyed this neat interplay of algebra and geometry.

Third proof. We now offer a third proof which draws upon the midpoint theorem for its central logic (just like the first proof shown above) but in a very different way. It is taken from an article that appeared in *The Mathematical Gazette*, in the November 2001 issue, written by Mowaffaq Hajja and Peter Walker, and titled "Why Must the Triangle's Medians Be Concurrent?" The central idea is ingenious and subtle.

Figure 8(a) shows a triangle ABC and points D, E, F which are the midpoints of its sides BC, CA, AB; the median AD has been drawn, and the medial triangle DEF shown shaded. Here's how we argue.

- ⋆ The midpoint theorem implies that figure $DEAF$ is a parallelogram. Since the diagonals of a parallelogram bisect each other, we deduce that AD bisects side EF of $\triangle DEF$. So the medial line through A of $\triangle ABC$ is the same line as the medial line through D of $\triangle DEF$.

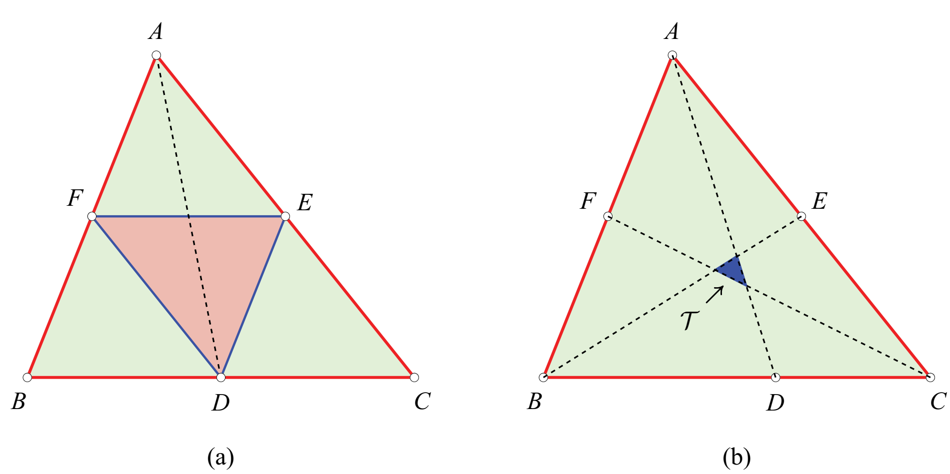


Figure 8.

- ⋆ It follows, by symmetry, that: *the medial lines of $\triangle DEF$ coincide with the medial lines of $\triangle ABC$*. We now make use of this observation in a surprising way.
- ⋆ Next, we note that $\triangle DEF$ is similar to $\triangle ABC$: it is a copy of $\triangle ABC$ but with half its scale. It stands to reason, surely, that any genuine geometric property exhibited by $\triangle ABC$ will also be exhibited by $\triangle DEF$. In particular we can be sure that if the medians of $\triangle ABC$ concur, then so do the medians of $\triangle DEF$; and if the medians of $\triangle ABC$ do not concur, then neither do the medians of $\triangle DEF$.
- ⋆ Let us now suppose now that the medians of $\triangle ABC$ do *not* meet in a point, and rather that the situation is as depicted in Figure 8 (b). In that case, it must be that the medians AD, BE, CF enclose a triangle $\mathcal{T}$ (shown with a heavy blue filling).
- ⋆ Since $\triangle DEF$ is similar to $\triangle ABC$, we expect that the triangle $\mathcal{T}'$ created by the medial lines of $\triangle DEF$ is similar to $\mathcal{T}$ but has half its dimensions. This implies that $[\mathcal{T}'] = \frac{1}{4}[\mathcal{T}]$.
- ⋆ On the other hand, we just noted that the medial lines of $\triangle DEF$ coincide with those of $\triangle ABC$. This means that $\mathcal{T}'$ coincides with $\mathcal{T}$. Therefore we have: $[\mathcal{T}'] = [\mathcal{T}]$.
- ⋆ How can we reconcile these two statements? There is clearly just one way: it must be that both $[\mathcal{T}] = 0$ and $[\mathcal{T}'] = 0$. In other words, both $\mathcal{T}$ and $\mathcal{T}'$ have zero area.
- ⋆ But this is just another way of saying that the medians AD, BE, CF concur!

Closing remark. One proposition, but three entirely different proofs .... How would you contrast them?

In a future column, we will discuss an important tool in the study of concurrence, which allows us to deduce a vast number of concurrence results in one stroke.

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Factor Fun…

Activities and Questions Around Factorisation

An Exercise in Factorisation

A. Ramachandran

Here is an interesting exercise in factorisation with boundary conditions, for students of upper primary school – classes 5, 6 and 7.

We generally record dates by writing the day first, followed by the month order and ending with the last two digits of the year ("common era"); i.e., we record the date as *DD-MM-YY*. For instance 17-11-32 would represent the 17^{th} of November of the year '32. In some cases the product of the first two numbers of the triad equals the third number. An example is 15-04-60. The question we pose is: *How many such dates are possible?*

We need to consider the years of only one century as the whole pattern would repeat in other centuries. We can ignore the year ending '00.

This exercise can be approached in three ways.

One is the strictly chronological. We start from the year '01 and proceed till '99. In each year we find the various possibilities (this is basically an exercise in factorisation), keeping in mind the restrictions on the possible dates and month orders. One would need to arrange the dates pertaining to a particular year chronologically again.

The second approach is to go down the month order (1-12), taking one month at a time and writing down all possible combinations of date/month/year satisfying the given condition. Ignore situations that give products exceeding 99.

The third approach is to take the dates in order (1-31), taking one date at a time and listing the possible date/month/year combinations that satisfy the given condition.

Keywords: *Factorisation, HCF, LCM, relatively prime*

The latter two approaches may be described as 'quasi-chronological'.

In practice, different groups of students could work on these approaches and the results compared. The following questions could be posed at the end:

1. How many dates *DD-MM-YY* satisfy the condition $DD \times MM = YY$?
2. Which year has the largest number of such dates?
3. How many years do not have any such dates?
4. Are there any interesting personal or historical connections with these dates?

Further Questions on Factorisation

Here are some additional questions about factorisation and hints to solve them.

a) **How many zeroes are there at the units' end of the number 100! (when written in its normal decimal expanded form)?** (Here, 100! is the product of the numbers 1, 2, 3, …, 99, 100.)

HINT: Every zero at the end of a number is contributed by a combination of 2 and 5 as factors. So we need to consider the number of 2's and 5's in the fully factorised form of 100!. Now the 2's must outnumber the 5's, i.e., 5 is the 'limiting factor' one needs to consider.

b) **Which 3-digit number has the largest number of factors (prime or otherwise)?**

HINT: Use the following rule here: If $N = p^a \times q^b \times r^c \times \cdots$ where $p, q, r, \ldots$ are primes and $a, b, c, \ldots$ are positive integers, then the number of factors of N is equal to $(a+1)(b+1)(c+1)\cdots$. Here the list of factors includes 1 and the number itself.

c) **Which 4-digit number has the largest number of factors?**

HINT: Use a similar approach to the earlier question. There seem to be two contenders for this problem. Perhaps the smaller of the two should be given the honours.

d) **Which is the smallest number with exactly 100 factors?**

HINT: This could be considered as a 'converse' to the earlier two questions.

e) **Let a and b be two given positive integers. How many pairs of integers can be found such that their HCF is a and their LCM is b? (The count could include the trivial case of the given numbers themselves forming the pair.)**

HINT: Two numbers can be expressed as the products CX and CY, C being a common factor and X, Y mutually prime. Then their HCF is C and LCM is CXY. If we divide the LCM by the HCF we get XY. Now the task is to express XY as the product of two mutually prime numbers. What matters now is the number of *distinct* prime factors of XY, which we denote by d. The required number of pairs is to be expressed in terms of d.

Answers

a) 24
b) 840, with 32 factors
c) 7560 and 9240, each with 64 factors
d) 45360
e) 2^{d-1}

Afterword

The 3-digit number with the greatest number of factors has 32 factors. The corresponding 4-digit number has 64 factors. The 5-digit number with the greatest number of factors has 128 factors. The number in question is 83160. There seems to be a pattern here. Unfortunately it does not carry to the next level. The corresponding 6-digit number (720720) has 240 factors.

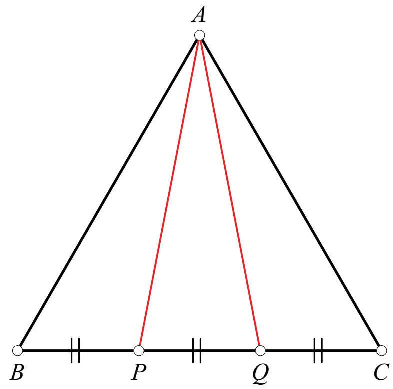
A RAMACHANDRAN has had a long standing interest in the teaching of mathematics and science. He studied physical science and mathematics at the undergraduate level, and shifted to life science at the postgraduate level. He taught science, mathematics and geography to middle school students at Rishi Valley School for over two decades, and now stays in Chennai. His other interests include the English language and Indian music. He may be contacted at archandran.53@gmail.com.

Trisection of a 60 degree angle? Not Quite!

C⊗MαC

Many students on first hearing that "Trisection of a general angle is not possible using only compass and straight-edge" immediately set about trying to disprove this assertion! Curiously, many among them hit upon the following method (illustrated for a 60° angle).

In the figure, $\triangle ABC$ is equilateral, and P and Q are points of trisection of BC (so $BP = PQ = QC$). Segments AP and AQ are drawn. *Question*. Do these two segments trisect $\measuredangle BAC$? Many students believe that they do. How do we check whether they are right? Noting that $\measuredangle BAP = \measuredangle CAQ$ by symmetry, we only need to compare $\measuredangle BAP$ and $\measuredangle PAQ$.



We choose to make the comparison using coordinates. Let $B = (0,0)$, $C = (6,0)$, $A = (3, 3\sqrt{3})$, $P = (2,0)$ and $Q = (4,0)$. Then the slopes of AB, AP, AQ and AC are as follows:

$$\text{slope}(AB) = \tan 60^\circ = \sqrt{3},$$

$$\text{slope}(AC) = \tan 120^\circ = -\sqrt{3}.$$

$$\text{slope}(AP) = \frac{3\sqrt{3} - 0}{3 - 2} = 3\sqrt{3},$$
$$\text{slope}(AQ) = \frac{3\sqrt{3} - 0}{3 - 4} = -3\sqrt{3}.$$

Hence, using the 'angle between two lines' formula, we get, for $\measuredangle BAP$ and $\measuredangle PAQ$:

$$\tan \measuredangle BAP = \frac{3\sqrt{3} - \sqrt{3}}{1 + 3\sqrt{3} \cdot \sqrt{3}} = \frac{2\sqrt{3}}{10} = \frac{1}{5} \times \sqrt{3},$$

$$\tan \measuredangle PAQ = \frac{-3\sqrt{3} - 3\sqrt{3}}{1 - 3\sqrt{3} \cdot 3\sqrt{3}} = \frac{-6\sqrt{3}}{-26} = \frac{3}{13} \times \sqrt{3}.$$

We see right away that $\measuredangle BAP$ and $\measuredangle PAQ$ are unequal (since $1/5$ and $3/13$ are unequal). But we can say more: since $1/5 < 3/13$, it follows that $\measuredangle BAP < \measuredangle PAQ$. (Here we make implicit use of the fact that for acute angles x and y, if $x < y$ then $\tan x < \tan y$, and vice versa. Differently expressed, $\tan \theta$ is an increasing function of θ for $0 \le \theta < \pi/2$.)

Thus, $\measuredangle PAQ$ exceeds both $\measuredangle BAP$ and $\measuredangle QAC$. Here are the actual magnitudes of the angles:

$$\measuredangle BAP = \measuredangle QAC \approx 19.1066^\circ, \quad \measuredangle PAQ \approx 21.7868^\circ.$$

So $\measuredangle PAQ$ exceeds $\measuredangle BAP$ by a fair bit. The method doesn't quite work ....

Can we prove this without computation?

Is there a *non-computational way* of proving that $\measuredangle BAP < \measuredangle PAQ$? It is a nice challenge to find such a proof. Note that if we do find one, it will not tell us by how much the two angles differ.

Here is a possible approach. Consider $\triangle ABP$ and $\triangle APQ$. The two triangles have equal bases ($BP = PQ$) and the same altitude (namely: the altitude of $\triangle ABC$). So they have equal area.

