Azim Premji University

A publication of Azim Premji University together with Community Mathematics Centre, Rishi Valley

COMMUNITY MATH CENTRE · RISHI VALLEY SCHOOL (KFI)

At Right Angles

A RESOURCE FOR SCHOOL MATHEMATICS

Volume 3, No. 2
July 2014

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Mathematics and Kolam

The images on the cover highlight the close connection between Art and Mathematics – two ancient pursuits of humankind, which for vast numbers of people have nothing to do with each other. A typical artefact which brings out this hidden connection is Kolam, which is a form of line drawing using white or coloured powder (typically made from rice); it consists of straight and curved lines drawn around dots, forming a complex design with many symmetries.

In many parts of the country, it is a tradition to draw a fresh kolam in front of the house each morning. There is a deep symbolism concealed in this: in some traditions there is even an insistence on 'closing' the shape, leaving no openings, so that evil is prevented from entering the house. Over the day the patterns get eroded under numerous passing feet. One also sees ants, birds and squirrels eating the rice powder; a delightful sight! By the end of the day, the artefact is no more – only to be renewed the next morning. In this too, one sees a beautiful symbolism.

To the discerning eye, kolam patterns have a rich and elegant mathematical content, illustrating aspects of symmetry and connectedness which are conveniently studied in the subject called "graph theory". We shall feature this topic in a future issue of the magazine.

From the Chief Editor's Desk . . .

Greetings to all readers. Let us hope that by the time this issue appears before your eyes, the rains will have covered the country. We have, yet again (we hope; please tell us if you agree!), a rich fare in store for the reader. In the Features section we have the first part of an article by V G Tikekar, dwelling on a truly remarkable theorem in plane geometry, aptly called "Morley's Miracle". This is followed by the second part of Haneet Gandhi's article on Tessellations, and following that is a piece by Shailesh Shirali that connects with the "Fair Division" article that appeared in the March 2014 issue of At Right Angles, showcasing a result in Euclidean geometry. Its simplicity and unexpectedness reveals the astonishing richness that exists in the field of plane geometry.

In the Classroom section, Punya Mishra and Gaurav Bhatnagar lead us through another episode of "Art and Mathematics", underscoring yet again the close connection between these two ancient pursuits of mankind — a link which is so rarely dwelt upon in our classrooms. Shikha Takker and Rossi tell us about the rich potential of puzzles as a source of mathematics, and how negative results are just as positive as 'positive' results. In a similar vein, Gautham Dayal showcases the potential that 'toys' like the pentomino set hold for mathematics; this is the first part of what we hope will be a many-part series. A Ramachandran tells us about some symmetries of fourth order magic squares. Sneha Titus continues the series on the pedagogy of the CCE (continuous and comprehensive evaluation), while Swati Sircar draws out new meanings of the word 'Unfold'. The latest episode of "How To Prove It" dwells on a property of the prime numbers, and on the distinction between direct and indirect proof.

Continuing to the other parts of this issue: Sangeeta Gulati writes about the free cloud application "Desmos", one of many such online applications that are beginning to appear these days, which promise to enrich the teaching-learning of mathematics at the secondary and higher secondary levels. (We will feature more such applications in subsequent issues of this magazine.) Keshav Mukunda writes about one of James Gleick's best-selling books: "The Information: A History, A Theory, A Flood" and in the Pullout, Padmapriya Shirali tells us about strategies to teach Division to children in primary school.

— Shailesh Shirali

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At Right Angles is a publication of Azim Premji University together with Community Mathematics Centre, Rishi Valley School and Sahyadri School (KFI). It aims to reach out to teachers, teacher educators, students & those who are passionate about mathematics. It provides a platform for the expression of varied opinions & perspectives and encourages new and informed positions, thought-provoking points of view and stories of innovation. The approach is a balance between being an ‘academic’ and 'practitioner’ oriented magazine.

At Right Angles
A Resource for School Mathematics

Azim Premji University

A Publication of Azim Premji University together with Community Mathematics Centre, Rishi Valley and Sahyadri School, Pune

Contents

Features

This section has articles dealing with mathematical content, in pure and applied mathematics. The scope is wide: a look at a topic through history; the life-story of some mathematician; a fresh approach to some topic; application of a topic in some area of science, engineering or medicine; an unsuspected connection between topics; a new way of solving a known problem; and so on. Paper folding is a theme we will frequently feature, for its many mathematical, aesthetic and hands-on aspects. Written by practising mathematicians, the common thread is the joy of sharing discoveries and the investigative approaches leading to them.

In the Classroom

This section gives you a 'fly on the wall' classroom experience. With articles that deal with issues of pedagogy, teaching methodology and classroom teaching, it takes you to the hot seat of mathematics education. 'In The Classroom' is meant for practising teachers and teacher educators. Articles are sometimes anecdotal; or about how to teach a topic or concept in a different way. They often take a new look at assessment or at projects; discuss how to anchor a math club or math expo; offer insights into remedial teaching etc.

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Contents contd.

Tech Space

'Tech Space' is generally the habitat of students, and teachers tend to enter it with trepidation. This section has articles dealing with math software and its use in mathematics teaching: how such software may be used for mathematical exploration, visualization and analysis, and how it may be incorporated into classroom transactions. It features software for computer algebra, dynamic geometry, spreadsheets, and so on. It will also include short reviews of new and emerging software.

Problem Corner

Reviews

Pullout

A thing of beauty …

Covering the Plane with Repeated Patterns - Part II

… is a pattern forever

Haneet Gandhi

In Part I of this article (reference 1) we had noted how some regular polygons fit with each other to cover the plane without either gaps or overlaps, in arrangements called tilings. During our bus tour around the historic monuments of Delhi (described in the same article), we had seen many patterns based on simple rules, resulting in intricate tilings with great aesthetic appeal. Such patterns have been of interest to humans from ancient times, perhaps dating to the time when we started making shelters and used the logic of fitting rocks and weaving leaves to cover space while minimizing gaps. Over time, such endeavours took on artistic forms. Societies made use of tiles and patterns to emphasize different aspects of their culture. For example, Romans and some Mediterranean people portrayed human figures and natural scenes in their mosaic; the artistic impulse of the Arab artisans showed in their use of shape and colour to build complex geometric designs (as seen in the tiling patterns at Red Fort, Jama Masjid, Qutab Minar and Chandni Chowk; the Alhambra Palace in Granada, Spain, is another rich site of such patterns). Now, in Part II, we look at simple ways by which regular tessellations can be modified to make appealing patterns. We use simple techniques of colouring, shading or modifying a polygon to make interesting designs. The examples taken in this article are basic but lead to many possibilities that the readers can explore.

Keywords: *Pattern, tessellation, tiling, symmetry, art, architecture*

Art work using modified tessellations

We now revisit the regular tessellations made of congruent copies of one kind of regular polygon: equilateral triangles, squares and regular hexagons. An examination of their properties and their symmetries will help us deconstruct and recast the polygons to produce irregular motifs that will tile. We shall see that the art and architecture of tiling needs technical skills and imaginative ideas.

Modifying the regular quadrilateral

We start with the square. This shape has many symmetries that help us make irregular tessellating units. Some basic properties of this shape are: it has two pairs of equal, opposite and parallel sides; its adjacent sides are equal; it has rotational symmetry of order 4. We will now use combinations of these properties to modify our fundamental square tile.

To create the modified tile, carve out a piece along the length and translate it to the opposite parallel side. To make the design more interesting, chop off another piece along the width and translate it to the side opposite it. The pattern in Figure 1(a)-1(b) is based on translational symmetry of a square.

Figure 1(a)

The modified shape tessellates with translation symmetry. When embellished with some imaginative designs,the effect is enhanced.

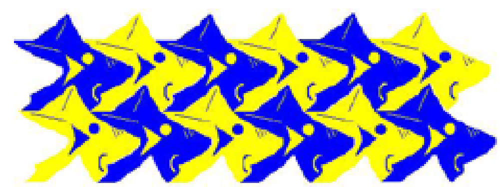

Figure 1(b)

Likewise, motifs can built on pairs of adjacent sides of a square to create tiles that show four-fold rotation symmetry, akin to that of the parent tile. Figures 1(c)-1(e)are some such patterns.

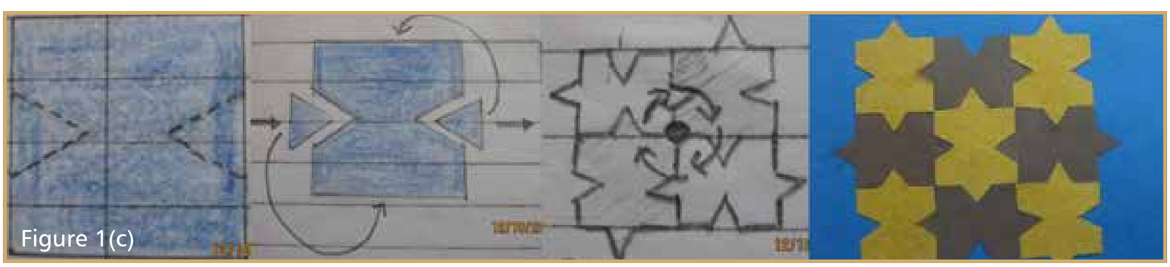

Figure 1(c)

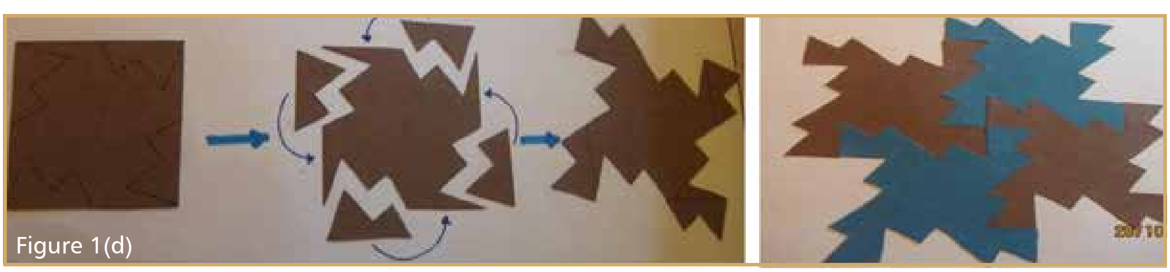

Figure 1(d)

Figure 1(e)

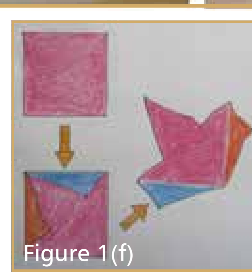

Figure 1(f)

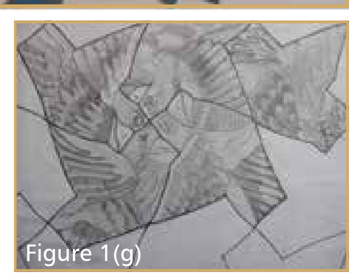

Figure 1(g)

By ornamenting the modified tile produced from a square (Figure 1(f)) with some enterprising art work, we get an interesting bird pattern (Figure 1(g)).