Now we invoke another formula: *area of a triangle equals half the product of any two sides and the sine of the included angle*. Applying this to $\triangle ABP$ and $\triangle APQ$, which we know have equal area, we get:

$$\frac{1}{2} AB \times AP \times \sin \measuredangle BAP = \frac{1}{2} AP \times AQ \times \sin \measuredangle PAQ,$$
$$\therefore\ AB \times \sin \measuredangle BAP = AQ \times \sin \measuredangle PAQ.$$

Hence $AB/AQ = \sin \measuredangle PAQ / \sin \measuredangle BAP$. Now which is greater, AB or AQ? Clearly, it is AB which is larger. This can be seen from $\triangle ABQ$, in which $\measuredangle AQB > \measuredangle ABQ$ (proof: $\measuredangle AQB > \measuredangle ACQ$, which equals $\measuredangle ABQ$). Invoking the fact that the larger angle in a triangle has the larger side opposite it, we deduce that $AB > AQ$ and so $AB/AQ > 1$.

Therefore $\sin \measuredangle PAQ / \sin \measuredangle BAP > 1$, and it follows that $\measuredangle BAP < \measuredangle PAQ$. (Once again, we implicitly make use of a fact from trigonometry: that over the domain of acute angles, sine is an increasing function of the angle.)

The reader is invited to find other non-computational proofs showing that $\measuredangle BAP < \measuredangle PAQ$.

The **COMMUNITY MATHEMATICS CENTRE** (CoMaC) is an outreach arm of Rishi Valley Education Centre (AP) and Sahyadri School (KFI). It holds workshops in the teaching of mathematics and undertakes preparation of teaching materials for State Governments and NGOs. CoMaC may be contacted at shailesh.shirali@gmail.com.

techspace

Hill Ciphers

Jonaki B Ghosh

Introduction

Cryptography is the science of making and breaking codes. It is the practice and study of techniques for secure communication. Modern cryptography intersects the disciplines of mathematics, computer science, and electrical engineering. Applications of cryptography include ATM cards, computer passwords, and e-commerce. In this article we explore an interesting cryptography method known as the *Hill Cipher*, based on matrices. We explore the method using the spreadsheet MS Excel to perform operations on matrices.

Hill Ciphers

Hill ciphers are an application of matrices to cryptography. Ciphers are methods for transforming a given message, the *plaintext*, into a new form unintelligible to anyone who does not know the key – the transformation used to convert the plaintext to the *ciphertext*. The inverse key is required to reverse the transformation to recover the original message. To use the key to transform plaintext into ciphertext is to *encipher* the plaintext. To use the inverse key to transform the ciphertext back into plaintext is to *decipher* the ciphertext.

In order to understand Hill ciphers, we must first understand *modular arithmetic.*

Definition 1: A *Hill n-cipher* has for its key a given $n \times n$ matrix whose entries are non-negative integers from the set $\{0, 1, 2, 3, \ldots, m-1\}$, where m is the number of characters used for the encoding process. Suppose we wish to use all 26 alphabets from A to Z and three more characters, say '.', '-' and '?'.

This means we have 29 characters with which to write our plaintext. These have been shown in Table 1, where the 29 characters have been numbered from 0 to 28.

A	**B**	**C**	**D**	**E**	**F**	**G**	**H**	**I**	**J**	**K**	**L**	**M**	**N**	**O**	**P**
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Q	**R**	**S**	**T**	**U**	**V**	**W**	**X**	**Y**	**Z**	**.**	**_**	**?**			
16	17	18	19	20	21	22	23	24	25	26	27	28			

Table 1: The substitution table for the Hill Cipher

Let us apply this to an example of a Hill 2-cipher corresponding to the substitution scheme shown in Table 1 with 29 characters. Let the key be the 2 × 2 matrix

$$E = \begin{bmatrix} 1 & 4 \\ 2 & 9 \end{bmatrix}$$

We can also refer to E as the encoding matrix. We will use E to encipher groups of two consecutive characters. Suppose we have to encipher the word **GO**. The alphabets G and O correspond to the numbers 6 and 14, respectively from our substitution table. We represent it by a 2 × 1 matrix.

$$\begin{bmatrix} 6 \\ 14 \end{bmatrix}$$

To encipher **GO**, we pre-multiply this matrix by the encoding matrix E.

$$\begin{bmatrix} 1 & 4 \\ 2 & 9 \end{bmatrix}\begin{bmatrix} 6 \\ 14 \end{bmatrix} = \begin{bmatrix} 62 \\ 138 \end{bmatrix}$$

The product is a 2 × 1 matrix with entries 62 and 138. But what characters do the numbers 62 and 138 represent? These are not in our substitution table! What we do is as follows:

We divide these numbers by 29 and consider their respective remainders after the division process is done. Thus when we divide 62 by 29, the remainder is 4 and when we divide 138 by 29, the remainder is 22.

To express this in the language of *modular arithmetic*, we write:

$$62 \equiv 4 \ (mod\ 29),$$
$$138 \equiv 22 \ (mod\ 29).$$

Equivalence and residues modulo an integer

Definition 2: Given an integer $m > 1$, called the *modulus*, we say that the two integers a and b are *congruent* to one another *modulo* m if $a - b$ is an integral multiple of m. To denote this, we write: $a \equiv b \ (mod\ m)$.

We read this as: 'a is congruent to b modulo m'.

In other words, $a \equiv b \ (mod\ m)$ means that $a = b + km$ for some integer k which could be positive, negative or zero.

For our cipher, we have: $62 \equiv 4\ mod\ \ 29$ and $138 \equiv 22\ mod\ 29$. Note: $62 - 4 = 2 \times 29$, or $62 = 4 + 2 \times 29$, and $138 - 22 = 4 \times 29$, or $138 = 22 + 4 \times 29$.

The numbers 4 and 22 correspond to the letters E and W respectively from our substitution table. Thus, the word **GO** is enciphered to **EW**!

In order to use this method of sending secret messages, the sender has to encrypt the plaintext **GO** and send the encrypted form. This means the sender sends **EW** instead. The receiver gets the message **EW**. The secret key, that is, the encoding matrix E is known only to the sender and the receiver. Now let us see how the receiver deciphers what **EW** stands for.

In order to decipher the message **EW**, we begin by looking for the numbers corresponding to E and W in our substitution table. These are 4 and 22 respectively. We represent this in the form of a 2 × 1 matrix

$$\begin{bmatrix} 4 \\ 22 \end{bmatrix}$$

We pre-multiply this matrix by the inverse of the matrix E, that is, by $E^{-1} = \begin{bmatrix} 9 & -4 \\ -2 & 1 \end{bmatrix}$. So:

$$E^{-1}W = \begin{bmatrix} 9 & -4 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 4 \\ 22 \end{bmatrix} = \begin{bmatrix} -52 \\ 14 \end{bmatrix}$$

We need another definition. Note that any arbitrary integer a can be divided by m to yield a quotient q and remainder r; that is, $a = q\,m + r$. Then we say $a \equiv r\ (mod\ m)$.

Definition 3: Let m be any integer exceeding 1. For an arbitrary integer a, the *residue of a modulo m* is the unique integer r in the set $\{0,\ 1,\ 2,\ 3,\ \ldots,\ m-1\}$ such that $a \equiv r\ (mod\ m)$.

Thus: $23 \equiv 7\ (mod\ 8)$, since $23 = 2 \times 8 + 7$. Here $a = 23, m = 8$ and $r = 7$. What about negative integers? For example, what is r when $a = -18$ and $m = 8$? Clearly r has to be an integer between 0 and 7. Note that $-18 = -3 \times 8 + 6$. Thus $r = 6$ and we can write $-18 \equiv 6\ (mod\ 8)$.

Now coming back to deciphering our Hill cipher, we need to find r for −52 and 14 for $m = 29$. Note that $-52 = -2 \times 29 + 6$ and $14 = 0 \times 29 + 14$.Thus: $-52 \equiv 6\ (mod\ 29)$ and $14 \equiv 14\ (mod\ 29)$.

Thus we may write $\begin{bmatrix} -52 \\ 14 \end{bmatrix} \equiv \begin{bmatrix} 6 \\ 14 \end{bmatrix} (mod\ \ 29)$

Isn't this great! 6 represents G and 14 represents O from our substitution table. Hence we have deciphered **EW** to obtain **GO** the original characters or the plaintext!

Let us now see how to encipher a longer message or plaintext using the key $\begin{bmatrix} 1 & 4 \\ 2 & 9 \end{bmatrix}$.

We shall encipher the plaintext **MATH_IS_FUN.** The steps are indicated below. For matrix computations we use MS Excel. In Excel, the commands for multiplying matrices and finding the inverse of a matrix are MMULT and MINVERSE respectively. For reducing a number modulo a divisor the required command is MOD.

Encoding or enciphering the plaintext

Step 1: Convert the plaintext **MATH_IS_FUN.** to the corresponding substitution values from the substitution table. The values are

12 0 19 7 27 8 18 27 5 20 13 26

We need to make a 2 × n matrix using these values

Step 2: Form pairs of these numbers as follows

12 0   19 7   27 8   18 27   5 20   13 26

Note: In case the message has an odd number of characters, a full stop or underscore (a '.' or a '_') may be added at the end to complete the pair. For example, if the plaintext is "LET_US_GO", then we have 9 characters. So we add a '.' at the end to make the message "LET_US_GO."

Each pair will form a column of a 2 × 6 matrix (as there are 6 pairs). Let us call this matrix P (the plaintext matrix)

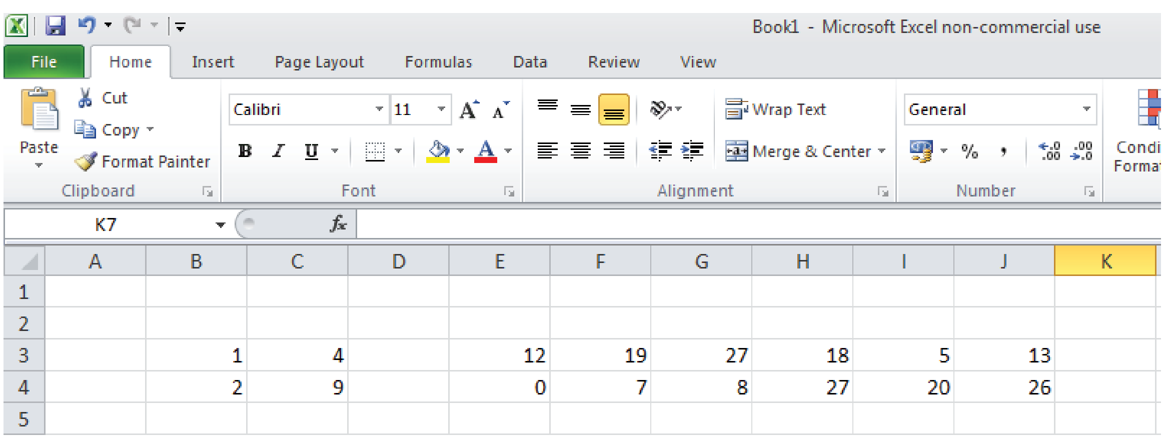
$$P = \begin{bmatrix} 12 & 19 & 27 & 18 & 5 & 13 \\ 0 & 7 & 8 & 27 & 20 & 26 \end{bmatrix}$$

Step 3: Compute the product EP

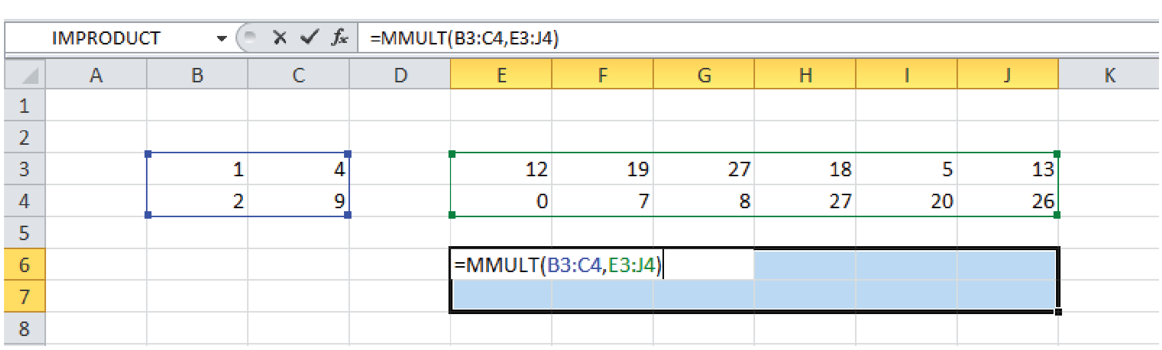
$$EP = \begin{bmatrix} 1 & 4 \\ 2 & 9 \end{bmatrix} \begin{bmatrix} 12 & 19 & 27 & 18 & 5 & 13 \\ 0 & 7 & 8 & 27 & 20 & 26 \end{bmatrix} = \begin{bmatrix} 12 & 47 & 59 & 126 & 85 & 117 \\ 24 & 101 & 126 & 279 & 190 & 260 \end{bmatrix}$$

In order to perform this computation in Excel we proceed as follows

Enter the 2 × 2 matrix E and the 2 × 6 matrix P as separate arrays as shown. Each entry of a matrix may be entered by typing a number in a cell and pressing Enter. The arrow keys may be used to move to the next appropriate cell.



To obtain the product, select a blank 2 × 6 array and type **=MMULT(** in the top leftmost cell of the closed array. Within the parentheses, first select the array for matrix E and then the array for matrix P separated by a comma. Press **Crtl + Shift** followed by **Enter** to obtain the product. (Note that you need to press Crtl and Shift simultaneously and then press Enter.)



G11											
	A	B	C	D	E	F	G	H	I	J	K
1											
2											
3		1	4		12	19	27	18	5	13	
4		2	9		0	7	8	27	20	26	
5											
6					12	47	59	126	85	117	
7					24	101	126	279	190	260	
8											

Step 4: Reduce the product modulo 29 to obtain the Hill 2-cipher values. This means we have to divide each number by 29 and find the remainder. In Excel we can reduce the entire matrix modulo 29 in one go!

$$EP = \begin{bmatrix} 12 & 47 & 59 & 126 & 85 & 117 \\ 24 & 101 & 126 & 279 & 190 & 260 \end{bmatrix} \equiv \begin{bmatrix} 12 & 18 & 1 & 10 & 27 & 1 \\ 24 & 14 & 10 & 18 & 16 & 28 \end{bmatrix} (mod\ 29)$$

To do this in Excel proceed as follows

Select a blank 2 $\times$ 6 array and type **=MOD(** in the top leftmost cell of the array. Within the parentheses, select the array of the product matrix EP and type 29 for the divisor.

5										
6				12	47	59	126	85	117	
7				24	101	126	279	190	260	
8										
9				=MOD(E6:J7,29)						
10										
11										

5										
6				12	47	59	126	85	117	
7				24	101	126	279	190	260	
8										
9				12	18	1	10	27	1	
10				24	14	10	18	16	28	
11										

Step 5: Write out the columns of the matrix in a sequence

$$\begin{bmatrix} 12 & 18 & 1 & 10 & 27 & 1 \\ 24 & 14 & 10 & 18 & 16 & 28 \end{bmatrix}$$

These are

12 24 18 14 1 10 10 18 27 16 1 28

Replace these values by the characters from the substitution table to which these values correspond.

The encrypted message or ciphertext is **MYSOBKKS_QB?**

Decoding or deciphering the ciphertext

In this section we will try to decipher the ciphertext **MYSOBKKS_QB?**

Step 1: Convert the characters to their respective Hill-2-cipher values from the substitution table

$$12 \quad 24 \quad 18 \quad 14 \quad 1 \quad 10 \quad 10 \quad 18 \quad 27 \quad 16 \quad 1 \quad 28$$

Form a 2 × 6 matrix using these values. Make pairs of these numbers as follows

$$12 \;\; 24 \qquad 18 \;\; 14 \qquad 1 \;\; 10 \qquad 10 \;\; 18 \qquad 27 \;\; 16 \qquad 1 \;\; 28$$

Each pair will form a column of a 2 × 6 matrix (since there are 6 pairs). Let us call this matrix C (the ciphertext matrix)

$$C = \begin{bmatrix} 12 & 18 & 1 & 10 & 27 & 1 \\ 24 & 14 & 10 & 18 & 16 & 28 \end{bmatrix}$$

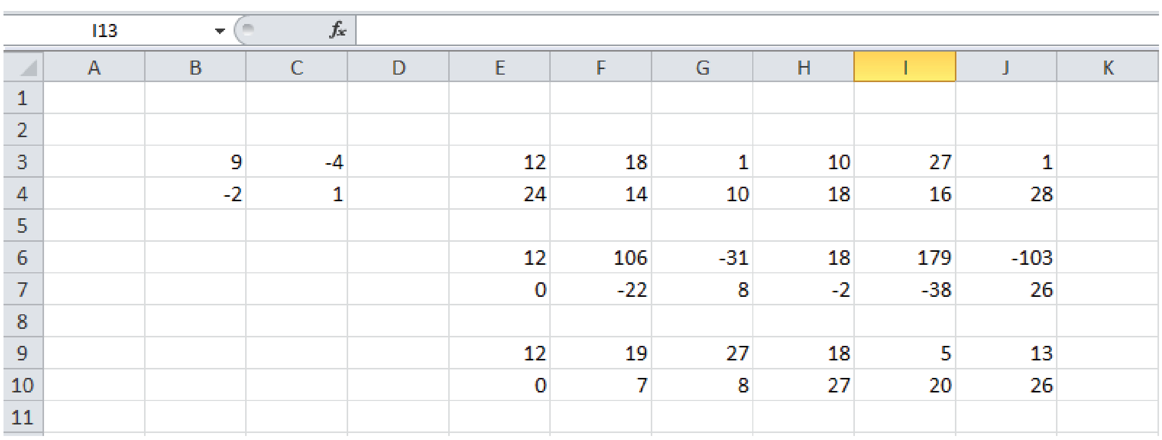
Step 2: Compute the product E^{-1} C

$$E^{-1}C = \begin{bmatrix} 9 & -4 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 12 & 18 & 1 & 10 & 27 & 1 \\ 24 & 14 & 10 & 18 & 16 & 28 \end{bmatrix} = \begin{bmatrix} 12 & 106 & -31 & 18 & 179 & -103 \\ 0 & -22 & 8 & -2 & -38 & 26 \end{bmatrix}$$

Step 3: Reduce the product modulo 29 to obtain the substitution values.

$$\begin{bmatrix} 12 & 106 & -31 & 18 & 179 & -103 \\ 0 & -22 & 8 & -2 & -38 & 26 \end{bmatrix} \equiv \begin{bmatrix} 12 & 19 & 27 & 18 & 5 & 13 \\ 0 & 7 & 8 & 27 & 20 & 26 \end{bmatrix} (mod\ 29)$$

The reader may perform these computations using Excel. The screenshot of the Excel sheet is as follows.


I13

	A	B	C	D	E	F	G	H	I	J	K
1											
2											
3		9	-4		12	18	1	10	27	1	
4		-2	1		24	14	10	18	16	28	
5											
6					12	106	-31	18	179	-103	
7					0	-22	8	-2	-38	26	
8											
9					12	19	27	18	5	13	
10					0	7	8	27	20	26	
11											

Step 4: Write out the columns of the matrix in a sequence

$$\begin{bmatrix} 12 & 19 & 27 & 18 & 5 & 13 \\ 0 & 7 & 8 & 27 & 20 & 26 \end{bmatrix}$$

These are

$$12 \quad 0 \quad 19 \quad 7 \quad 27 \quad 8 \quad 18 \quad 27 \quad 5 \quad 20 \quad 13 \quad 26$$

Replace these values by the characters from the substitution table to which these values correspond.

The decrypted message or plaintext is **MATH_IS_FUN.**

The method works! Observe that **MATH_IS_FUN** has been encrypted as **MYSOBKKS_QB?**

Note that there are two '_' (underscores) in the original plaintext message. But these do not get encrypted to the same characters. The first one is encrypted to B and the second one gets encrypted to S. Can you explain why?

So far we have learnt how to encrypt a plaintext using a Hill 2-cipher. This means that our encoding matrix is a 2×2 matrix. If we choose a 3×3 matrix, the plaintext will have to be converted to a $3 \times n$ matrix (here the number of columns 'n' depends on the length of the message).

The reader is urged to try to decode the messages in the next few exercises to practice the method. All computations may be done on Excel. Note that the substitution table remains the same as before.

EXERCISES

1. Decode the secret message **FY O. KI ZT WA QC** which was encrypted using the encoding matrix

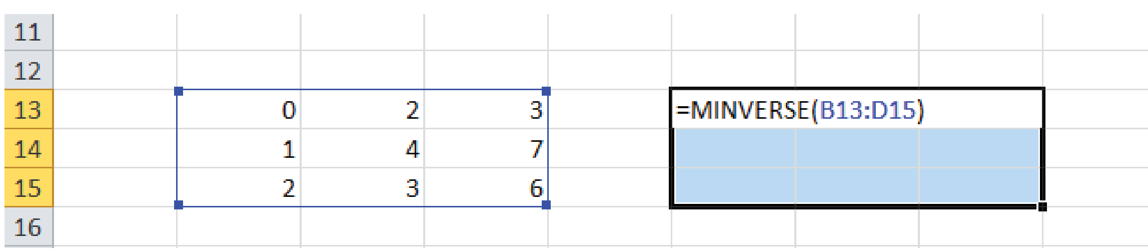
$$E = \begin{bmatrix} 1 & 4 \\ 2 & 9 \end{bmatrix}$$

2. Decode the secret message **ITS DGN STX SJK DVO JHE TCB** which was encrypted using the encoding matrix

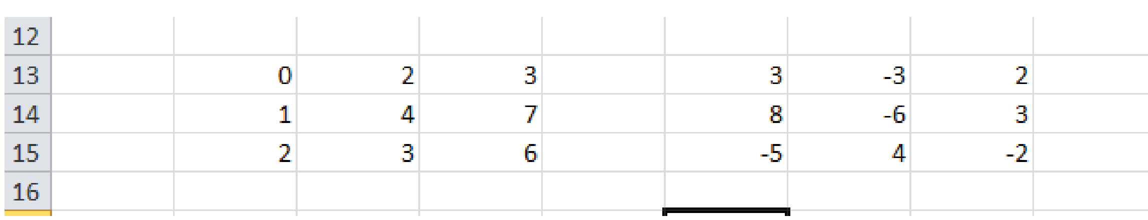
$$E = \begin{bmatrix} 0 & 2 & 3 \\ 1 & 4 & 7 \\ 2 & 3 & 6 \end{bmatrix}$$

   This is an example of a Hill 3-cipher.

Note that the command **=MINVERSE()** may be used in Excel to find the inverse of the given matrix E. Enter the matrix in a 3×3 array. Then select a blank 3×3 array and type the command as shown in the screenshot

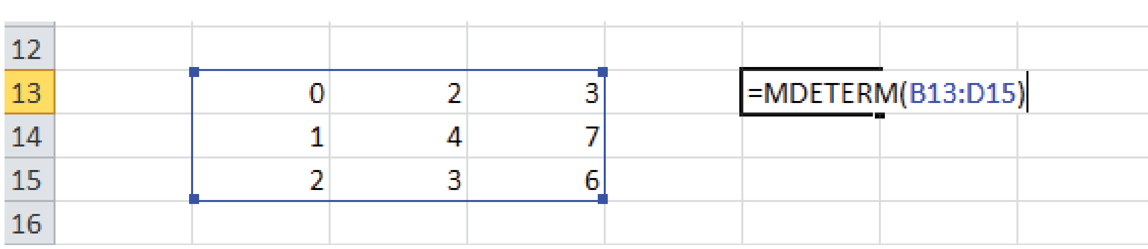


11							
12							
13		0	2	3	=MINVERSE(B13:D15)		
14		1	4	7			
15		2	3	6			
16							



12							
13		0	2	3	3	-3	2
14		1	4	7	8	-6	3
15		2	3	6	-5	4	-2
16							

Also if you want to compute the determinant of the matrix, then select any cell and type the command **=MDETERM()** and press Enter.



12							
13		0	2	3	=MDETERM(B13:D15)		
14		1	4	7			
15		2	3	6			
16							

12							
13		0	2	3		1	
14		1	4	7			
15		2	3	6			
16							

Note that the given 3 × 3 matrix has determinant 1.

3. Decode the secret message **UAR CR? WBQ BYW WBL LCD RWD** which was encrypted using the encoding matrix

$$E = \begin{bmatrix} 0 & 2 & 3 \\ 1 & 4 & 7 \\ 2 & 3 & 6 \end{bmatrix}$$

4. Find a 4 × 4 matrix whose determinant is equal to 1 and use it to encrypt your own message.

Solutions to these exercises are given on page 74.