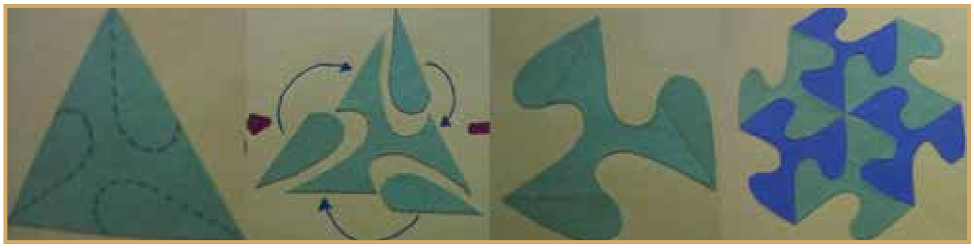

Figure 2(a)

Figure 2(b)

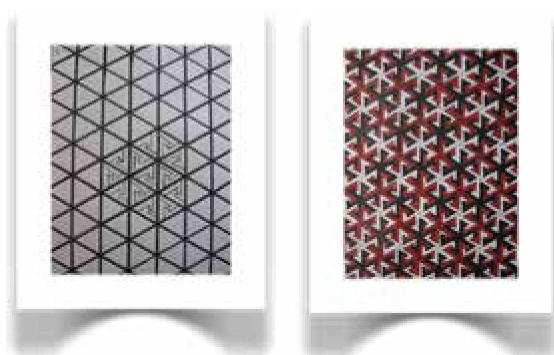

Figure 2(c)

Figure 3(a)

Modifying the regular triangle

The second set of Euclidean tessellations can be created from equilateral triangles. In this section we will explore one of the ways that was used to modify an equilateral triangle (Figure 2(a)-2(b)).

Draw curved lines inside the triangle, from a vertex to the midpoint of a side. Carve out a section with this as a boundary, give it a half-turn about the midpoint, and paste the piece on the outside of the other half length. Repeat this on the other two sides. Use the resulting tile as a stencil and create more tiles. We can now fit these tiles with one other, giving us a tessellation with rotational symmetry like that of the parent tile (the tessellation extends to the entire plane).

Modifying the regular hexagon

Regular hexagons can also be modified using their pairs of equal sides. The modification shown in Figure 3(a) is self-explanatory.

Decoding the tile

An interesting exercise with tessellations is to deconstruct the modified tile and to work out what must have been the parent tile. To do this, you will have to do the reverse of what was described above. Let's begin with a commonly used tile, often seen on the pedestrian paths of Delhi Metro stations (Figure 4(a)). Note that at any vertex three tiles interlock with each other, presenting the same rotational symmetry as that of a regular hexagon. By doing some basic modifications on the pairs of opposite and parallel sides, we can reshape it to get the desired motif.

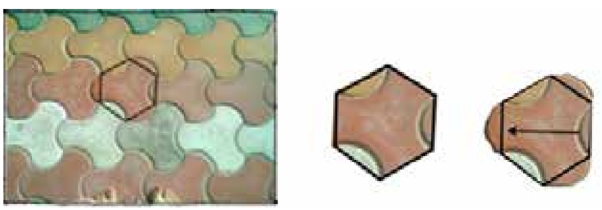

Figure 4(a)

Figure 4(b) is trickier. We see that at each vertex three of the lizards meet, exhibiting three-fold rotational symmetry. Now we must untangle the shape. To do this, trace out the irregular shape (the lizard) and circumscribe it with the associated regular polygon (here it is a hexagon). Now you must identify the portions sliced off and shift them to get the desired shape. There is no loss of any portion, so the area of the modified tile is the same as that of the parent tile.

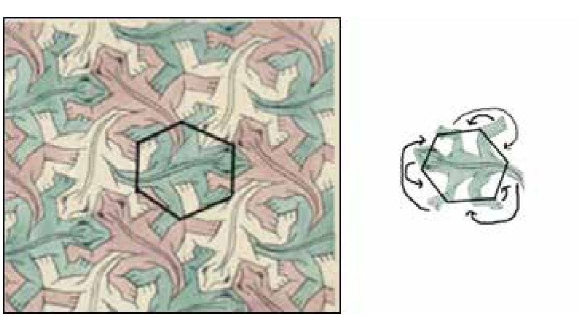

Figure 4(b)

In Figure 4(c) we see four fish interlocking at each vertex, showing rotational symmetry of order 4. It is thus easy to decode that the parent shape must have been a square.

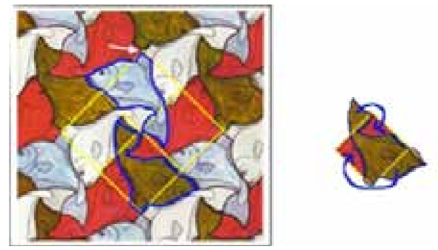

Figure 4(c)

The tessellation in Figure 4(d) was produced using equilateral triangles. Note the art work!

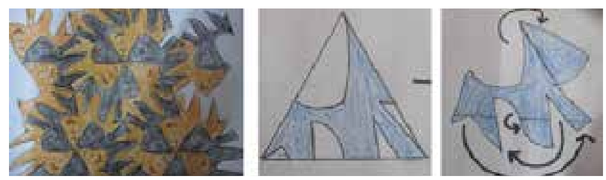

Figure 4(d)

Time for the thinking cap: some exercises

Can you guess the parent tile that has been modified to make the given tessellations (Figure 5(a)-5(e))? What properties of the parent tile helped in the modifications?

Figure 5(a)

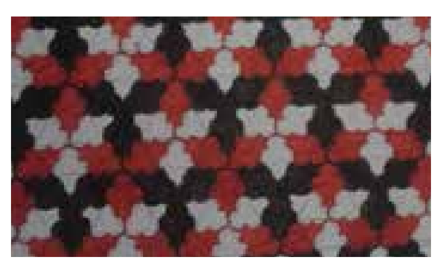

Figure 5(b)

The best way to deconstruct these tessellations is to use tracing paper to draw the outline of the repeating unit. By placing the traced figure on its look-alike, we deduce the underlying symmetry. The symmetries then help reveal the parent shape and the modifications done to it.


Figure 5(c)


Figure 5(d)


Figure 5(e)

Exploring further

Tessellations offer a huge range of mathematical explorations. One can only scratch the surface in a short article like this one. Some topics we have not explored relate to tessellations in which the units do not maintain the kind of repetition that equilateral triangles, squares and regular hexagons show. For example, there are tilings based on non-regular pentagons. Modifications in these polygons also make for intricate and artistic designs (Islamic star and the Penrose patterns) and can be explored further.

Middle grade teachers can use some of these examples for introductory work in tessellations. Concepts such as angle, area, perimeter, symmetry, closest packing, inscribed and circumscribed circles and more can be integrated and consolidated through such project work.

Further Reading

i. Grünbaum, B. & Shephard, G. C (1986), *Tilings and Patterns.* This is a comprehensive text on tessellations. It is a rich source of ideas that can be integrated in school geometry.

ii. Steinhaus, H. (1999), *Mathematical Snapshots* (Dover). Elementary yet engrossing.

iii. Critchlow, K (1970), *Order in space: A Design Source Book,* New York: Viking Press

Acknowledgements

The photographs have been taken by me and my B.Ed. students (batches of 2012-13 and 2013-14). I acknowledge my students' efforts in making intricate tessellations and extend my thanks to them for allowing me to use their designs for this article. Some pictures were taken from open sources such as Wikipedia.

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feature

Lurking within any triangle . . .

Morley's Miracle - Part I

. . . is an equilateral triangle

In this three-part series we study one of the most celebrated and beautiful theorems of Euclidean geometry, discovered at the dawn of the twentieth century. It has justly become known as 'Morley's Miracle'. It happens to be uncommonly challenging to prove!

The inaugural July 2012 issue of this magazine had displayed a figure of the theorem on the cover, and we had promised to present a proof in a later issue.
It is appropriate that we are making good this promise now.

V G Tikekar

Frank Morley (1860–1937) was British by birth (though he lived much of his life in USA) and an expert geometer. After getting a Sc.D. degree from Cambridge, he became Chairman of the Math Dept of the Johns Hopkins University in 1900. Prior to this, while working as a Professor of Mathematics at Haverford College (Pennsylvania), he had found a theorem in the field of Euclidean geometry (but using methods far removed from those of synthetic geometry). It was regarded by many as "marvelous but strange" because it related to the lines that trisect the internal angles of a triangle. Early in the nineteenth century it had been shown that it is not possible to trisect a general angle using straightedge and compass (thus settling a question that had been lying unresolved since Greek times), leading to a sort of psychological barrier in exploring any matter related to angle trisection. Morley's theorem thus came as a shock to many lovers of mathematics. Over time the result became known as 'Morley's Miracle'.

Keywords: *Angle trisectors, equilateral, cardioid*

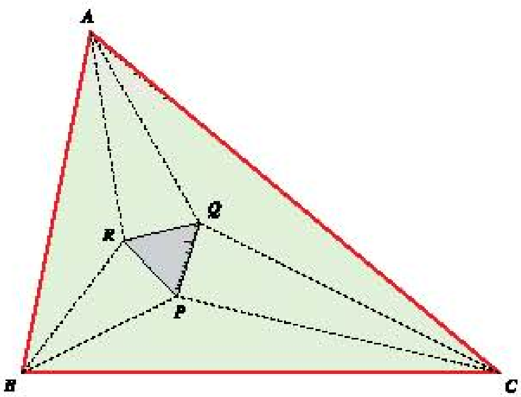

FIGURE 1. The angle trisectors closest to each side intersect in points which are the vertices of an equilateral triangle

FIGURE 2. A manifestation of the coffee-cup caustic curve. Source: http://m.today.duke.edu/2009/04/caustics.html

Displayed below is the statement of the theorem.

Theorem 1 (Morley's Miracle). *In any triangle, the three points of intersection of the adjacent angle trisectors closest to each side are the vertices of an equilateral triangle.*

The theorem is illustrated in Figure 1: the trisectors of $\angle A$ are lines AQ and AR; those of $\angle B$ are BP and BR; and those of $\angle C$ are CP and CQ. The trisectors nearest to side BC meet at P, those nearest to side CA meet at Q, and those nearest to side AB meet at R. The claim now is ΔPQR is always equilateral, regardless of the shape of ΔABC. Readers are encouraged to make trial drawings of their own (this is very conveniently done using *GeoGebra*) to see that the claim does seem to be true.

The theorem is indeed a gem; but it is also a great challenge to prove. In this three-part series we describe a few of the many known proofs.

It appears that Morley discovered (and proved) the result around the year 1899, during his investigations on the differential geometry of curves, but did not publish it anywhere. He did mention it to a few people, though, and it gradually became known. His proof did not use the methods we associate with Euclidean geometry; it was based on a study of the set of cardioids that touch all the three sides of a triangle! (A cardioid is a heart-shaped curve generated by each point on a circle that rolls without slipping on a fixed circle of equal radius. You will see a portion of a cardioid on the surface of a cup of milk or coffee when light shines upon it; see Figure 2. So it is sometimes called the 'coffee cup curve'.)

The first 'pure geometry' proof to be published was in 1909, by M.T. Naraniengar (who, incidentally, was president of the Indian Mathematical Society from 1930 to 1932, and Editor of the Journal of the Indian Mathematical Society from 1909 to 1927). In 1914 one more such proof appeared, by Marr and Taylor. Over the years a large number of beautiful proofs have been found, of which special mention must be made of one by John Conway (see [4]).

Naraniengar's proof This beautiful proof follows an unusual strategy, one not seen too often in geometric proofs.

The trisectors closest to side BC meet at P (as stated). Let the remaining two trisectors of $\angle B$ and $\angle C$ (respectively) meet at S. Join SP (see Figure 3). In ΔSBC, rays BP and CP are internal angle bisectors, so P is the incentre of ΔSBC; therefore, SP bisects $\angle BSC$.

Now, locate points R' on BS and Q' on CS such that $\angle SPR' = 30°$ and $\angle SPQ' = 30°$. Then $\Delta SPR' \cong \Delta SPQ'$ (angle-side-angle congruence), so $PR' = PQ'$. Hence $\Delta PQ'R'$ is an isosceles triangle with a $60°$ angle; but this implies that it is equilateral. Hence to prove Morley's theorem, it suffices to show that AR' and AQ' are trisectors of $\angle A$.

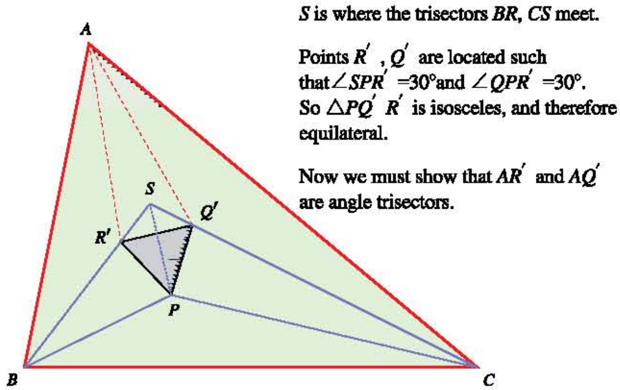

FIGURE 3. Construction and strategy of Naraniengar's proof

To this end, Naraniengar uses the following lemma (see Figure 4):

Lemma. *If four points* $\boldsymbol{M, R', Q', N}$ *satisfy the following conditions:*

(i) $\boldsymbol{MR' = R'Q' = Q'N}$, *(ii)* $\boldsymbol{\angle MR'Q' = \angle R'Q'N = 180^\circ - 2x > 60^\circ}$, *for some value of* $\boldsymbol{x}$ *less than 60°, then the four points* $\boldsymbol{M, R', Q', N}$ *lie on a circle. Further, if point* $\boldsymbol{A}$ *on the side of* $\boldsymbol{MN}$ *away from* $\boldsymbol{R'}$*, is so situated that* $\boldsymbol{\angle MAN = 3x}$, *then* $\boldsymbol{A}$ *lies on the same circle.*

To see why this is true, draw the bisectors of $\boldsymbol{\angle MR'Q'}$ and $\boldsymbol{\angle R'Q'N}$, and let them meet at $\boldsymbol{O}$. Then we have $\boldsymbol{\Delta OMR' \cong \Delta OR'Q' \cong \Delta OQ'N}$ (side-angle-side congruence), hence $\boldsymbol{OM = OR' = OQ' = ON}$. Therefore the four points $\boldsymbol{M, R', Q', N}$ lie on a circle $\boldsymbol{\omega}$ centred at $\boldsymbol{O}$.

Hence $\angle OMR'$, $\angle OR'M$, $\angle OR'Q'$, $\angle OQ'R'$, $\angle OQ'N$ and $\angle ONQ'$ are all equal to $90^\circ - x$. But this implies that $\angle MOR'$, $\angle R'OQ'$ and $\angle Q'ON$ are all equal to $2x$, and hence that $\angle MON = 6x$.

Hence to show that A lies on ω, it suffices to: (i) note that A lies on the same side of MN as O, and (ii) show that $\angle MAN = 3x$.

To use this result we refer to Figure 5 (which is Figure 3, redrawn) and locate points M on side AB and N on side AC such that $BM = BP$ and $CN = CP$. Then $\Delta PBR' \cong \Delta MBR'$, so $MR' = PR'$ and so $MR' = R'Q'$. In the same way we have $NQ' = R'Q'$.

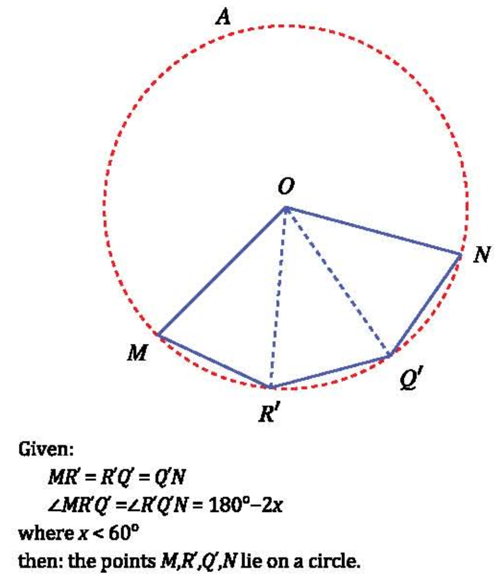

FIGURE 4. Lemma used by Naraniengar

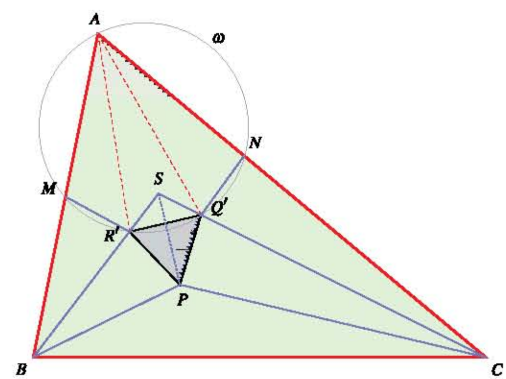

FIGURE 5. Applying the lemma

Now we must compute the measures of $\boldsymbol{\angle MR'Q'}$ and $\boldsymbol{\angle NQ'R'}$.

Let $\boldsymbol{\angle A = 3x}$, $\boldsymbol{\angle B = 3y}$, $\boldsymbol{\angle C = 3z}$. Since $\boldsymbol{\angle BSC = 180^\circ - 2y - 2z = 60^\circ + 2x}$ (remember that $\boldsymbol{3x + 3y + 3z = 180^\circ}$), we get:
$\boldsymbol{\angle PSR'} = 30^\circ + \boldsymbol{x}$. Since $\boldsymbol{\angle SPR'} = 30^\circ$, we get $\boldsymbol{\angle BR'P = 60^\circ + x}$ and $\boldsymbol{\angle MR'B = 60^\circ + x}$ as well. Hence
$\boldsymbol{\angle MR'Q'} = 360^\circ - (120^\circ + \boldsymbol{2x + 60^\circ}) = 180^\circ - \boldsymbol{2x}$.
In the same way we have $\boldsymbol{\angle NQ'R'} = 180^\circ - \boldsymbol{2x}$.

The conditions of the lemma now hold, so we can assert that points $\boldsymbol{A, M, R', Q', N}$ lie on a circle $\boldsymbol{\omega}$ as shown in Figure 4. Since chords $\boldsymbol{MR', R'Q', Q'N}$ of this circle have equal length, they subtend equal angles at $\boldsymbol{A}$, implying that $\boldsymbol{AR'}$ and $\boldsymbol{AQ'}$ are angle trisectors. This now establishes what we set out to prove, and so Morley's theorem is proved.

Remarks. Naraniengar's proof follows an unusual strategy: it *starts* with an equilateral triangle, then sets up a configuration similar to that constructed in the theorem. This clearly implies that the triangle constructed by Morley is equilateral. It is curious that a similar strategy is pursued in many pure-geometry proofs of Morley's theorem: start with an equilateral triangle, then reconstruct a configuration similar to the original one. Perhaps the most spectacular of these is the proof by John Conway. In the next part we examine proofs that use trigonometry.

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PROF. V.G. TIKEKAR retired as the Chairman of the Department of Mathematics, Indian Institute of Science, Bangalore, in 1994. He has been actively engaged in the field of mathematics research and education and has taught, served on textbook writing committees, lectured and published numerous articles and papers on the same. Prof. Tikekar may be contacted at vgtikekar@gmail.com.

feature

In mathematics, breaking up . . .

Equalizers of a Triangle

. . . is not hard to do!

In the March 2014 issue of At Right Angles, the article "A Fair Division" presented a study of a problem involving a geometrical division. A plot of land in the form of a scalene triangle is to be divided, as per the dictates of a whimsical will, into two parts having equal area as well as equal perimeter, using a straight dividing line. A simple argument shows that there always exists such a line; see [2]. In the mathematical literature, such a line has been called the equalizer of the triangle. It is known that any triangle has 1, 2 or 3 equalizers; see [4]. In this article we prove two results related to the equalizers.

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The results mentioned in the preamble above are not only beautiful but remarkable as well, packing a good deal of 'surprise value'. Here they are:

Theorem 1. An equalizer of a triangle necessarily passes through its incentre.

Theorem 2. A line passing through the incentre of a triangle divides its perimeter and area in the same ratio.

Theorem 1 is a known result (see [1], [3], [5]). We have not seen Theorem 2 anywhere in the literature. The proofs of both the theorems are easy to find. We invite you to find your own proofs before reading ahead.

Keywords: *Triangle, perimeter, area, ratio, incentre, equalizer, quadratic, roots*

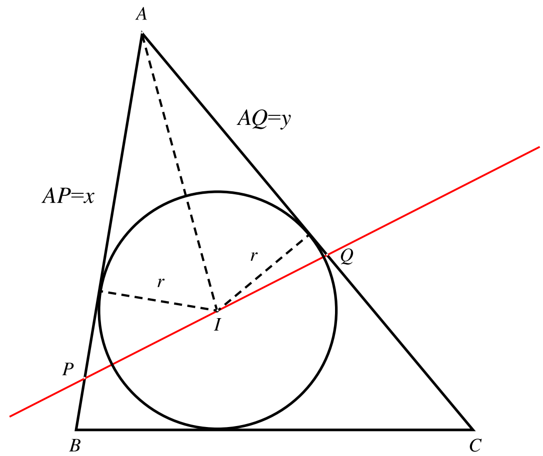

FIGURE 1

Figure 1

Proof of Theorem 2. The two results are proved in nearly the same way, but we choose to prove Theorem 2 first. Figure 1 shows a triangle ABC with incentre I, with an arbitrary line ℓ through I. This line must pass through some two sides of the triangle, and we shall suppose them to be AB and AC. Let the points of intersection of ℓ with AB and AC be P and Q respectively; let $AP = x$ and $AQ = y$. The theorem then claims the following:

$$\frac{\text{Area of } \triangle APQ}{\text{Area of quadrilateral } PBCQ} = \frac{AP + AQ}{PB + BC + CQ}.$$

This may be written in the following equivalent form:

$$\frac{\text{Area of } \triangle APQ}{\text{Area of } \triangle ABC} = \frac{AP + AQ}{AB + BC + CA}.$$

Let $2s = a + b + c$ be the perimeter of $\triangle ABC$. Then we must show the following:

$$\frac{\text{Area of } \triangle APQ}{\text{Area of } \triangle ABC} = \frac{x + y}{2s}.$$

Consider the fraction on the right side. Multiplying both the numerator and denominator by the in-radius r, we get the following:

$$\frac{x + y}{2s} = \frac{r(x + y)}{2rs} = \frac{\frac{1}{2}rx + \frac{1}{2}ry}{rs}.$$

In the last expression, note that $\frac{1}{2}rx$ is the area of $\triangle API$ (because if we treat $x = AP$ as the base, then its altitude is r) and, similarly, $\frac{1}{2}ry$ is the area of $\triangle AQI$. Hence $\frac{1}{2}rx + \frac{1}{2}ry$ is the area of $\triangle APQ$. Also, rs is the area of $\triangle ABC$. (This is a known formula. To prove it, note that the area of $\triangle ABC$ is the sum of the areas of $\triangle IBC$, $\triangle ICA$ and $\triangle IAB$. Now treat BC, CA and AB as the bases of these triangles, and note that all three triangles have the same altitude, r; now fill in the rest of the proof.) Hence the expression is equal to the ratio

$$\frac{\text{Area of } \triangle APQ}{\text{Area of } \triangle ABC}.$$

But that is just what we wanted to show! Hence, Theorem 2 is proved.