Conclusion

The Hill Cipher presents an interesting application of matrices and number theory to cryptography. It is open to exploration and students find it exciting to use this method. This is an example of a practical situation where performing matrix operations such as matrix multiplication and finding the inverse are actually required. It helps the student to understand the need and importance of matrix operations and also explore the method by using different keys (that is, encoding matrices). Any computing tool which can perform matrix operations will be helpful, as the computations may be tedious and time consuming (especially when the plaintext or ciphertext are lengthy). In this article we have chosen square matrices whose determinant is equal to 1 as the key. However this is not necessary. Any invertible matrix may be chosen. In case the determinant of the encoding matrix is anything other than 1, the computations are slightly different, and this case will be discussed in another article.

References

1. http://en.wikipedia.org/wiki/Hill_cipher
2. http://practicalcryptography.com/ciphers/hill-cipher/

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techspace

Hill Ciphers: Solutions to the Exercises

JONAKI B GHOSH

1. Using the substitution table the encrypted message (**FY O. KI ZT WA QC**) is converted to the following 2×6 matrix:

$$\begin{bmatrix} 5 & 14 & 10 & 25 & 22 & 16 \\ 24 & 26 & 8 & 19 & 0 & 2 \end{bmatrix}$$

We pre-multiply this with the inverse of the matrix $\begin{bmatrix} 1 & 4 \\ 2 & 9 \end{bmatrix}$ which is $\begin{bmatrix} 9 & -4 \\ -2 & 1 \end{bmatrix}$. Thus:

$$\begin{bmatrix} 9 & -4 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 5 & 14 & 10 & 25 & 22 & 16 \\ 24 & 26 & 8 & 19 & 0 & 2 \end{bmatrix}$$
$$= \begin{bmatrix} -51 & 22 & 58 & 149 & 198 & 136 \\ 14 & -2 & -12 & -31 & -44 & -30 \end{bmatrix}$$

Reducing the product modulo 29 we get:

$$\begin{bmatrix} 7 & 22 & 0 & 4 & 24 & 20 \\ 14 & 27 & 17 & 27 & 14 & 28 \end{bmatrix}$$

Converting the numbers to characters, column wise, we obtain the original message:

HO W_ AR E_ YO U?

That is: **HOW_ARE_YOU?**

2. The secret message **ITS DGN STX SJK DVO JHE TCB** is first converted to a 3 × 7 matrix using the substitution table. We get:

$$\begin{bmatrix} 8 & 3 & 18 & 18 & 3 & 9 & 19 \\ 19 & 6 & 19 & 9 & 21 & 7 & 2 \\ 18 & 13 & 23 & 10 & 14 & 4 & 1 \end{bmatrix}$$

We pre-multiply this matrix with the inverse of the encoding matrix $\begin{bmatrix} 0 & 2 & 4 \\ 1 & 4 & 7 \\ 2 & 3 & 6 \end{bmatrix}$ which is $\begin{bmatrix} 3 & -3 & 2 \\ 8 & -6 & 3 \\ -5 & 4 & -2 \end{bmatrix}$. Thus

$$\begin{bmatrix} 3 & -3 & 2 \\ 8 & -6 & 3 \\ -5 & 4 & -2 \end{bmatrix}\begin{bmatrix} 8 & 3 & 18 & 18 & 3 & 9 & 19 \\ 19 & 6 & 19 & 9 & 21 & 7 & 2 \\ 18 & 13 & 23 & 10 & 14 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 17 & 43 & 47 & -26 & 14 & 53 \\ 4 & 27 & 99 & 120 & -60 & 42 & 143 \\ 0 & -17 & -60 & -74 & 41 & -25 & -89 \end{bmatrix}$$

In Excel, the entry in position (3,1) of the matrix is seen to be a very small number. It may be treated as 0. Reducing the matrix modulo 29 we get the following:

$$\begin{bmatrix} 3 & 17 & 14 & 18 & 3 & 14 & 24 \\ 4 & 27 & 12 & 4 & 27 & 13 & 27 \\ 29 & 12 & 27 & 13 & 12 & 4 & 27 \end{bmatrix}$$

(Note that 29 is equivalent to 0 in modulo 29.) We convert the numbers to characters, column wise and obtain the original message: **DEA R_M OM_ SEN D_M ONE Y_ _**

That is: **DEAR_MOM_SEND_MONEY_ _**

Note that the number of characters in the original message is 19, which is not a multiple of 3. Hence two underscores have been added at the end of the message so that the 3 × 7 matrix could be completed.

3. The same process as shown in Exercise 2 may be used to decode the message. The details are left to the reader. The original message is: **HILL_CIPHERS_ARE_FUN.**

4. Here is a 4 × 4 matrix whose determinant is equal to 1:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 5 & 0 & 2 & 3 \\ 1 & 1 & 4 & 7 \\ 8 & 2 & 3 & 6 \end{bmatrix}$$

There are clearly many more such matrices (in fact, infinitely many).

problem corner

Two Combinatorial Problems

C⊗MαC

'Good' subsets

The following problem was posed in a recent mathematics contest:

> ***Problem.*** *Consider all three-element subsets of the set S given by*
>
> $$S = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}.$$
>
> *Call a subset 'good' if the sum of its elements is a multiple of* 3*. Thus,* $\{2, 3, 7\}$ *is good, but not* $\{2, 3, 8\}$*. Find the number of good three-element subsets of S.*

We shall find the answer in two ways — a brute-force way, using complete enumeration, and then by a more subtle approach, using ideas from number theory.

It is always a good idea to have an idea what the answer will be, roughly, when we are computing any quantity. Here, the total number of three-element subsets of S is $\binom{10}{3}$ which equals

$$\frac{10 \times 9 \times 8}{1 \times 2 \times 3} = 120.$$

For each subset, if we add the elements of that subset and divide by 3, the remainder must be 0, 1 or 2. We wish to count the number of times the remainder 0 occurs. It seems reasonable to suppose that there will not be a big difference between the number of occurrences of each of the remainders, 0, 1 and 2. Accordingly we expect the answer to the problem to be close to one-third of 120, i.e., close to 40. Let us see if this is so.

Brute-force way. Let $\{a, b, c\}$ denote a potential good subset of S, with $a < b < c$. Then a can be any of the numbers 0, 1, 2, ..., 7. Let a take each of these values in turn. We shall count the number of pairs $\{b, c\}$ with $a < b < c$ such that $\{a, b, c\}$ is good, and then find the total number from these individual numbers.

- Suppose $a = 0$. Then $b + c$ must be a multiple of 3, so the possibilities for $\{b, c\}$ are: $\{1, 2\}$, $\{1, 5\}$, $\{1, 8\}$, $\{2, 4\}$, $\{2, 7\}$, $\{3, 6\}$, $\{3, 9\}$, $\{4, 5\}$, $\{4, 8\}$, $\{5, 7\}$, $\{6, 9\}$, $\{7, 8\}$. Hence there are **12** possibilities.

- Suppose $a = 1$. Then $b + c$ must be 1 less than a multiple of 3, so the possibilities for $\{b, c\}$ are: $\{2, 3\}$, $\{2, 6\}$, $\{2, 9\}$, $\{3, 5\}$, $\{3, 8\}$, $\{4, 7\}$, $\{5, 6\}$, $\{5, 9\}$, $\{6, 8\}$, $\{8, 9\}$. Hence there are **10** possibilities.

- Suppose $a = 2$. Then $b + c$ must be 1 more than a multiple of 3, so the possibilities for $\{b, c\}$ are: $\{3, 4\}$, $\{3, 7\}$, $\{4, 6\}$, $\{4, 9\}$, $\{5, 8\}$, $\{6, 7\}$, $\{7, 9\}$. Hence there are **7** possibilities.

- Suppose $a = 3$. Then $b + c$ must be a multiple of 3, so the possibilities for $\{b, c\}$ are: $\{4, 5\}$, $\{4, 8\}$, $\{5, 7\}$, $\{6, 9\}$, $\{7, 8\}$. Hence there are **5** possibilities.
- Suppose $a = 4$. Then $b + c$ must be 1 less than a multiple of 3, so the possibilities for $\{b, c\}$ are: $\{5, 6\}$, $\{5, 9\}$, $\{6, 8\}$, $\{8, 9\}$. Hence there are **4** possibilities.
- Suppose $a = 5$. Then $b + c$ must be 1 more than a multiple of 3, so the possibilities for $\{b, c\}$ are: $\{6, 7\}$, $\{7, 9\}$. Hence there are **2** possibilities.
- Suppose $a = 6$. Then $\{b, c\}$ must be $\{7, 8\}$. Hence there is just **1** possibility.
- Suppose $a = 7$. Then $\{b, c\}$ must be $\{8, 9\}$. Hence there is just **1** possibility.

So the required number is $12 + 10 + 7 + 5 + 4 + 2 + 1 + 1 = \mathbf{42}$. Note that the answer is close to 40, as anticipated.

A subtler approach. Let's see if we can do better. We categorize the numbers from 0 to 9 according to the remainder left when they are divided by 3. We get three sets:

- Remainder 0: $A_0 = \{0, 3, 6, 9\}$
- Remainder 1: $A_1 = \{1, 4, 7\}$
- Remainder 2: $A_2 = \{2, 5, 8\}$

Now there are two ways in which the sum of three numbers can be a multiple of 3 (a bit of thinking will convince you why this must be true):

(i) The three numbers leave the same remainder under division by 3; for example, a sum like $2 + 5 + 8$ or $0 + 6 + 9$. The underlying reason is, of course, that the sum $1 + 1 + 1$ is a multiple of 3 (and hence also $2 + 2 + 2$; the sum $0 + 0 + 0$ is clearly a multiple of 3).

(ii) The three numbers leave three different remainders under division by 3; for example, a sum like $1 + 2 + 6$ or $2 + 6 + 7$. The underlying reason is, of course, that the sum $0 + 1 + 2$ is a multiple of 3.

It follows that there are just two kinds of three-element subsets which are good: (i) those for which the three elements are all from A_0, all from A_1, or all from A_2; (ii) those which have one element each from A_0, A_1 and A_2. Let us count these separately.

Since $|A_0| = 4$, $|A_1| = 3$ and $|A_2| = 3$, the number of three-element subsets of the first kind is the sum of the number of three-element subsets of A_0, the number of three-element subsets of A_1 and the number of three-element subsets of A_2; that is, the sum of $\binom{4}{3}$, $\binom{3}{3}$ and $\binom{3}{3}$. This equals $4 + 1 + 1 = 6$.

The number of three-element subsets of the second kind is even simpler to compute: it is equal to the product $4 \times 3 \times 3 = 36$. Note the use of the 'multiplication principle of counting' here.

Hence the total number of good three-element subsets is $6 + 36 = \mathbf{42}$. We have got the same answer as earlier.

There are yet other ways of solving this problem, but we leave them for you to find.

Greatest odd divisor

Each positive integer has a *greatest odd divisor*. For example:

- $10 = 5 \times 2$, so the greatest odd divisor of 10 is 5.
- $48 = 3 \times 2^4$, so the greatest odd divisor of 48 is 3.

It should be clear that each positive integer n can be written in just one way as the product of an odd integer and a power of 2, and that odd integer is the largest odd divisor of n.

Here is a curious problem posed in a recent Regional Math Olympiad, pertaining to the greatest odd divisor function:

> ***Problem.*** *Consider the following set S of n numbers:*
>
> $$S = \{n + 1, n + 2, n + 3, \ldots, 2n - 1, 2n\}.$$
>
> *For each number in this set, find its greatest odd divisor. Show that the sum of these numbers is n^2.*

The result looks quite astonishing, doesn't it? However a closer look reveals that it is nothing but an old friend in a very clever disguise: namely, the statement that the sum of the first n odd numbers equals n^2. For example, consider the case when $n = 6$. We have:

$$7 = 7 \times 2^0, \quad 8 = 1 \times 2^3, \quad 9 = 9 \times 2^0,$$
$$10 = 5 \times 2^1, \quad 11 = 11 \times 2^0, \quad 12 = 3 \times 2^2.$$

So the odd numbers which we have to sum are $7, 1, 9, 5, 11, 3$. Note that these are simply the numbers $1, 3, 5, 7, 9, 11$ in a permuted order, and their sum is 5^2.

But how do we show this in general? It is easier than it looks. Consider the numbers in $S = \{n+1, n+2, n+3, \ldots, 2n-1, 2n\}$. For $i = 1, 2, \ldots, n-1, n$, let $n+i$ be written as

$$n + i = a_i \times 2^{b_i},$$

that is, $a_i \times$ some power of 2, where a_i is some odd positive integer and b_i a non-negative integer. Obviously, we must have $a_i \leq n + i$. Also, $n + i \leq 2n$; therefore $a_i \leq 2n$. Since a_i is odd, it follows that a_i is one of the numbers $1, 3, 5, \ldots, 2n-1$.