Proof of Theorem 1. We adopt a very similar strategy. Let the line ℓ bisect the perimeter as well as the area of $\triangle ABC$. As earlier, we argue that ℓ must intersect some two sides of the triangle; let them be AB and AC, and let the points of intersection of ℓ with these two sides be P and Q respectively. Let $AP = x$ and $AQ = y$.

The fact that ℓ is an equalizer implies that $x + y = s$ and $xy = \frac{1}{2}bc$. Let the internal bisector of $\angle BAC$ meet ℓ at J. We must then show that J is the incentre of $\triangle ABC$. (See Figure 2.)

From J, drop perpendiculars JU and JV to AB and AC respectively. Since J lies on the bisector of $\angle A$, it follows that $JU = JV$; let their common length be r'. To show that J is the incentre of $\triangle ABC$ is equivalent to showing that r' equals the in-radius r of $\triangle ABC$, and this is what we shall now show.

The areas of $\triangle AJP$ and $\triangle AJQ$ are $\frac{1}{2}r'x$ and $\frac{1}{2}r'y$ respectively, so the area of $\triangle APQ$ is $\frac{1}{2}r'(x + y)$. Since $x + y = s$, it follows that the area of $\triangle APQ$ is $\frac{1}{2}r's$. But since ℓ is an equalizer, the area of $\triangle APQ$ is half the area of $\triangle ABC$; hence the area of $\triangle ABC$ is $r's$. But the area of $\triangle ABC$ is also equal to rs. It follows that $r' = r$ and hence that J is the incentre of the triangle. Thus the equalizer passes through the incentre of the triangle, as claimed.

Locating the Equalizers. A candidate line ℓ for the post of equalizer of a triangle ABC must pass through some two sides of the triangle, say AB & AC. Let ℓ cut these two sides at P and Q respectively,

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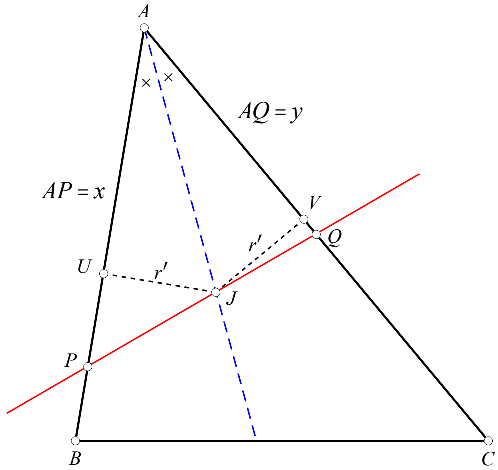

Figure 2

and let $AP = x$, $AQ = y$. As ℓ is an equalizer, we have $x + y = s$ (where $2s = a + b + c$ is the perimeter of the triangle) and $xy = \frac{1}{2}bc$. Hence an equalizer passing through sides AB and AC exists if and only if the equations $xy = \frac{1}{2}bc$, $x + y = s$ yield values for x, y satisfying the inequalities $0 \le x \le c$ and $0 \le y \le b$.

Now if $x, y \ge 0$ and $x + y = s$, the range of possible values of xy is $0 \le xy \le \frac{1}{4}s^2$; the least possible value is taken when one of x, y is 0, and the maximum possible value is taken when $x = y = \frac{1}{2}s$ (because if the sum of two numbers is held fixed, their product is largest when the numbers are equal). For a solution to exist, a *necessary* condition is that $\frac{1}{2}bc$ lies within this interval. So we must have $\frac{1}{2}bc \le \frac{1}{4}s^2$, i.e., $s^2 \ge 2bc$. If this inequality is strict, there is a possibility of two solutions (x, y), while if equality holds ($s^2 = 2bc$), there is just one solution. Note that we say 'possibility' —because we also need the inequalities $0 \le x \le c$ and $0 \le y \le b$ to hold (for P, Q must lie on sides AB, AC respectively). The actual values of x, y (got by solving the equations $x + y = s$, $xy = \frac{1}{2}bc$) are:

$$x, y = \frac{s \pm \sqrt{s^2 - 2bc}}{2}.$$

Case study–I: Triangle with sides 3, 4, 5 We take each pair of sides in turn to be candidates for $\{b, c\}$, and check for feasible solutions. Here $s = 6$, so $s^2 = 36$.

- $\{b, c\} = \{3, 4\}$. Here $2bc = 24$, so $s^2 > 2bc$. Solving for x, y, we get:

  $$x, y = \frac{6 \pm \sqrt{36 - 24}}{2} = 3 \pm \sqrt{3}$$

  Neither choice of sign works, because $3 + \sqrt{3} > 4$. So we do not get any equalizer associated with this pair of sides.

- $\{b, c\} = \{3, 5\}$. Here $2bc = 30$, so $s^2 > 2bc$. Solving for x, y, we get:

  $$x, y = \frac{6 \pm \sqrt{36 - 30}}{2} = 3 \pm \sqrt{1.5}$$

  Since $3 - \sqrt{1.5} < 3$ and $3 < 3 + \sqrt{1.5} < 5$, we get one equalizer here (but only one).

- $\{b, c\} = \{4, 5\}$. Here $2bc = 40$, so $s^2 < 2bc$. This does not yield any equalizers.

So for the 3, 4, 5 triangle, there exists just one equalizer. Figure 3 gives a sketch of the situation. The sole equalizer PQ has been drawn, with $CP = 3 - \sqrt{1.5}$ and $CQ = 3 + \sqrt{1.5}$. The equalizer passes through the incentre I, as it should. Observe that $CP + CQ = 6 = s$, and $CP \times CQ = 9 - 1.5 = 7.5 = \frac{1}{2}(3 \times 5)$.

Case study–II: Triangle with sides 7, 8, 9
As earlier, we take each pair of sides in turn to be candidates for $\{b, c\}$, and check for feasible solutions. Here $s = 12$, so $s^2 = 144$.

- $\{b, c\} = \{7, 8\}$. Here $2bc = 112$, so $s^2 > 2bc$. Solving for x, y, we get:

  $$x, y = \frac{12 \pm \sqrt{144 - 112}}{2} = 6 \pm 2\sqrt{2}.$$

  Neither choice of sign works, because $6 + 2\sqrt{2} > 8$. So we do not get any equalizer associated with this pair of sides.

- $\{b, c\} = \{7, 9\}$. Here $2bc = 126$, so $s^2 > 2bc$. Solving for x, y, we get:

  $$x, y = \frac{12 \pm \sqrt{144 - 126}}{2} = 6 \pm \sqrt{4.5}.$$

  Since $7 < 6 + \sqrt{4.5} < 9$, we get one equalizer (but only one).

- $\{b, c\} = \{8, 9\}$. Here $2bc = 144$, so $s^2 = 2bc$. Solving for x, y, we get:

  $$x, y = \frac{12 \pm \sqrt{144 - 144}}{2} = 6.$$

  Since $6 < 9$, we get an equalizer here. Since $x = y$ in this case, the two equalizers are coincident.

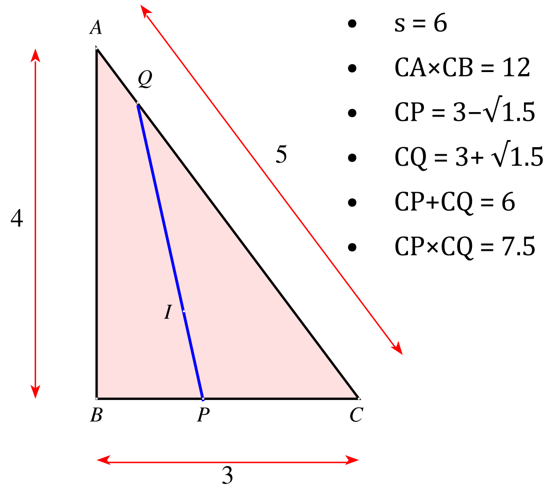

Figure 3. Equalizer for a 3,4,5 triangle; I is the incentre (there is just one equalizer)

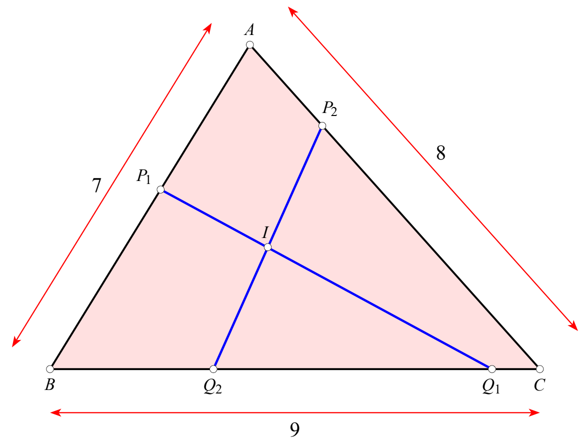

Figure 4. Equalizers for a 7, 8, 9 triangle: P_1Q_1, with $BP_1 = 6 - \sqrt{4.5}$ and $BQ_1 = 6 + \sqrt{4.5}$; P_2Q_2, with $CP_2 = 6 = CQ_2$; I is the incentre

So for the 7, 8, 9 triangle, there exist two equalizers. Both of them have been sketched in Figure 4 (segments $P_1 Q_2$ and $P_2 Q_2$).

An equilateral triangle obviously has three equalizers (all three medians). So we may anticipate that as the triangle changes in shape from a high degree of scalene-ness towards equilateral-ness, the number of equalizers changes from 1 to 3. A complete analysis of how this change happens is given in [4]. However, we do not try to prove this here.

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in the classroom

No dead-ends

Learning mathematics through puzzles

No solution is also a good solution

Mathematical puzzles are generally perceived to be at the periphery of mathematics, and not part of the core of the discipline. This may be related to the fact that we sometimes come across puzzles that have no solution. The general expectation among mathematics practitioners and school children seems to be that problems in mathematics should have a solution. In this paper, we argue that puzzles can be an important source of learning some core mathematical ideas. We present an exemplar to justify our claim that problems with no solutions are as significant to the teaching and learning of mathematics as problems with solutions.

Rossi & Shikha

Introduction

Often puzzles are considered as games or 'time-off' from routine mathematics in school classrooms. The objective is generally to find who solves the puzzle, how quickly and what is the shortest and probably the quickest way of solving it. But despite knowing that they are mathematical in nature, they are not seen as 'connected with hard core mathematics'.

Many of these puzzles contain all the ingredients and themes of mathematics: proof, generalization, pattern recognition, assuming the truth of a statement and arriving at a contradiction, non-existence of a solution, etc. In this article, we present a puzzle which was used to discuss some core mathematical ideas with participants at two public programs. The participants ranged from fourth standard students to B Ed graduates. For these programs, we designed a few puzzles which required unconventional methods of problem solving.

Keywords: *Puzzle, game, solution, parity, invariance, proof, validity*

By 'unconventional' we mean that solving these puzzles was not necessarily confined to elementary mathematics, a well defined syllabus, or an algorithmic solution. Central to our considerations was also that these puzzles should be able to create an opportunity for students to learn some important concepts in mathematics.

Puzzle: Circles and counters

In [1, p. 125] we came across the following puzzle.

A circle is divided into six sectors and each of them has a counter or button in it (see Figure 1). *You have to get all the counters into one sector using jumps, following these two rules:* (*i*) *In a single jump, a counter can be moved only to an adjacent sector.* (*ii*) *Each move consists of two jumps.*

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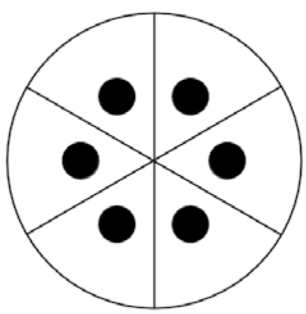

Figure 1. Circle divided into 6 sectors with a counter in each sector

The focus of this puzzle is on parity and invariance, and the solution proposed is that *there is no solution*. The proof for this assertion goes as follows.

Number the sectors from 1 to 6. For each sector, find the product of its number and the number of counters in it. Let the sum of these products be s. Call this sum the 'score'. At the start of the game, each sector has one counter, so the score is $1+2+3+4+5+6=21$.

After a move in which one counter jumps from, say, 2 to 3 and the other counter jumps from, say, 4 to 5, the new score would be $1+0+(2\times3)+0+(2\times5)+6=23$. Note that the score has changed by 2. A little thought will reveal that no matter how the moves are made, the score always changes by an even number; more specifically, by 0, 2, 4 or 6.

Since the starting value of *s* is 21, an odd number, the value of *s* will be odd at every stage. But if all the counters reach one sector, the value of *s* will surely be a multiple of 6 at that stage, and hence even. Therefore the puzzle can have no solution.

We found it interesting that the puzzle did not have a solution. Then we wondered how children and teachers would look at a mathematical problem like this which does not have a solution. Often, such problems are rejected as being wrong or as having inadequate information. However, realising that "no solution is also a valid solution", mathematically, is interesting and important for teachers and students to know.

Also, we can assert that a problem does not have a solution only when we have a clear proof for such a claim.

These are both important mathematical ideas: first, that "no solution" is a valid state of affairs in mathematics, and second, the need for a proof for the claim that a problem does not have a solution.

Both these matters are currently not a part of our understanding of teaching mathematics especially at the elementary level. Proofs are considered to be abstract and are introduced very late in the school curriculum. Also, problems with no solution or multiple solutions are rarely discussed.

As we were working with the puzzle with these considerations in mind, we started thinking about the different variables involved and how changing them would change the problem. The variables included the number of counters and its positions and of jumps.

For example, in the puzzle, there is just one counter in each sector. What if we have two or more counters instead? What if the number of jumps in each move is increased?

When we worked through these variations we found neither of them so interesting. Let us explain why. Placing more counters in each sector does not make it more challenging but rather a tiring exercise. Also, this would not be a suitable extension for very young children. (We were targeting young children because they had not yet been exposed to ideas of 'proof and proving'.) So we were doubtful whether this would make the puzzle more engaging.

For the second, we found that number of jumps can be either odd or even. The *odd* number of jumps will make it similar to moving in jumps of 1, and *even* would be in terms of jumps of 2. There would be nothing puzzling here!

However, there was another extension we found to be more interesting: varying the *number of sectors* in the circle. We tried to see if there is a way to complete the task for a circle divided into *n* sectors, for *n* between 2 and 10. Then we looked for a pattern. While working on the extended puzzle, we noted some interesting mathematical processes. This included playing and identifying a generalised pattern, finding the values of *n* for which the solution exists and for which a solution does not exist, finding proofs in each case, etc. This is where we experienced all the ingredients of mathematical engagement.

Modified problem and students' solutions

We posed the modified problem to students of different age levels. We then examined their strategies. The problem posed was:

A circle is divided into n sectors and each of them has a counter or button. You must get all of them into a single sector using jumps, following these rules: (i) In a single jump, a counter can be moved only to an adjacent sector. (ii) Each move consists of two jumps. For which values of n can we get all the counters into a single sector? Why do you say so?

Being able to interact with a large group of students from diverse classrooms was a great advantage, and we got this opportunity from two fora.

One occasion was the 'National Science Day' at the Homi Bhabha Centre for Science Education, Mumbai, and the other was a session in the popular lecture series called 'Chai and Why'. (This is an outreach public activity conducted by TIFR. It consists of talks by members of TIFR and is held every second and fourth Sunday of each month at Prithvi Theatre, Juhu, and at Ruparel College, Matunga, Mumbai respectively. The aim of 'Chai and Why' is essentially to popularize mathematics and science.)

’ Images from ‘Chai and Why

We interacted with a wide range of participants, from children to adults (including mathematicians, physicists, etc.). We were keen to see what kinds of proofs would come forth from learners of different age groups. We share two interesting and representative solutions.

Given enough time, almost all students, even those in grade 4, came up with a generalised pattern. They figured out that a solution to the problem exists for all odd numbers and for numbers divisible by 4. But many of them could not state confidently that a solution does not exist for even numbers not divisible by 4, such as 10. Also, when asked how they could say that a solution exists for all numbers divisible by 4, most students just stuck to examples with smaller numbers. Their general way of solving the problem was trial and error: looking for a solution for $n < 10$, then stating the generalisation. None of them could explain comfortably why a solution does not exist for $n = 2$ and $n = 6$.

One student came close to finding a proof. He came up with the following observation for what made $n = 6$ different from the rest. He said: "Suppose we choose one sector (the target) as the place where all buttons must end up. This implies that the button on that sector requires 0 jumps to reach there. Similarly, the buttons in its (two) adjacent sectors require 1 jump each; the buttons in the sectors adjacent to those require 2 jumps each; etc. Working this way we find that the total number of jumps required to reach the final sector is odd for $n = 6$ or 2, but even for other values of n." Of course, the sector that requires 1 jump to reach the final sector would require 5 jumps if moved in the opposite direction. But this maintains the parity of the total. ('Parity' refers to whether an integer is odd or even. Note that adding an even number to an integer maintains its parity, and adding an odd number reverses its parity. 'Parity invariance' is a commonly used theme in solving problems and in constructing proofs.)

Although he stopped there, it turns out that his observation was a good starting point for a legitimate proof; given more time, he would probably have completed it. To understand and extend his proof, we continue the argument. For odd n, if a sector requires an odd number of jumps to reach the target in one direction, it requires an even number of jumps if moved in the opposite direction. Hence we can always end up in the target sector by choosing for each button an appropriate direction so as to ensure an even number of jumps. For n divisible by 4, the sector diametrically opposite the target requires an even number of jumps to reach the target. The remaining sectors are symmetrically placed. Thus, every such sector has a corresponding sector with the same number (of jumps to reach the target).

Our proof

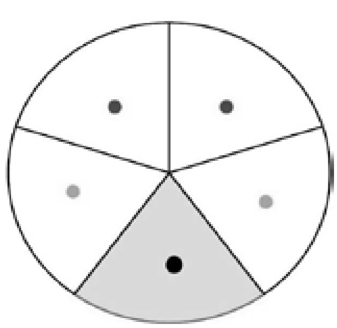

Figure 2. Circle divided into $n = 5$ sectors with bottom sector as 'target'

Our proof was on similar lines as that of the student. Let n be the number of sectors of the circle. Then n is either odd, or divisible by 4, or even but not divisible by 4.

The case when n is odd: Let us choose one sector as the target (see Figure 2). Now we have an even number of buttons to be moved to that sector. These sectors are placed symmetrically and can thus be moved to the target in steps of two jumps: one jump of a button towards the target, followed by another jump by its corresponding button in a symmetric manner. The status of the puzzle after the first move is shown in Figure 3. Therefore, we can always end with all the buttons in a common sector if n is odd.

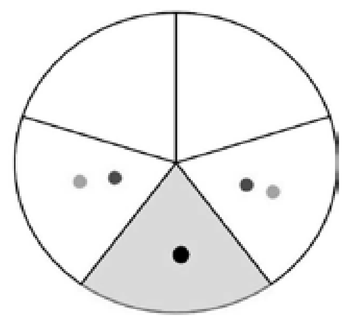

Figure 3. State of the puzzle for $n = 5$ after one move (i.e., two jumps)

The case when *n* is divisible by 4: Designate one sector as the target. Since n is a multiple of 4, the counter diametrically opposite to the target would require, and can be moved by, an even number of jumps to reach there. Now again, we are left with an even number of buttons placed symmetrically. These can be moved to the target using the same method as described above for the case of the circle divided into an odd number of sectors. Therefore we have a solution for n divisible by 4 (similar to the student's proof).

The case when *n* is even but not divisible by 4: For $n = 6$, we cannot use the above argument. Of course, this is not a proof. To prove that we cannot have a solution for such values of n, we assign a number to each sector, in the following way. Beginning with any sector, we number them 0 and 1 in alternation. Since n is even, this is possible; further, each '1' will only have a '0' as its neighbour, and vice versa.

Now for each sector we find the product of its number and the number of counters in it, and find the sum (s) of the products. At the start, since each of the sectors has one counter, s would be 1+0+1+0+1+0+... = $n/2$, an odd number. With each move, we make two jumps — in each jump, a button would move from 0 to 1 or from 1 to 0. So, every jump changes *s* from an even number to an odd number or the reverse; that is, *it reverses its parity.*

Hence two jumps maintain the parity. Since the number of sectors is not divisible by 4, we have an odd number of zeros and an odd number of ones. Hence we begin with *s* being odd. The parity stays invariant. However, a solution to the problem would imply that all buttons come to a common sector thereby making *s* even. Therefore a solution does not exist.

Implications of this activity

The aim of the activity was to use 'concept based puzzles' to create challenges that would encourage the development of formal proof among children.

The fact that some students articulated justifications behind the non-existence of a solution gave us some evidence that mathematical puzzles can drive students to perform problem solving activities that are consistent with the nature of mathematics.

The process of drawing upon a puzzle to identify important mathematical ideas and using them to create the need for proof was interesting and insightful. We are finding it possible to engage even very young children in the idea of proof, and engaging them with the centrality of proof in mathematics. The scope for learners to come up with their own proofs would create a legitimate participation in the culture of doing mathematics and such an environment can make them appreciate the significance of rigour in mathematics.