Next we ask: if it is possible for $a_i = a_j$ for a pair of unequal indices i and j? Suppose this is the case; say $a_i = a_j$ for some pair i, j (here $i \neq j$). Let a denote the common value of a_i, a_j. Then we have, by supposition:

$$n + i = a \times 2^{b_i},$$
$$n + j = a \times 2^{b_j}.$$

It cannot be that $b_i = b_j$, for this would mean that $n + i = n + j$, i.e., $i = j$; but we had supposed that $i \neq j$. Hence $b_i \neq b_j$. This means that one of the b's is larger than the other one.

Suppose that $b_j > b_i$. Then b_j exceeds b_i by at least 1, as both b_i and b_j are integers. This implies that

$$2^{b_j} \geq 2 \times 2^{b_i},$$

and hence that

$$n + j \geq 2(n + i).$$

On the other hand, the least number in S is $n + 1$, and the largest number is $2n$, and $2n$ is strictly less than twice $(n + 1)$. So it *cannot* happen that $n + j \geq 2(n + i)$ for some pair of numbers $i, j \in \{1, 2, \ldots, n\}$.

Therefore it cannot happen that $a_i = a_j$ for some pair $i \neq j$. In other words, the a_i's are all distinct from one another. Where does this leave us? The n odd numbers $a_1, a_2, \ldots, a_n$ all lie between 1 and $2n - 1$, and no two are the same. Hence it must be that the string $(a_1, a_2, \ldots, a_n)$ is simply a permutation of the string $(1, 3, \ldots, 2n-1)$! Therefore the sum of the a_i's is the same as the sum

$$1 + 3 + \cdots + (2n - 1)$$

and we know that this is equal to n^2.

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A Nines Multiples Problem

C⊗MαC

problem corner

The following is adapted from a problem that appeared in an online column on the Math Forum site (see http://mathforum.org/wagon/fall13/p1185.html; the original source: Felix Lazebnik, "Surprises", *Math Mag*. **87** (2014) 212–221).

> *Let N be an integer whose decimal digits are non-decreasing from left to right and having distinct digits as its two rightmost digits; for example,* 113445889*. Let s be the sum of the digits of* $9N$*. How large can s be?*

Let us try out different possible values of N and study the outcome.

- Let $N = 347$. Then $9N = 3123$, therefore $s = 3 + 1 + 2 + 3 = 9$.
- Let $N = 3557$. Then $9N = 32013$, therefore $s = 3 + 2 + 0 + 1 + 3 = 9$.
- Let $N = 126679$. Then $9N = 1140111$, therefore $s = 9$.
- Let $N = 1123566679$. Then $9N = 10112100111$, therefore $s = 9$.

Each time we get $s = 9$. Well! Is it possible that $s = 9$ no matter what the choice of N (provided only that it satisfies the stated conditions)? Let us see how we might go about investigating such a claim.

As a first step, let us ask: What significance can the conditions "non-decreasing from left to right" and "having distinct digits as its two rightmost digits" have? What happens if we relax these two conditions? Let's try ...: Take $N = 231$ (this violates the first condition); we get $9N = 2079$ and $s = 18$. Or take $N = 122$ (this violates the second condition); we get $9N = 1098$ and $s = 18$. Or take $N = 12322$ (this violates both the conditions); we get $9N = 110898$ and $s = 27$. In all these cases we get $s > 9$. What these examples show is that if we relax either of the conditions, or both of them, we may not get $s = 9$. Henceforth, let us assume that the two conditions are satisfied.

Here is an immediate and crucial consequence of the conditions placed on the number: N can be expressed as a sum of numbers whose digits are all 1s, in the following way,

$$N = 111 \ldots 111 + 11 \ldots 111 + 11 \ldots 11 + \cdots + 1 + 1 + \cdots + 1,$$

where each successive number has fewer than or the same number of 1s as the number preceding it, and there is at least one solitary 1 at the end (the number of solitary 1s is equal to the difference between the units digit and the tens digit of N).

For convenience, we write R_k to denote the number $111 \dots 11$ with k repetitions of 1 (e.g., $R_3 = 111$). Thus we may write: $224 = 111 + 111 + 1 + 1 = R_3 + R_3 + R_1 + R_1$. Here are a few more examples in support of the claim made above:

- $347 = R_3 + R_3 + R_3 + R_2 + R_1 + R_1 + R_1$
- $3557 = R_4 + R_4 + R_4 + R_3 + R_3 + R_1 + R_1$
- $126679 = R_6 + R_5 + R_4 + R_4 + R_4 + R_4 + R_2 + R_1 + R_1$
- $1123566679 = R_{10} + R_8 + R_7 + R_6 + R_6 + R_5 + R_2 + R_1 + R_1$

It follows that the number $9N$ can be expressed in the form

$$9N = 999 \dots 999 + 99 \dots 999 + 99 \dots 99 + \cdots + 9 + 9 + \cdots + 9,$$

where each successive number has fewer than or the same number of 9s as the number preceding it, and there is at least one solitary 9 at the end. The number of these 9s is equal to the difference between the units digit and the tens digit of N, so it is one of the numbers 1, 2, 3, 4, 5, 6, 7, 8.

Now we shall formulate a simple and interesting result. Let M_1 be a positive integer whose units digit is not 0. To M_1 we add a number M_2 made up of only 9s, the number of 9s in M_2 being at least equal to the number of digits in M_1. Then: *The sum of the digits of M_1 is equal to the sum of the digits of $M_1 + M_2$.* For example:

- Let $M_1 = 215$ and $M_2 = 999$. Then $M_1 + M_2 = 1214$. Observe that 215 and 1214 have the same sum of digits.
- Let $M_1 = 215$ and $M_2 = 9999$. Then $M_1 + M_2 = 10214$. Observe that 215 and 10214 have the same sum of digits.
- Let $M_1 = 329$ and $M_2 = 99999$. Then $M_1 + M_2 = 100328$. Observe that 329 and 100328 have the same sum of digits.

There is no mystery in this. Since M_2 is of the form $10^k - 1$ where k is at least equal to the number of digits in M_1, adding M_2 to M_1 results in a number with 1 and a few 0s appended to the left of M_1, and with units digit 1 less than that of M_1. No other digit changes occur since the units digit of M_1 is positive. So, naturally, the numbers M_1 and $M_1 + M_2$ have the same sum of digits.

Now we apply this result to the number

$$999 \dots 999 + 99 \dots 999 + 99 \dots 99 + \cdots + 9 + 9 + \cdots + 9.$$

We infer that no matter how many extra terms we add to the left (i.e., numbers of the form $999 \dots 9$, with the number of 9s never decreasing), the sum of digits of the number is always the same as that of the rightmost number. Since the sum of digits of the number at the extreme right is 9, the sum of digits always remains fixed at 9, which is exactly what we had observed.

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How To …

Solve A Geometry Problem — III

A Three Step Guide

Ajit Athle

problem corner

In this edition of 'Geometry Corner' we solve the following challenging problem.

Problem. In triangle ABC (Figure 1), D is a point on CB extended such that $AB = BD$. The bisector of $\measuredangle ABC$ meets AC at R. The midpoint of AC is M, and DM intersects BR in P. Prove that $\measuredangle BAP = \measuredangle C$.

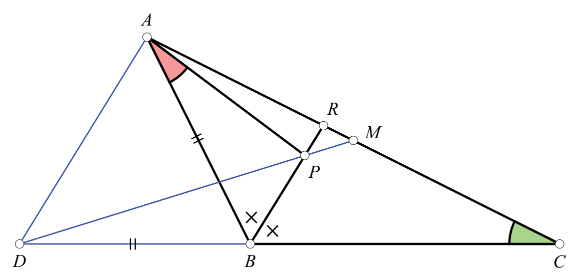


Figure 1.

Here we are required to prove a proposition, and this opens a new window of opportunity; for we can *assume* the proposition to be true and then study its implications, and this may give us a hint of how we should proceed; indeed, this is often an important first step in the exploration of a problem. In Figure 2 the given information has been recorded, and line AP has been extended to meet BC in Q. This kind of construction (a line in this case) is known as an 'auxiliary construction' which helps in the proof.

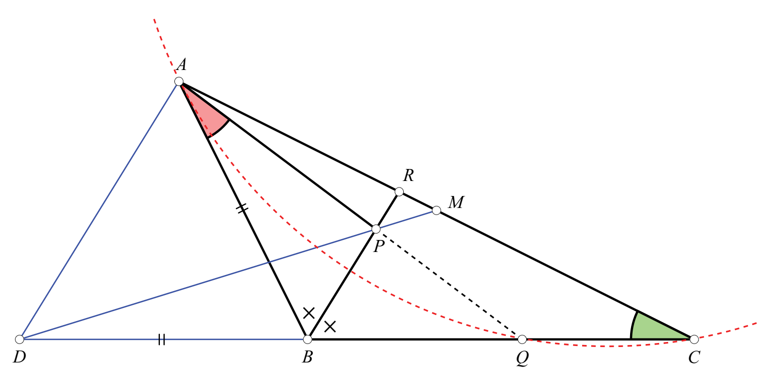


Figure 2.

The figure makes it evident that **if** the given proposition is true, then BA is tangent to the circumcircle of $\triangle AQC$ (shown dashed). Thus, what we really need to prove is $BA^2 = BQ \cdot BC$. For if we show this, then the

Power theorem ensures that BA is tangent to circle (AQC) and hence that $\measuredangle BAP = \measuredangle ACB$, by the Alternate Segment theorem. We shall now prove this relation, invoking several important theorems along the way. (See the Appendix for statements of these theorems.)

First we note that BP bisects $\measuredangle ABQ$, and hence (by the Angle Bisector theorem) that:

$$\frac{AB}{BQ} = \frac{AP}{PQ}. \tag{1}$$

We further observe that DPM is a transversal cutting through the sides of $\triangle AQC$. Hence we have by the theorem of Menelaus (see the Appendix for a statement of this theorem):

$$\frac{AP}{PQ} \cdot \frac{QD}{DC} \cdot \frac{CM}{MA} = -1, \quad \therefore \frac{AP}{PQ} \cdot \frac{DQ}{DC} \cdot \frac{CM}{MA} = 1. \tag{2}$$

(Note that $QD = -DQ$; we are using *directed line segments* here.) But $CM = MA$; hence by using (1) we get:

$$\begin{aligned} \frac{AB}{BQ} &= \frac{DC}{DQ} \\ &= \frac{DB + BC}{DB + BQ} = \frac{AB + BC}{AB + BQ}, \end{aligned}$$

since $DB = AB$. On simplification we obtain:

$$AB^2 + AB \cdot BQ = AB \cdot BQ + BQ \cdot BC,$$

$$\therefore AB^2 = BQ \cdot BC.$$

This implies that AB must be tangent to the circle passing through points A, Q, C, and hence $\measuredangle BAQ = \measuredangle BAP = \measuredangle C$ by the Alternate Segment theorem.

Note how we have achieved our objective with the use of many different theorems — the Angle Bisector theorem, the theorem of Menelaus, the Alternate Segment theorem and the Power theorem.

It is evident that the student needs to be familiar with many postulates regarding lines, angles, triangles and circles, for one can never know which theorem or property of a figure may come in handy in solving a given problem. Further, the importance of construction of auxiliary lines needs to be understood well since this often paves a way to a solution which is otherwise not readily evident.

Appendix: Some standard theorems of plane geometry

Power theorem: *If P is the intersection point of two chords AB and CD of a circle, extended if needed, then $PA \cdot PB = PC \cdot PD$.*

Converse: *If A, B, C, D are four distinct points, no three of which are collinear, and P is the intersection point of lines AB and CD, then the equality $PA \cdot PB = PC \cdot PD$ implies that A, B, C, D lie on a circle.* (See Figure 3 (a).)

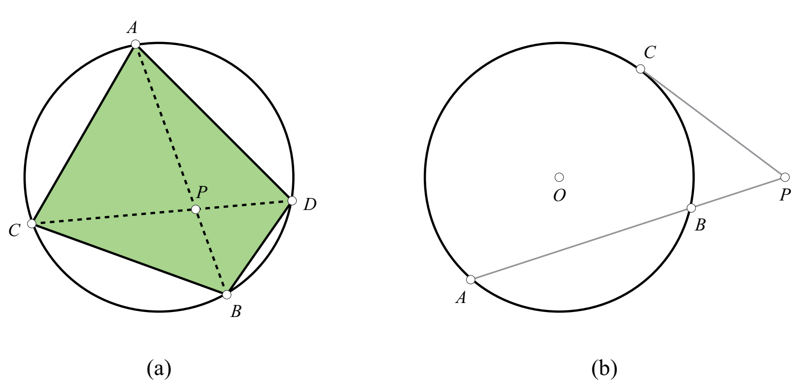


Figure 3.