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Of Art and Math

Self-Similarity

You have wakened not out of sleep, but into a prior dream, and that dream lies within another, and so on, to infinity, which is the number of grains of sand. The path that you are to take is endless, and you will die before you have truly awakened
*— **Jorge Luis Borges***

Punya Mishra
Gaurav Bhatnagar

The Argentinian author, Jorge Luis Borges, often wrote about his fear of *infinity*—the idea that space and/or time could continue forever. Though Borges' response may appear somewhat overblown, who amongst us has not felt a frisson of excitement when thinking of the infinite and our relative insignificance in front of it. Borges' quote of reality being a dream within a dream within a dream ad infinitum reminds us of the hall of mirrors effect—the seemingly infinite reflections one generates when one places two mirrors in front of each other—the same object over and over and over again.

This idea of infinite reflections can be seen in the chain ambigram in Figure 1 for the word *reflect*. In this design the "RE" and "FLECT" are written in a mirror-symmetric manner, which means that if we repeat this design over and over again it will read the same when held up against a mirror. (For a different ambigram for reflect, see *Introducing Symmetry,* in the March 2014 issue of *At Right Angles).*

Figure 1. An ambigram of "reflect," reflecting the infinite reflections in a pair of mirrors

This idea of repeating a similar shape (often at a different scale) over and over again, is called *self-similarity*. In other words, a self-similar image contains copies of itself at smaller scales. A simple example appears in Figure 2: a repeated pattern for a square that is copied, rotated and shrunk by a factor of $1/\sqrt{2}$.

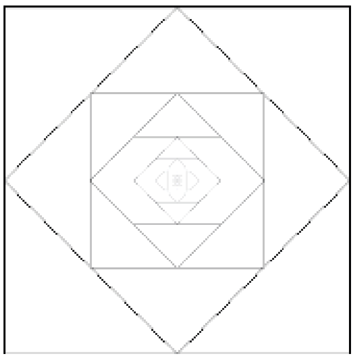

Figure 2. A self–similar design

Of course you can do this with typographical designs as well, such as the design for the word "Zoom" in Figure 3.

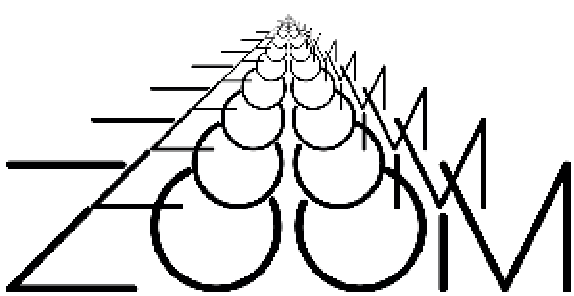

Figure 3. A self-similar ambigram for ZOOM

Examining self-similarity leads to a discussion of infinity, iteration and recursion, some of the ideas we discuss in this article.

Before taking a serious look at self-similarity, we present (see Figure 4) a rotational-ambigram of "self-similarity," which is *not* self-similar. However, below it is another version of the same design, where the word "self" is made up of little rotationally symmetric pieces of "self" and similarity is made up of little ambigrams of "similarity" and, most importantly the hyphen between the words is the complete ambigram for "self-similarity." So this leads to the question: What do you think the hyphen in the hyphen is made of?

Self similarity and Fractals

Self-similar shapes are all around us, from clouds to roots, from branches on trees to coastlines, from river deltas to mountains. The idea of self-similarity was popularized by Benoit B. Mandelbrot, whose 1982 book "The Fractal Geometry of Nature" showed how self-similar objects known as 'fractals' can be used to model 'rough' surfaces such as mountains and coast-lines. Mandelbrot used examples such as these to explain how when you measure a coastline the length of the line would increase as you reduced the unit of measurement. Such convoluted folds upon folds that lead to increased length (or in the case of 3-d objects, increased surface area) can be seen in the structure of the alveoli in the human lungs as well as in the inside of our intestines. The volume does not increase by much, while the surface area increases without limit.

Figure 5 is an ambigram of "Fractal" which illustrates Mandelbrot's own definition of fractals: *A fractal denotes a geometric shape that breaks into parts, each a small scale model of the original*.

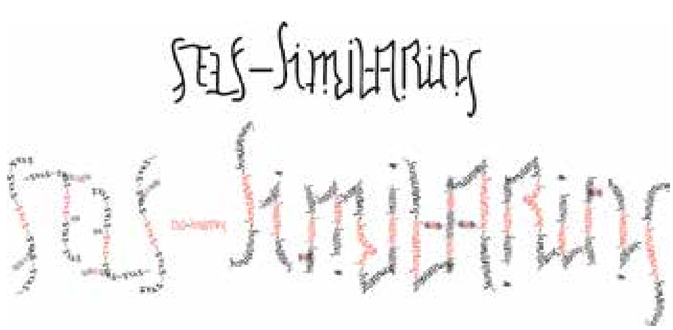

Figure 4: (Top) A rotational ambigram for "self-similarity." (Bottom) The strokes in the first ambigram are now replaced by words. The "self" is made up of tiny versions of "self" and "similarity" of smaller versions of "similarity" (each of which are ambigrams of course). That is not all, the hyphen is made up of a tiny version of the entire design!

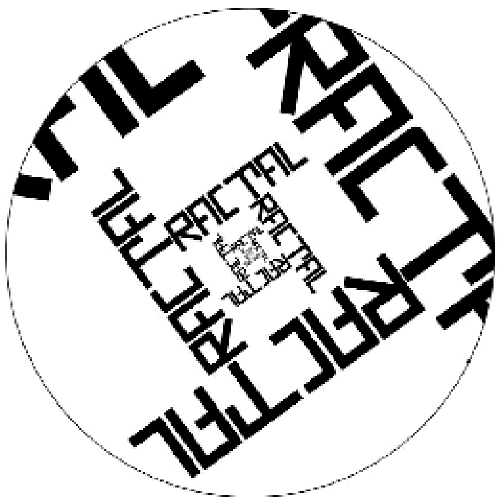

Figure 5: A self-similar, fractal ambigram for "Fractal"

In other words, fractals are geometrical shapes that exhibit invariance under scaling i.e. a piece of the whole, if enlarged, has the same geometrical features as the entire object itself. The design of Figure 6 is an artistic rendition of a fractal-like structure for the word "Mandelbrot".

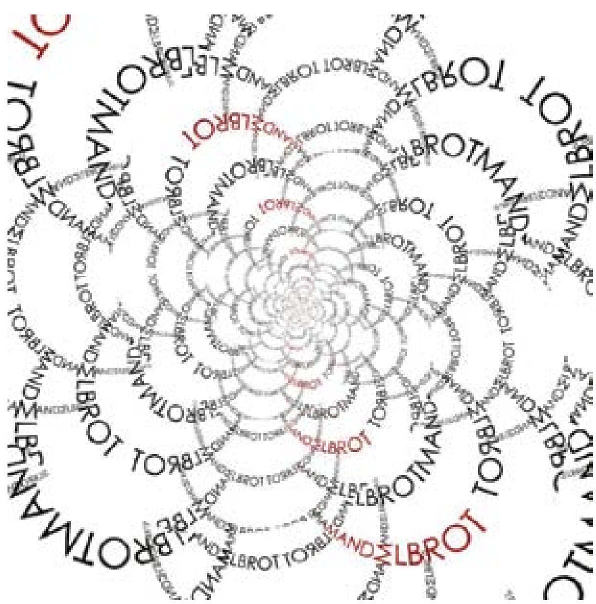

Figure 6. A fractal ambigram for "Mandelbrot"

Speaking of Mandelbrot, what *does* the middle initial "B" in Benoit B. Mandelbrot stand for? A clue is provided in Figure 7.

Figure 7. Puzzle: What does the B in "Benoit B Mandelbrot" stand for?
Answer at the end of the article.

It is clear that the idea of infinity and infinite processes are an important aspect of fractals and self-similarity. We now examine the concept of infinity typographically and mathematically.

Infinity

Infinity means *without end*, or limitless. Mathematically speaking, a finite set has a definite number of elements. An infinite set is a set that is not finite. The word infinity is also used for describing a quantity that grows bigger and bigger, without limit, or a process which does not stop.

Figure 8 has two designs for "infinity" subtly different from each other. Notice how in the first design the chain is created by "in" mapping to itself and "finity" mapping to itself. In contrast the second design breaks the word up differently, mapping "ity" to "in" and "fin" to itself.

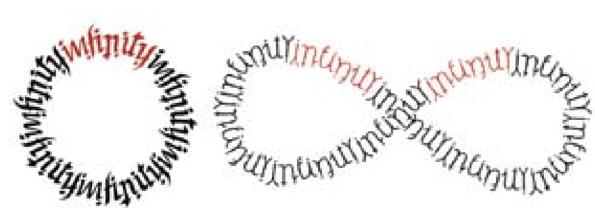

Figure 8. Two ambigrams for "infinity".
The first wraps around a circle and the second says infinity by word and symbol!

The first design wraps “infinity” around a circle. You can go round and round in a circle, and keep going on, so a circle can be said to represent an infinite path but in a finite and understandable manner. The second design is shaped like the symbol for infinity!

In keeping with the idea of self-similarity here are two other designs of the word “infinite”. In fact there is a deeper play on the word as it emphasizes the *finite* that is *in* the *infinite.* The two designs in Figure 9 capture slightly different aspects of the design. The first focuses on mapping the design onto a sphere while the second is a self-similar shape that can be interpreted in two different ways. Either being made of an infinite repetition of the word “finite” *or* the infinite repetition of the word “infinite” (where the shape that reads as the last “e” in the word “finite” can be read as “in” in the word “infinite” when rotated by 90 degrees).

Infinities are difficult to grasp and when we try to apply the rules that worked with finite quantities things often go wrong. For instance, in an infinite set, *a part of the set can be equal to the whole*! The simplest example is the set of natural numbers, and its subset, the set of even numbers.

The set $\mathbb{N} = \{1, 2, 3, 4, ...\}$ of natural numbers is infinite. Now consider the set of even numbers $\mathbb{E} = \{2, 4, 6, 8, ...\}$ Clearly, the set of even numbers has half the number of elements of the set of natural numbers, doesn’t it?

But not so quick! Things are tricky when it comes to infinite sets. We need to understand what it means for two sets to have an equal number of elements. Two sets have an equal number of elements when they can be put in one to one correspondence with each other. Think of children sitting on chairs. If each child can find a chair to sit on, and no chair is left over, then we know that each child *corresponds to* a chair, and the number of children is the same as the number of chairs.

Returning to the natural numbers, each number n in $\mathbb{N}$ corresponds to the number $2n$ in $\mathbb{E}$. So every element of $\mathbb{N}$ corresponds to an element of $\mathbb{E}$ and vice versa.

Thus though one set may intuitively look like it is half the other it is in fact not so! Our intuition is wrong, the sets $\mathbb{E}$ and N have the same number of elements. Since $\mathbb{E}$ is a part of $\mathbb{N}$, you can see that when it comes to infinite sets, *a part can be equal*

Figure 9. Two ambigrams for “infinite”, a play on the finite in infinite. Is the second design an infinite repetition of the word “finite” or “infinite?”

to the whole. In fact, this part-whole equivalence has sometimes been used to define an infinite set.