Special case: *If A, B, C are distinct non-collinear points, and P is a point on the extension of AB such that $PC^2 = PA \cdot PB$, then the circle through A, B, C is tangent to line PC at C.* (See Figure 3 (b).)

Alternate Segment theorem: *Let line ABC be tangent to a circle ω at B; let BD be a chord of ω. Let E be a point on the circle, on the same side of line BD as A. Then $\measuredangle DBC = \measuredangle BED$.* (See Figure 4.)

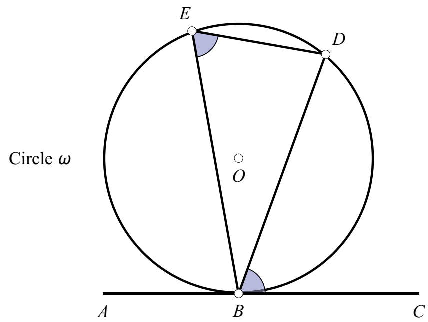


Figure 4.

Theorem of Menelaus: *Let line ℓ pass through triangle ABC, intersecting its sides BC, CA, AB (extended, if needed) at points D, E, F respectively. Then* (Figure 5):

$$\frac{BD}{DC} \times \frac{CE}{EA} \times \frac{AF}{FB} = -1.$$

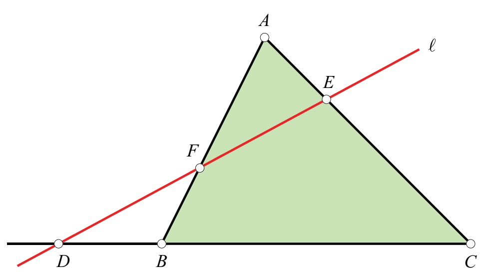
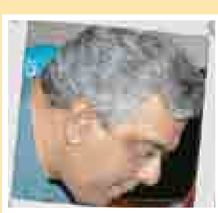


Figure 5.

Remark: The theorem makes use of *directed line segments*. Here, $BC = -CB$ for any pair of points B, C; and if D is a point on line BC, then $BD + DC = BC$, regardless of whether or not D lies between B and C.



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problem corner

Problems for the Middle School

Problem Editor : R. ATHMARAMAN

Problems for Solution

Problem III-3-M.1
Amar, Basil, Celia and Dharam are four children. Basil's age is greater than twice Amar's age; the sum of Amar's and Celia's ages is less than Basil's age. Dharam is older than Basil. If Celia is 6 years old, and Dharam is 9 years old, find Basil's age. (All ages are in whole numbers).

Problem III-3-M.2
Mary's teacher notes the test scores of 32 students in her class. She finds that the median score is 80 and the range of the scores is 40. ('Range' is the difference between the highest and lowest score.) The teacher then tells the class that their average score is 58. Mary contends that her teacher has gone wrong somewhere. Who is right, Mary or her teacher? [*Fryer Contest, 2003*]

Problem III-3-M.3
Select 50 distinct integers from the first 100 natural numbers, such that their sum is 2900. What is the least possible number of even integers amongst these?

Problem III-3-M.4
Find the digits A and B if the product $2AA \times 3B5$ is a multiple of 12. (Find all the possibilities.)

Problem III-3-M.5
One of the altitudes of a triangle is tangent to its circumcircle. Prove that some angle of the triangle has measure larger than 90° but less than 135°.

Solutions of Problems in Issue-III-2 (July 2014)

Solution to problem III-2-M.1 *What is the least multiple of* 9 *which has no odd digits?*

The digital sum of such a number must be a multiple of 9. A sum of even numbers cannot be odd, so the sum must be at least 18. The smallest number with digital sum 18 is clearly 99. But 99 has no even digits and does not satisfy our requirement.

Hence the smallest required number will have at least three digits. It cannot start with 1, as 1 is odd; hence we commence searching for a three-digit number starting with 2. The first such number is 288, which satisfies the given condition. So this must be the answer.

Solution to problem III-2-M.2 *Which number is larger:* 31^{11} *or* 17^{14}*?*

We have:

$$31^{11} < 32^{11} = (2^5)^{11} = 2^{55},$$
$$17^{14} > 16^{14} = (2^4)^{14} = 2^{56}.$$

Therefore, 17^{14} is the larger number.

Solution to problem III-2-M.3 *What is the remainder when* 2015^{2014} *is divided by* 2014*?*

Since $2015 = 2014 + 1$, a power of 2015 is of the form

$$(2014+1)\times(2014+1)\times(2014+1)\times\cdots\times(2014+1).$$

If we imagine this product expanded out, it must have the form

$$(\text{some multiple of } 2014) + 1.$$

Hence the remainder is 1.

Solution to problem III-2-M.4 *Find the least natural number larger than* 1 *which is simultaneously a perfect square, a perfect cube, a perfect fourth power, a perfect fifth power and a perfect sixth power. How many such numbers are there?*

We are given an equation

$$a^2 = b^3 = c^4 = d^5 = e^6 = n,$$

where a, b, c, d, e, n are natural numbers. It is clear that $n = m^{60}$ for some natural number m. Here, note that 60 is the least common multiple (LCM) of 2, 3, 4, 5, 6. Thus the numbers of the required form are

$$1,\ 2^{60},\ 3^{60},\ 4^{60},\ \ldots.$$

It follows that the least such number exceeding 1 is 2^{60}.

Solution to problem III-2-M.5 Apologies to the reader: there was an error in the statement of this problem. Here's how it should read:

*A group of ten people (men and women), sit side by side at a long table, all facing the same direction. In this particular group, ladies always tell the Truth while the men always lie. Each of the ten people announces: "There are more men on my left, than **ladies** on my right." How many men are there in the group? (This problem has been adapted from the Berkeley Math Circle, Monthly Contests.)* (The word in **bold** had been missed out in the version given in the July 2014 issue.)

Label the people from 1 to 10, from left to right ('left' or 'right' according to their own perspective). We argue in alternation from the two ends of the line, as follows.

- The person labeled 1 is lying, as there is no one to his/her left. Hence it is a Man.
- The person labeled 10 has no one to the right, and at least one Man (label 1) to the left. Hence this person is telling the truth and so must be a Lady.
- The person labeled 2 has a Man (label 1) to his/her left, and at least one Lady (label 10) to the right. So this person's statement is false. Hence it must be a Man.
- The person labeled 9 has exactly one Lady to the right (label 10) and at least two Men (labels 1 and 2) to the left. So this person's statement is true. Hence it must be a Lady.

Arguing this way, we find that labels 3, 4, 5 stand for Men while 6, 7, 8 stand for Ladies. Hence there are five Men in the line.

Problems for the Senior School

Problem Editors : PRITHWIJIT DE & SHAILESH SHIRALI

Problems for Solution

Problem III-3-S.1
If $(x - y + z)^2 = x^2 - y^2 + z^2$ then prove that either $x = y$ or $z = y$.

Problem III-3-S.2
Prove that the numbers of the form $10017, 100117, 1001117, \ldots$ are all divisible by 53.

Problem III-3-S.3
Let $ABCD$ be a parallelogram. Let the bisector of $\measuredangle ABD$ meet CD produced at X and let the bisector of $\measuredangle CBD$ meet AD produced at Y. Prove that the bisector of $\measuredangle ABC$ is perpendicular to XY.

Problem III-3-S.4
Prove that if $a_1 \le a_2 \le a_3 \le \ldots \le a_{10}$, then

$$\frac{a_1 + \ldots + a_6}{6} \le \frac{a_1 + \ldots + a_{10}}{10}.$$

Solutions of Problems in Issue-III-2 (July 2014)

Solution to problem III-2-S.1 *Let n be a positive integer not divisible by 2 or by 5. Prove that there exists a positive integer k, depending on n, such that the number $111\ldots1$, where the digit 1 is repeated k times, is divisible by n.*

Let $R_k = 111\ldots11$, with 1 repeated k times. Let the numbers $R_1, R_2, R_3, \ldots$each be divided by n. Each of these divisions yields a remainder which is one of the numbers $\{0, 1, 2, \ldots, n-1\}$. Since there are only n possible remainders, it must happen that some remainder is repeated for the first time. Suppose that R_a and R_b leave the same remainder on division by n (with $b > a$). Then $R_b - R_a$ is divisible by n. But

$$R_b - R_a = \underbrace{111\ldots1}_{(b-a)\ \text{ones}} \times 10^a = R_{b-a} \times 10^a.$$

To see why this is so, it helps to study individual cases. Consider for example:

$$R_5 - R_3 = 11111 - 111 = 11000 = 11 \times 10^3,$$
$$R_6 - R_2 = 111111 - 11 = 111100 = 1111 \times 10^2.$$

Hence $R_{b-a} \times 10^a$ is divisible by n. But we know that n is not divisible by either 2 or 5; so n is coprime to 10. Hence it must be that R_{b-a} is divisible by n. Therefore, R_k is divisible by n for $k = b - a$. This proves the claim.

Solution to problem III-2-S.2 *Let $\mathbb{R}$ be the set of all real numbers, and let $b \neq \pm 1$ be a real number. Determine a function $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) + bf(-x) = x + b$, for all $x \in \mathbb{R}$.*

Writing $-x$ for x in the given functional equation we get $f(-x) + bf(x) = -x + b$. Solving the system of simultaneous equations,

$$f(x) + bf(-x) = x + b,$$
$$f(-x) + bf(x) = -x + b.$$

we get

$$f(x) = \frac{x}{1-b} + \frac{b}{1+b}, \qquad x \in \mathbb{R}.$$

It is easy to verify that $f(x)$ obtained above does satisfy the given equation.

Solution to problem III-2-S.3 *Determine all three-digit numbers N such that: (i) N is divisible by 11, (ii) $N/11$ is equal to the sum of the squares of the digits of N.* (This problem appeared in the International Mathematical Olympiad 1960.)

Let the digits of N be a, b, c, so that $N = 100a + 10b + c$. Since $11|N$, either $b = a + c$ or $b + 11 = a + c$, giving $N = 110a + 11c$ or $N = 110a + 11c - 110$; so either $N/11 = 10a + c$ or $N/11 = 10a + c - 10$. Now consider condition (ii). Assuming that $b = a + c$ we get $10a + c = a^2 + (a + c)^2 + c^2$, and so:

$$2a^2 + 2a(c-5) + 2c^2 - c = 0.$$

Looking at each term in succession, we deduce that c is even; so $c \in \{0, 2, 4, 6, 8\}$. Viewing the above equation as a quadratic equation in a, its discriminant is:

$$4(c-5)^2 - 8(2c^2 - c) = -4(3c^2 + 8c - 25).$$

For $c = 0, 2, 4, 6, 8$ the values taken by the discriminant are $100, -12, -220, -524, -924$. As the equation must have integer roots, the discriminant must be a perfect square. There is just one square value in the above list (namely: 100), taken when $c = 0$. For this value the quadratic equation takes the form $a^2 - 5a = 0$, which yields $a = 0$ or $a = 5$. The former value yields $N = 0$, which is not of interest. The latter value yields $N = 550$. We may verify that this does satisfy the given conditions.

Now consider the other possibility, that $b + 11 = a + c$. This yields the equation $10a + c - 10 = a^2 + (a + c - 11)^2 + c^2$, which simplifies to:

$$2a^2 + a(2c - 32) + 2c^2 - 23c + 131 = 0.$$

Looking at the parity of each term, we deduce that c is odd; so $c \in \{1, 3, 5, 7, 9\}$. Next, viewing the above equation as a quadratic equation in a, its discriminant is $-4(3c^2 - 14c + 6)$, whose values for $c = 1, 3, 5, 7, 9$ are $20, 36, -44, -220, -492$. There is just one square value in this list (namely: 36), taken when $c = 3$. For this value the equation simplifies to $a^2 - 13a + 40 = 0$, which yields $a \in \{5, 8\}$. The first value yields $b < 0$, so we discard it. The second value yields $N = 803$, which does satisfy the given conditions.

It follows that there are two solutions to the given equation: $N = 550$ and $N = 803$.

Solution to problem III-2-S.4 *You are given a right circular conical vessel of height H. First, it is filled with water to a depth $h_1 < H$ with the apex downwards. Then it is turned upside down and it is observed that water level is at a height h_2 from the base. Prove that*

$$h_1^3 + (H - h_2)^3 = H^3.$$

Can h_1, h_2 and H all be positive integers?

Let α be the semi-vertical angle of the cone. If the cone is held upside down, then at a distance x from the apex the radius of the circular cross-section is $x \tan\alpha$. Thus the volume of a conical section of height x is proportional to x^3. Equating the volume of water in the two different positions we get

$$h_1^3 = H^3 - (H - h_2)^3.$$

Rearranging we get $h_1^3 + (H - h_2)^3 = H^3$.

All three of h_1, h_2 and H cannot be integers, for if they were then Fermat's Last Theorem will be violated (the cubic case).

Solution to problem III-2-S.5 *Let $\{a_n\}_{n=1}^{\infty}$ be a sequence defined as follows: $a_1 = 3$, $a_2 = 5$ and:*

$$a_{n+1} = |a_n - a_{n-1}|, \quad \textit{for all } n \geq 2.$$

Prove that $a_k^2 + a_{k+1}^2 = 1$ for infinitely many positive integers k.

The first few entries of the sequence are

$$3, 5, 2, 3, 1, 2, 1, 1, 0, 1, 1, 0, 1, 1, 0, 1, 1, 0, \ldots.$$

Note that $a_7 = 1$, $a_8 = 1$, $a_9 = 0$, which leads to $a_{10} = 1$, $a_{11} = 1$, $a_{12} = 0$, and the pattern now repeats $(1, 1, 0, 1, 1, 0, \ldots$, endlessly). The conclusion follows.

review

Review of Kim Plofker's *Mathematics in India*

Plofker offers a coherent historical account of mathematics in India, found in Sanskrit, starting from the Vedic period, until the seventeenth century. She carefully adheres to documentary historical evidence, and points to controversies on dating some of the literature. The book is very well written and provides a resource that mathematics teachers in India can greatly benefit from.

R. RAMANUJAM

I. Between ignorance and pride

As a child growing up in Tiruchirappalli, I used to marvel at the *Rājagōpuram*, the majestic tower that welcomed us into the Srirangam temple. I recall my teacher in school asking us to think about how we might determine the height of the tower. He had the hidden agenda (which we easily saw through) of discussing trigonometry. While I used to admire the thousands of sculptures, the music played daily, and the beauty of the Tamil and Sanskrit verses we heard, in that and other temples around me, it never occurred to me to ask how mathematics might have been used in the construction of the temples.

Indeed, I saw Sanskrit and Tamil as classical languages that I was trained in, with great literature in them. If someone had told me that this literature included significant scientific and mathematical knowledge, I would not have believed any of it. Nobody did tell me either.

On the other hand, I had been raised on a staple diet of 'modern' mathematics that apparently had Greek origins, but was mostly developed in Europe in the last six hundred years or so, spiced with mysterious references to a 'glorious past' in India. Ancient India had come up with the number zero (presumably because of its philosophical predilections) and thus the positional number system, and thus was *really* responsible for **all** mathematical knowledge in an essential sense. Indian mathematicians had been great in algebra (though I had no clue in what way), but seriously speaking, I thought, India contributed to the foundation and little else, Europeans actually built the edifice that I came to love. I knew famous names like Aryabhaṭa, Bhāskara and Brahmagupta, but little of their actual work. The mathematics I studied, especially at the higher secondary level or at University, was clearly 'modern'. I could not conceive of Indian scholars, writing in Indian languages, producing any such mathematics.

Why, you may wonder, this public confession of such appalling historical ignorance: only because I later discovered that scores of teachers of mathematics that I interacted with were similarly afflicted. A history of the development of ideas in mathematics is usually not considered a prerequisite for teaching the subject, and cursory acquaintance provided in textbooks is of little help. University mathematics is studded with X's theorem and Y's formula, where X and Y are almost never Mishra or Chung. Popular books like E. T. Bell's *Men of Mathematics* reinforce this view, so it is not surprising. (*Comment.* Whether mathematics teachers *need* a historical perspective to teach mathematics well is a relevant question, but it will take us too far afield to discuss it here.)

For people of this history-deprived ilk, let me just mention two pearls from Plofker's account. One is Madhava's early 15th century discovery of infinite series for the sine, cosine and arctangent functions. Note that this was 200 years before Gregory's series $\left(\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \cdots\right)$ and Newton's calculus. The other is Brahmagupta's 7th century theorem on the area of a quadrilateral inscribed in a circle. This can be proved easily but involves significant algebraic manipulation. However, we have no idea whether Brahmagupta did use much algebra.

This is perhaps an appropriate place to also remark on another brand of pride in Indian mathematics, which tends to be altogether jingoistic. Typically, this covers 'ancient' Indian mathematics with gold and glory, attributing all kinds of mathematical knowledge to wise folk 'thousands of years ago'. Indeed, 'Vedic mathematics' is touted these days as a cure for all mathematical maladies among school students. This brand tends to be nationalistic and ahistorical, often reveling in false pride.

In this scenario, Kim Plofker's book is not only most welcome, but also needed in that it serves to provide a much needed cultural and historic perspective that is so clearly lacking in many of us. Reading it, one embarks on a fascinating journey into India, over centuries, and visits beautiful mathematical landscapes. It is but natural to feel a sense of pride in the achievements of ancestors from one's own geographical regions; the pleasure is much deeper when we understand which specific achievements we are admiring, placing them in their own historical context.

Kim Plofker's *Mathematics in India* was published in 2008, and has already achieved prominence as an authoritative text on its subject. However, it does not yet seem to be as well known among Indian teachers of mathematics as it deserves to be.

II. The Content

Throughout, Plofker uses the term India to denote the entire Indian subcontinent, and it is important to note this, since early Vedic culture flourished on the banks of the river Indus, a region that is in the confluence of modern day Afghanistan, Pakistan and India. I will first outline a chapter by chapter description of the book's contents.

The book opens with a discussion of the difficulties that any historian of mathematics faces in the Indian tradition: lack of documentation, conflicting accounts and claims, and intermingling of legends and factual records. The author offers a stunning instance (in the Preface, page vii) of two reputed authors, appearing side by side in the same volume, attributing the emergence of quantitative astronomy in Vedic India to periods that differ by as much as *two thousand years*! The author goes on to give a brief account of historiography in this context, and the form of Sanskrit literature in which knowledge was expressed.

Chapter 2 on Vedic India, discusses mathematical thought in the Vedas, which were canonical texts by the middle of the first millennium BCE. *(Note for readers*: 'BCE' stands for "Before the Common Era" and is used interchangeably with 'BC'. Similarly, 'CE' has the same meaning as 'AD'.) The cosmic significance of numbers and arithmetic in rituals and the geometry in different sizes and shapes of fire altars are discussed. *Śūlba-sūtras*, the rules of cords (or ropes), the earliest of which were composed by Baudhayana, offered many constructions in plane geometry, for instance, ways of transforming rectangles into squares or circles. The *sūtras* included a statement of the theorem we usually attribute to Pythagoras, and reasonably good approximations to $\sqrt{2}$ and π. Plofker discusses controversies relating to mathematical ideas in Vedic astronomy and astrology. What is striking is the systematization of large numbers, an elaborate system that is not positional but includes factorization.

The next chapter takes us into the centuries just before and just after the turn of the Common Era. Importantly, it discusses the work of *Pānini* (fifth century BCE) and *Pińgala* (third century BCE), in shaping Sanskrit grammar and the mathematical ideas contained therein. The chapter contains an attractive analysis of metric structure in poetry and its relation to binary representations. Important ideas like the use of rewrite rules and recursion (central to modern computer science) inform Pāninian grammar. The chapter also discusses trigonometry and controversy related to whether these were Greek transmissions (since Alexander's invasion brought such contact).

It is hard to separate mathematics and astronomy in Sanskrit texts and Chapter 4 takes up this linkage. The structure and content of *siddhāntas* is elaborated. *Āryabhatīyam*, dated to 499 CE, is a masterly treatise giving an analysis of planetary motion based on epicycles. Its notation is most intriguing, and should provide considerable amusement to readers. The use of linear interpolation techniques is the highlight of this chapter. In the calculation of sines, Indian astronomers recorded only 24 values in steps of 3.75 degrees, which made for easy memorization, and the rest were calculated by interpolation. The fascinating method of three dimensional projections, using right angles *inside* the sphere, is elucidated nicely by Plofker.

This chapter illustrates an important difference between Greek and Indian mathematicians. The latter were primarily applied mathematicians, who were principally interested in providing algorithms and their justifications. Memorizing short tables and learning techniques for generating longer ones shaped the way they approached problems. Moreover, they were not committed to any particular geometric model for astronomical phenomena, which allowed them wide experimentation.

The 'medieval period', with its connotation of 'dark ages' is considered one of intellectual stagnation in European accounts. Chapters 5 and 6, ranging through the work of *Āryabhata*, *Bhāskara I*, *Mahāvīra, Bhāskara II* (also known as Bhāskarachārya) and others, in a period from the sixth to the twelfth centuries CE, show fascinating mathematical development. It is no exaggeration to say that mathematics emerges as a discipline of some kind in this period. Mahāvīra's ninth century work *Gaṇita-sāra-sańgraha* is a text devoted solely to mathematics. The author summarizes verses and offers an account of trigonometry and also arithmetic and (extensive) algebra, in modern notation.

Every single section of these chapters (including expositions of the work of Mahāvīra above, Līlāvati, Bīja-gaṇita, the work of Nārāyana Pandita) is worth reading in detail. As an appetizer, let me mention the seventh century development of the arithmetic of negative numbers by Bhāskarachārya, which would not appear in Europe until a thousand years later. Another is Brahmagupta's formula for the area of a quadrilateral inscribed in a circle. A brilliant account is the work on 'Pell' equation by Brahmagupta: $x^2 - Ny^2 = c$, showing that solutions for c_1 and c_2 can be 'multiplied' to give one for $c_1 c_2$. The method of finite differences (for sines) developed led to a full theory of a calculus of polynomials, and for pretty much everything needed to solve problems related to spheres and circles in astronomy. Bhāskarachārya's division of the surface of the sphere using latitudes to calculate area and volume, can only be termed as sheer brilliance.

In these chapters, Plofker discusses not only calculation and symbolic manipulation but also the role of proof in the treatises, the mathematical culture, and its relation to society. Problem solving for various commercial applications and puzzle solving for amusement are illustrated. Yet again we see the primacy of problem posing and solving, as opposed to theory building, but also rootedness in social practice.

Chapter 7 presents the most precious jewel of Indian mathematics, namely the work of the Kerala school, from the 14th to the 17th centuries, which includes the discrete fundamental theorem of calculus. Mādhava (approximately 1350 to 1425), the founder of the school, is one of the great mathematicians of that time. The school provided power series expansions of sine, cosine and arctangent. Nīlakaṅtha's (fifteenth century) model of planets moving in eccentric circles is another major achievement of this school. The explanations and rationale they provide is also methodologically interesting.

India had extensive contacts with the Islamic world, and Islamic scholars were the principal intermediaries of transmission between India and Europe for much of the last millennium. Chapter 8 discusses this interaction, raising intriguing questions about why certain theories and methods were absorbed into Indian mathematics and not others. However, the discussion is too brief to get a clear understanding of the issues involved. The book then concludes in Chapter 9 with what we may call the colonial encounter. Direct relations with European culture and education brought both mutual interest and mistrust, shaped by power equations and imperialist attitudes to bringing India 'out of the dark ages'.

III. Some examples

It would be disappointing to talk about a book that presents mathematics and not talk mathematics at all. Let me present a few examples that illustrate not only the richness of content, but also the trouble taken by Plofker to present it in modern notation along with a translation of the original verses.

III.1. Transformations of squares and rectangles. This is from the Vedic period, and lets us calculate the side of a square whose area equals the sum of, or difference between, the areas of two given squares. Consider Figure 1(a) where we are given squares *ABCD* and *EFGH*, and $KA = LD = FE = GH$. Then *LA* is the side of the 'sum' square, and *MD* (obtained by locating point *M* on *AD* such that $LM = LK$) is the side of the 'difference' square.