Another interesting example where a part is equal to the whole, is provided by a fractal known as the Sierpinski Carpet.

The Sierpinski Carpet

The Sierpinski Carpet, like all fractals, is generated using the process of iteration. We begin with a simple rule and apply it over and over again.

Begin with a unit square, and divide the square into 9 equal parts. Remove the middle square. Now for each of the remaining 8 squares, we do the same thing. Break it into 9 equal parts and remove the middle square. Keep going on in this way till you get this infinitely filigreed Swiss-cheese effect. See Figure 10 for the first couple of steps and then the fifth stage of the carpet.

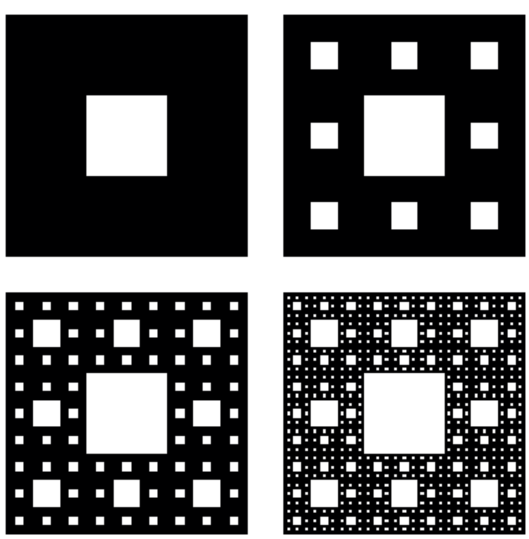

Figure 10. The Sierpinski Carpet

Which leads to the question: What is the total area of all the holes? Here is one way of computing the area of the holes in the Sierpinski Carpet. The first hole has area 1/9. In Step 2, you will remove 8 holes, each with area 1/9 th of the smaller square; so you will remove 8 holes with area $1/9^2$ or $8 \times \frac{1}{9^2} = 8/9^2$. In Step 3, for each of the smaller 8 holes, we remove 8 further holes with area $1/9^3$, so the area removed is $8^2/9^3$. In this manner it is easy to see that the total area of the hole is:

$$\frac{1}{9} + \frac{8}{9^2} + \frac{8^2}{9^3} + \frac{8^3}{9^4} + \cdots = \frac{1}{9}\left(1 + \frac{8}{9} + \frac{8^2}{9^2} + \frac{8^3}{9^3} + \cdots\right)$$

$$= \frac{1}{9} \times \frac{1}{1 - {}^{8}/_{9}} = 1.$$

To see why, we use the formula for the sum of the infinite Geometric Series:

$$1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}, \text{ for } -1 < x < 1.$$

How crazy is that! The area of the holes (taking away just 1/9th of a square at a time) is equal to the area of the unit square! *Thus the hole is equal to the whole!*

This seemingly contradictory statement has inspired the following design—where the words *whole* and *hole* are mapped onto a square – with the letter o representing the *hole* in the Sierpinski carpet. Of course as you zoom in, the *whole* and *hole* keep interchanging. We call this design *(w)hole in One* (in keeping with the idea the area of the *hole* is equal to the *whole* of the unit square).

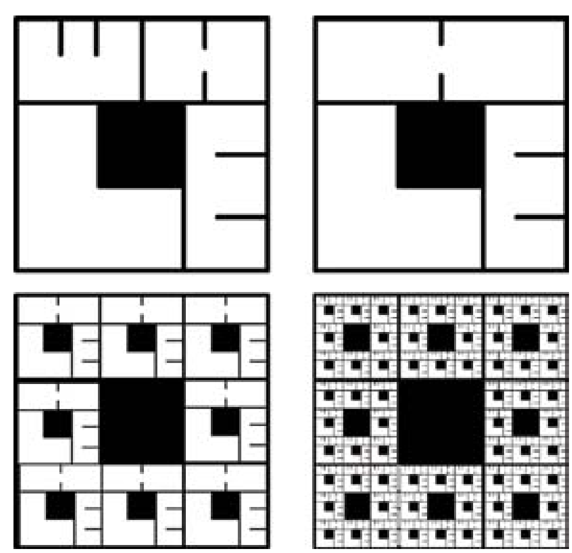

Figure 11. Fractal Ambigrams for "WHOLE" and "HOLE", a (w)hole in One.

The repetitive process of applying a set of simple rules that leads amazing designs like the Sierpinski carpet (and other fractal shapes) is called iteration.

Graphical interpretations of iteration

The process of iteration can be used to generate self-similar shapes. Graphically, we simply superimpose the original shape with a suitably scaled down version of the initial shape, and then repeat the process. The nested squares of Figure 2 is perhaps the simplest example of creating a self-similar structure using this process.

Essentially, such figures emerge from the repeated application of a series of simple steps—a program as it were, applied iteratively to the result of the previously applied rule. In this manner we can arrive at shapes and objects that are visually rich and complex.

Here is another, more creative, way to graphically interpret the idea of iteration.

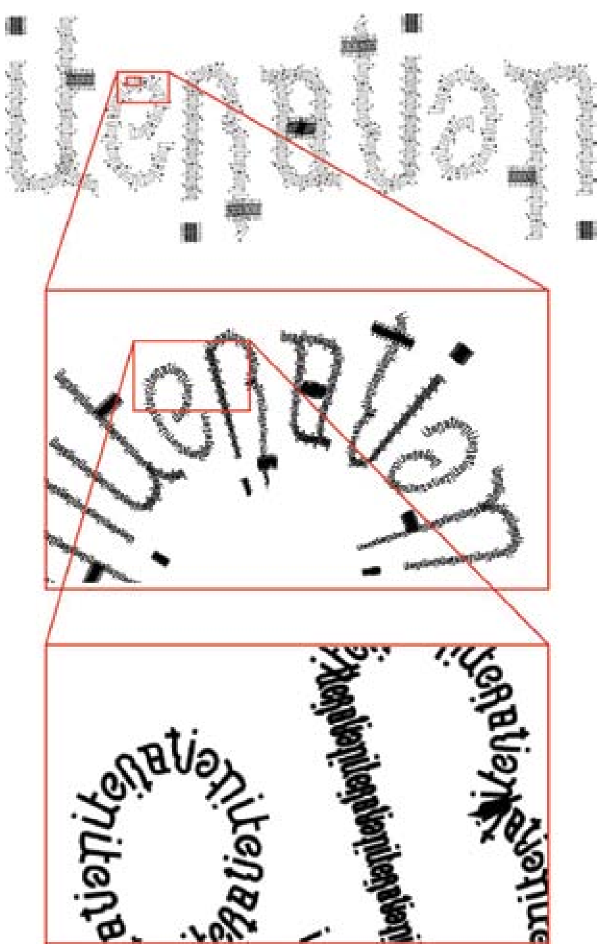

Figure 12: An ambigram of "iteration", illustrating a graphical approach to a part can be equal to the whole.

At one level the first ambigram in Figure 12 can be read as a rotational ambigram for the word "iteration." However if you zoom into the design (see zoomed figures below) you will see that each of the strokes is made of smaller strokes that in turn spell *iteration*.

In fact you can go down one more level and see "iteration" all over again. Theoretically we could do this forever, (within the limits of computational technology and visual resolution of screen, print and eye!). A similar idea is explored in the design of the word "self-similarity" (Figure 4) specifically in the design of the hyphen.

There are other fascinating examples of such iterative techniques, one of which we examine next.

The Golden Mean

Another example of a mathematically and visually interesting structure is the Golden Rectangle (and its close relative the Golden Mean). The Golden Mean appears as the ratio of the sides of a Golden Rectangle. A Golden Rectangle is such that if you take out the largest square from it, the sides of the resulting rectangle are in the same ratio as the original rectangle. Suppose the sides of the Golden Rectangle are a and b, where b is smaller than a. The ratio a/b turns out to be the Golden Mean (denoted by ϕ). The largest square will be of side b. Once you remove it, the sides of the resulting rectangle are b and $a - b$. From this, it is easy to calculate the ratio a/b and find that it equals .

$$\phi = \frac{1+\sqrt{5}}{2} = 1.618033988 \ldots.$$

If you begin with a Golden Rectangle and keep removing the squares, you will get a nested series of Golden Rectangles (see the underlying rectangles in Figure 14). The resulting figure shows self-similarity.

You may connect the diagonals using a spiral to obtain an approximation to what is called the Golden Spiral. Figure 13 shows an ambigram of "Golden Mean", placed in the form of a Golden Spiral inside a series of nested Golden Rectangles.

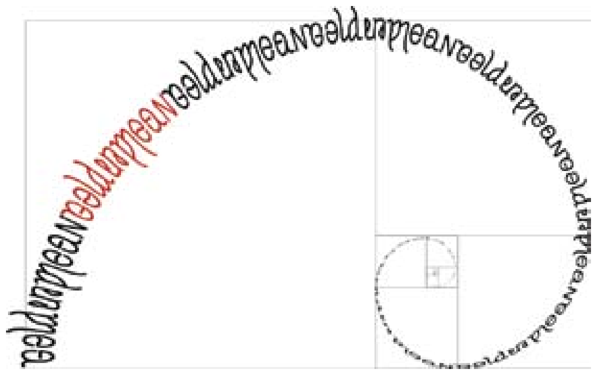

Figure 13. A rotationally symmetric chain-ambigram for the phrase "Golden Mean" mapped onto a Golden Spiral.

The Golden Mean appears in different contexts, in mathematics, in artistic circles, and even in the real world. It is closely related to the Fibonacci Numbers, namely 1, 1, 2, 3, 5, 8, 13, 21, ... Note that the Fibonacci Numbers begin with 1 and 1, and then each number the sum of the previous two numbers. If you take the ratio of successive Fibonacci numbers, the ratio converges to the Golden Mean.

The Fibonacci numbers are an example of a recursively defined sequence, where a few initial terms are defined, and then the sequence is built up by using the definition of the previous term (or terms).

Recursion and Pascal's Triangle

Recursion is similar to iteration. While iteration involves applying a simple rule to an object repeatedly, like in the creation of the Sierpinski Carpet, recursion involves using the results of a previous calculation in finding the next value, as in the definition of the Fibonacci numbers.

Fractals are usually obtained by iteration. Thus it is rather surprising that the fractal of Figure 14, called the Sierpinski triangle, may also be obtained using a recursive process.

The triangle in Figure 14 is a binary Pascal's triangle, where you use binary arithmetic (where $0 + 0 = 0$; $0 + 1 = 1$ and, $1 + 1 = 0$) to create the Pascal's triangle. The recursion is as follows: Each row and column begins and ends with a 1. Every other number is found by the (binary) addition of numbers above it. The formula for the recursion is

$$F(n+1, k) = F(n, k-1) + F(n, k)$$

where $F(n,k)$ is the term in the nth row and kth column, for $n = 0, 1, 2, 3, \ldots$ and $k = 0, 1, 2, 3, \ldots$ and the rules of binary arithmetic are used. In addition, we need the following values:

$$F(n, 0) = 1 = F(n,n).$$

This recurrence relation is the recurrence for generating Pascal's Triangle, satisfied by the Binomial coefficients.