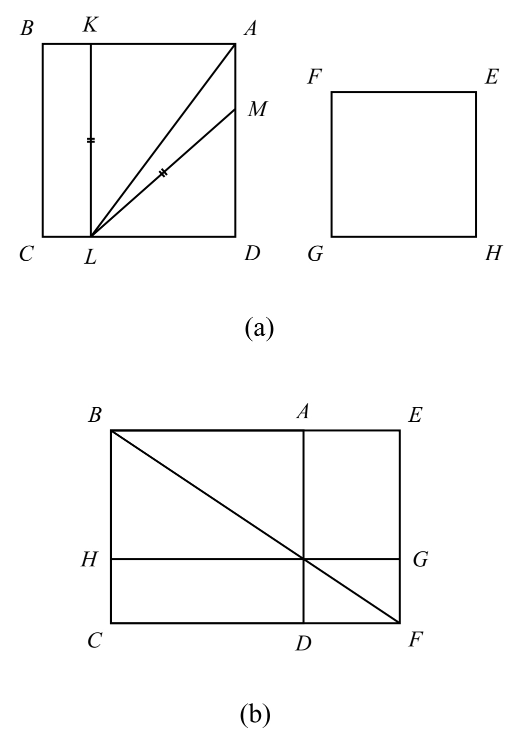

Figure 1. Adapted from Figure 2.3, page 22, of Kim Plofker's book

To transform a given square into a rectangle with a given side and having the same area, consider the illustration in Figure 1(b). Given the square *ABCD*, expand its side to the given length *BE*, forming the rectangle *BEFC*. Its diagonal *BF* defines the rectangle *BEGH* (with *GH* passing through the point of intersection of AD and BF), whose area equals that of $ABCD$.

III.2. Sine values. Consider values for the sine function from Varāhamihira's *Pańcasiddhāntikā*, around sixth century CE.

Let R be the radius of a circle. Consider an arc of the circle which subtends an angle θ at the centre of the circle. Let Sin θ refer to the length of the chord corresponding to this arc (so it is a length rather than a ratio). Then:

- $\text{Sin } 30° = \sqrt{R^2/4}$; $\text{Sin } 45° = \sqrt{R^2/2}$; $\text{Sin } 60° = \sqrt{R^2 - \text{Sin}^2 30°}$;
- $\text{Sin } (90° - \theta) = \sqrt{R^2 \text{ Sin}^2\theta}$;
- $\text{Sin } \theta = \sqrt{60(R - \text{Sin}(90° - 2 \cdot \theta))}$.

Comment. The third rule is accurate only when $R/2 = 60$; the verse itself offers a rough approximation of $D = \sqrt{(360)^2/10}$.

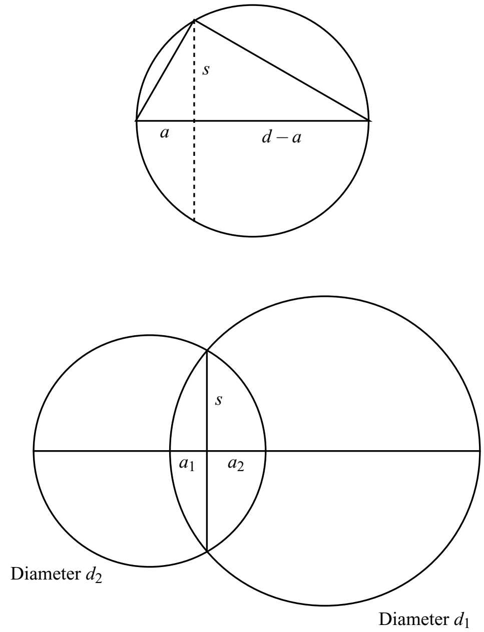

Figure 2. Adapted from Figure 5.3, page 131, of Kim Plofker's book

III.3. Āryabhaṭa. Consider the right triangles in Figure 2(a). It is easy to see by similarity of triangles that $\frac{d-a}{s} = \frac{s}{a}$. Now consider the two circles in Figure 2(b) with diameters d_1 and d_2, and let $a = a_1 + a_2$. We thus have $a_1 \cdot (d_1 - a_1) = s^2 = a_2 \cdot (d_2 - a_2)$. Now, with some easy manipulation, we get a rule given by Āryabhaṭa.

$$a_1 = \frac{(d_2 - a) \cdot a}{(d_1 - a) + (d_2 - a)},$$

$$a_2 = \frac{(d_1 - a) \cdot a}{(d_1 - a) + (d_2 - a)}$$

III.4. Brahmagupta. Consider quadrilateral $ABCD$ inscribed in a circle as in Figure 3, with diagonals AC and BD intersecting at H at right angles. Let GHJ be the line through H, perpendicular to CD. Then Brahmagupta's theorem states that J is the midpoint of AB. For proof, let BE and AF be the perpendiculars from B and A to CD. Then Brahmagupta argues that

$$JH = \frac{(BE - HG) + (AF - HG)}{2},$$

and from this we see that the 'height' of J is halfway between those of B and A, i.e., $JG = (BE + AF)/2$.

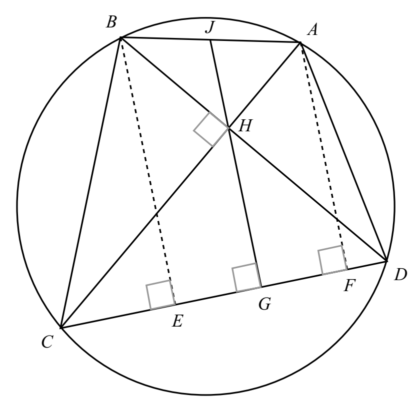

Figure 3. Adapted from Figure 5.6, page 146, of Kim Plofker's book

III.5. Bhāskara II. It is difficult to pick only one or two from the many beautiful results of Bhāskara II (from the 12th century).

Here is one from the *Līlāvatī*: suppose that we wish to find (rational) x, y such that $x^2 \pm y^2 - 1 = z^2$ for z an integer, or fraction. Assume a number a and let $x = 8a^4 + 1$, $y = 8a^3$. Then the right hand side becomes $16a^4(4a^4 \pm 4a^2 + 1)$, which is a perfect square. (This is from verses 56–72 of *Līlāvatī*, discussed on pages 186–187 of Plofker's book.)

Here is one from the *Bīja-gaṇita*. Suppose we want to solve the equation $Nx^2 + 1 = y^2$. When we already know a solution *a, b* for the 'auxiliary' equation $Na^2 + k = b^2$ for some *k*, we find an integer *m* such that $\frac{am+b}{k}$ is also an integer and $|m^2-N|$ is minimized. (Finding such *m* is called the *pulverizer technique*, going back to Āryabhaṭa.)

Now, $N.1^2 + (m^2 - N) = m^2$ trivially, so we can 'compose' these equations to get a new one:

$$N(am + b)^2 + k(m^2 - N) = (bm + Na)^2.$$

We then see that $(bm + Na)$ is also divisible by *k*, and hence we get yet another auxiliary equation $Na_1^2 + k_1^2 = b_1^2$, where

$$a_1 = \frac{am+b}{k}, \qquad k_1 = \frac{m^2-N}{k},$$
$$b_1 = \frac{bm+Na}{k},$$

and a_1, b_1, k_1 are integers. If k_1 is equal to any of the desired values 1, ±2, ±4, we are done. Otherwise, we have come a full circle, and start over again. Choose an integer m_1 such that $\frac{a_1m_1+b_1}{k_1}$ is also an integer, and so on. We repeat this until k_1 attains one of the desired values and we get a solution to $Nx^2 + 1 = y^2$. This is called the 'cyclic' method.

There is way too much in the *Līlāvatī* and the *jyotpatti* to pick any one, but here is a trigonometric identity, a real gem: let *R* be a radius of a circle and *α*, *β* be given arcs. Then

$$\operatorname{Sin}(\alpha \pm \beta) = \frac{\operatorname{Sin}\alpha \operatorname{Cos}\beta \pm \operatorname{Sin}\beta \operatorname{Cos}\alpha}{R}.$$

III.6. Mādhava. Below, let *C* denote the circumference of a circle, *D* its diameter, and *R* its radius.

- The Mādhava - Leibniz series for *π*

$$\pi \approx \frac{4D}{1} - \frac{4D}{3} + \frac{4D}{5} \ldots + (-1)^{n-1}\frac{4D}{2n-1} + (-1)^n \frac{4Dn}{(2n)^2+1}$$

- The Mādhava - Gregory series for the arctangent (Here we assume that Sin *θ* < Cos *θ*.)

$$\theta = \frac{R\operatorname{Sin}\theta}{1\operatorname{Cos}\theta} - \frac{R\operatorname{Sin}^3\theta}{3\operatorname{Cos}^3\theta} + \frac{R\operatorname{Sin}^5\theta}{5\operatorname{Cos}^5\theta} - \ldots$$

- The Mādhava - Newton series for the Sine:

$$\operatorname{Sin}\theta = \theta - \left(\frac{\theta^3}{R^2 \cdot 3!} - \left(\frac{\theta^5}{R^4 \cdot 5!} + \left(\frac{\theta^7}{R^6 \cdot 7!} - \ldots\right)\right)\right)$$

IV. Filling a huge gap

The story of mathematics in India, dating from Vedic times to the 1600s, is one that needed to be told in detail, and Plofker tells the tale admirably. Its chief virtue is that scholarly commitment is never compromised, and historical record is carefully adhered to. What it offers is an informed understanding of the tremendous achievements of mathematicians in India, their methods and style. There is much to be proud of, and there is no need to exaggerate claims either.

I was a little disappointed at the lack of reference to non-Sanskritic work in mathematics in India, but then historical sources on these are very few indeed. Another source of disappointment was the lack of any discussion on links with Chinese mathematics, another highly developed one close by, with many Chinese travellers carrying texts between the countries. Once again, one wonders what documentary evidence is available.

But these are relatively trivial complaints. For teachers, the book carries some important messages, in my opinion. One is a historical understanding of the development of key ideas. This is much needed if one is to see mathematics as a creative process (rather than as received wisdom). Another is contextualization, often claimed to be lacking in mathematics education. Seeing the context in which many methods arose can be of great help in this regard.

Yet another appeal of this story for teachers is that it provides an appreciation of how mathematics can develop in many different ways. A largely applied body of mathematical knowledge, with mostly informal and verbal justifications, can still lead to amazing discoveries of great mathematical richness, as evidenced in Indian mathematics. Perhaps this can help in removing the straitjacket that chokes many students of mathematics in schools.

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The Closing Bracket . . .

One of the peculiarities of mathematics as a discipline, shared to some extent with the sciences, is the fact that it is possible to teach and learn the subject as though it has no history — as though it emerged as a whole from the skies and landed on our table, neatly packaged. This is the way the author learned the subject through school; in all probability it is the way that a majority of our readers too learned it. One wonders why such a situation maintains. Is it because of our haste to advance the functional role of the subject, to focus only on that which has utility? Such thinking may play a role but the reality is surely more nuanced. Does it have something to do with the Platonic view of the subject, in which mathematics is viewed as an idealized perfect world (as different from the imperfect nature of man), and to learn about such perfection it is not necessary to learn about the follies and foibles of man? Again, one can only say “perhaps”.

It is difficult to assign a single cause to this phenomenon. But what matters is that it exists — and it need not! We do not need to argue whether giving a sense of the glorious history behind mathematics while teaching it will enhance the quality of learning of the subject. It may or may not, but what it will certainly do is bring a humanizing element into the curriculum, and that is badly needed: to internalize the fact that mathematics is above all a human endeavour with an amazing and impressive past, stretching back thousands of years. Here is what Manjul Bhargava, one of the Fields medalists of 2014, has to say about this matter: “My sense is that mathematics is sometimes not taught in India as a subject in itself . . . . It is taught to be a tool for engineering and an eventual engineering career. Students in India should be taught about the great Indian legacy of mathematicians, since ancient times, like Panini, Pingala, Hemachandra, Aryabhata, Bhaskara, Brahmagupta, Madhava, for example, and more recently Ramanujan, etc. Their stories and works inspired me, and I think they would inspire students across India. Many of these works were written in Indian languages in beautiful poetry with the flavor of Indian stories, and contain some of the most important breakthroughs in the history of mathematics. I think it would be beneficial if young Indian children were also exposed to their stories, just as I was as a child!”

The importance of problem solving in the pedagogy of mathematics is well known. If we marry this fact with the historical perspective, a wonderful new amalgam emerges: problem *studies rooted in history*. This is a rich classroom resource, highly underutilized, and simply asking to be used! Here are some examples that quickly come to mind: computation of square roots (ideas from Babylonian times, from the Bakhshali manuscript, and more); pre-calculus computation of the slope of a curve, using geometrical ideas (done independently by Fermat and Descartes); the occurrence of Fibonacci numbers in the works of Hemachandra and Bhāskarā. It would be wonderful if we could bring such inputs into the classroom. Let’s make it happen!

— Shailesh Shirali

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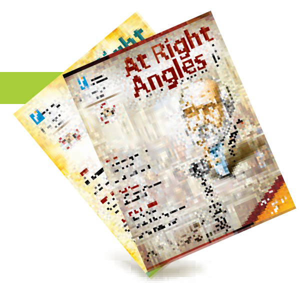

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