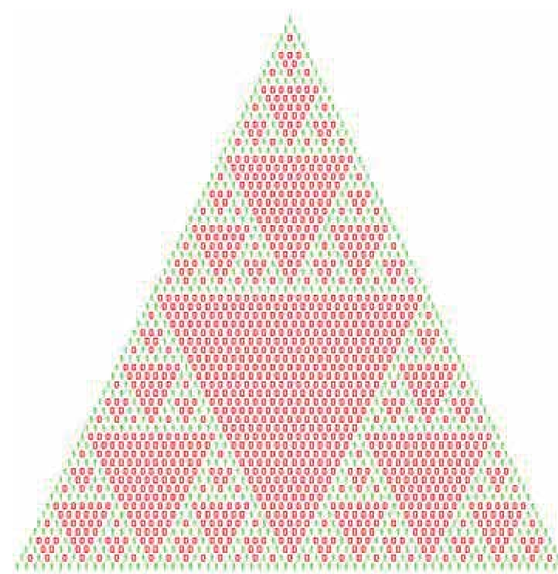

Figure 14. The binary Pascal's Triangle is also the Sierpinski Triangle

Of course, you can guess how to obtain the Sierpinski Triangle by iteration. Begin with a triangle, remove the middle triangle in step 1, which will leave behind three triangles to which you do the same! And just repeat this process forever.

The fact that Pascal's triangle is symmetric upon reflection, led to the design below (Figure 15)—made up of row over row of mirror-symmetric designs for the word "Pascal" increasing in size as we go down the rows. We call this design "*a* Pascals Triangle" (a triangle *made up of many* "Pascals") as opposed to "*the* Pascal's triangle" (the triangle *of or belonging to* Pascal). (Author's note: This design was created under psignificant work pressure. Can you guess why?)

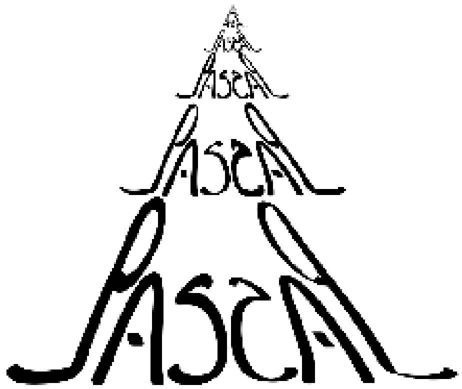

Figure 15. An ambigrammist's approach to Pascals Triangle (as opposed to Pascal's Triangle). What a difference an apostrophe makes!

Figure 16. An ambigram for "Hidden Beauty", whose beauty is not hidden from anyone!

In conclusion

We have explored many ideas in this article—self-similarity, iteration, recursion, infinity, and attempted to represent them graphically even while connecting them to deeper mathematical ideas. We hope that like us, you too experienced many feelings when you encountered these ideas—feelings of wonder, amusement, surprise, or the indescribable feeling when one finds something beautiful. We hope that these feelings make you wish to create something new, and perhaps dream up interesting things to share with your friends. As Borges eloquently said, "The mind was dreaming. The world was its dream." There is a lot of beauty one can find, hidden away in the world of ideas. We close with an ambigram for "Hidden Beauty" in Figure 16, where the word *hidden* becomes *beauty* when rotated 180 degrees!

Answer to the Puzzle in Figure 7: The "B" in "Benoit B Mandelbrot" stands for Benoit B Mandelbrot... and so on forever! Here is another way of representing the same idea, that we call, "Just let me B."

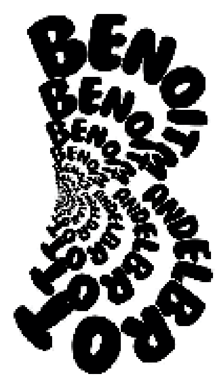

Figure 17. "Just let me B:" A fractal design for Benoit B Mandelbrot designed to answer the question "what does the B in Benoit B Mandelbrot stand for?"

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Over the years, they have shared their love of art, mathematics, bad jokes, puns, nonsense verse and other forms of deep-play with all and sundry. Their talents however, have never truly been appreciated by their family and friends.

Each of the ambigrams presented in this article is an original design created by Punya with mathematical input from Gaurav. Please contact Punya if you want to use any of these designs in your own work.

To you, dear reader, we have a simple request. Do share your thoughts, comments, math poems, or any bad jokes you have made with the authors. Punya can be reached at punya@msu.edu or through his website at http://punyamishra.com and Gaurav can be reached at bhatnagarg@gmail.com and his website at http://gbhatnagar.com/.

Math Exotica …

From magic squares to magic carpets

in the classroom

A. RAMACHANDRAN

Magic squares are a topic of interest to mathematicians, puzzlers and lay people alike. Apart from the mathematical properties, mystical qualities are often attributed to these in different cultures.

Magic squares are arrays of numbers (usually from 1 onwards) whose rows, columns and diagonals add up to the same 'magic' total. There is essentially just one 3 x 3 magic square with a magic sum of 15, but one could have obtained others by reflections and rotations.

There are a large number of different 4 x 4 magic squares (even excluding reflections and rotations.) In several of these, apart from the rows, columns and diagonals yielding the magic sum of 34, many other symmetrically located quartets of numbers give the same total.

Some 4 x 4 magic squares have the property that pairs of numbers symmetrically placed about the centre of the grid add up to 17. Two such pairs of numbers would form an interesting pattern, yielding the magic sum of 34. Let us for short refer to this property as the 'inversion symmetry'.

Keywords: *Magic squares, pattern, symmetry, inversion, quadrilaterals*

Source: http://en.wikipedia.org/wiki/File:Melencolia_I_(Durero).jpg

One such 4 x 4 magic square features in a celebrated work of art – an engraving titled Melancholia, executed by the German artist Albrecht Dürer in 1514. The square itself is shown below.

16	3	2	13
5	10	11	8
9	6	7	12
4	15	14	1

Observe that this has inversion symmetry. A straight line segment connecting the centres of a pair of squares thus related passes through the centre of symmetry and is bisected by it. With two such pairs of squares, therefore, we get two line segments that bisect each other. Hence the centres of the four squares in question form the corners of a parallelogram. To obtain such a parallelogram we must choose two squares out of the eight in one half of the 4 x 4 grid. The matching squares (their 'mates') get selected automatically. Now we have 28 ways of choosing 2 objects out of 8. One can identify these 28 parallelograms (with centres at the centre of the grid) and thereby obtain 28 quartets of numbers giving the magic total. Four of these shapes are actually squares, while four others are non-square rectangles, two are non-square rhombuses, sixteen are general parallelograms, and two are straight lines (the diagonals) which can be considered collapsed or 'degenerate' parallelograms. Some of these parallelograms are displayed below.

	10	11	
	6	7	

16			13
4			1

	3		
			8
9			
		14	

16			
		11	
	6		
			1

The equality of row and column sums is not a consequence of the inversion symmetry. They are independently contrived by a judicious distribution of the numbers 1 to 8 in the grid. (The other numbers then get assigned automatically.)

Each row and each column shares a symmetry axis with the entire grid. Eight other quartets giving the magic sum and sharing a symmetry axis with the entire grid are as follows: the four squares in each quadrant of the main grid and the corners of four 3 x 3 squares.

There are sixteen other quartets giving the magic sum, and with at least one element of symmetry, but not placed symmetrically in the main grid. These are eight 2 x 3 rectangles, four 'kites' (2 erect and 2 inverted) and four arrowheads (2 erect and 2 inverted). The latter two patterns appear only in the vertical sense and have no horizontal counterparts.

There are 86 ways of obtaining a sum of 34 by choosing four numbers from 1-16. (It would be an interesting but challenging exercise for the student to verify this.) Durer's magic square exhibits 60 of these in symmetrical patterns. (The student is invited to verify this as well.)

An Indian magic square

As a counterpoint to the magic square discussed above we look at a magic square of Indian origin.

7	12	1	14
2	13	8	11
16	3	10	5
9	6	15	4

It does not have the inversion property and so does not exhibit many properties that follow from it. However, apart from the row, column and diagonal property it has other interesting features.

Quartets of numbers forming the corners of eight isosceles trapeziums add to the magic sum. An example is given below.

	12	1	
16			5

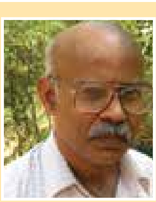
It has the 'pandiagonal' property, that is, quartets formed from numbers on the broken diagonals give the magic sum. An example follows.

		1	
	13		
16			
			4

Every 2 x 2 square gives the magic sum, as does the set of corners of each 2 x 4 rectangle.

The magic square can be extended by repetition in both East-West and North-South directions to give a 'Magic carpet' – an open 2-D array of numbers, where any four neighbouring numbers in a line (vertical, horizontal or diagonal) or forming a 2 x 2 square yield the magic sum. In addition, numbers at the corners of any 3 x 3 square and any 4 x 4 square yield the magic sum.

Further investigations in this area will surely prove to be a 'magic carpet ride' for a young mathematician or puzzle enthusiast.



A RAMACHANDRAN has had a long standing interest in the teaching of mathematics and science. He studied physical science and mathematics at the undergraduate level, and shifted to life science at the postgraduate level. He taught science, mathematics and geography to middle school students at Rishi Valley School for over two decades, and now stays in Chennai. His other interests include the English language and Indian music. He may be contacted at archandran.53@gmail.com.

in the classroom

Teacher's Diary on Classroom Assessment - III

Sneha Titus

In my last entry on CCE, I had included a project as part of the formative assessment. In order to allow for both individual and collaborative work, I had ambitiously planned to include a single project with both these components. It was now time to get real on my plans. In my search for suitable projects which encompassed a wide spectrum of arithmetic, geometric and algebraic components with a focus on mensuration, I naturally turned to tangrams. This topic is a favourite for both teachers and project designers. I wondered if I could get off the beaten path while taking advantage of the opportunities this material offered. That was when a colleague showed me how to make a tangram from a single A4 sheet using paper folding. This is the project I designed based on her input.

My rationale for developing this project was:

1. My overarching goal of enabling students to move from “Concrete to Abstract”. As students worked with paper cutting and then paper folding, they were able to see the dimensions change. For those comfortable with measuring the dimensions of their project at different stages, discussion with class mates would allow them to generalize and use algebraic terms instead of arithmetic quantities. I would of course aid this process with class discussions.

Keywords: *CCE, collaborative, project, paper folding, origami, tangram, quadrilateral, triangle, area, mensuration,concrete, abstract*

2. Algebraic simplification is never easy for students new to it. Very often, they simply don't see the point of it. Being able to calculate areas by algebraic simplification, and actually to verify the calculations, would help students tremendously.
3. The properties of quadrilaterals are often merely memorized. By asking students to create quadrilaterals, I hoped to make them understand and appreciate – by doing, not observing.
4. Working with paper and then generalizing in 2-dimensions would help students improve their spatial abilities.

Since the class was new to paper folding, I took time to explain the *valley fold* (inwards) and the *mountain fold* (outwards). As a group, we discussed the symmetries involved in folding. I was aware that this was not an easy project and that asking students to do independent work would result in them seeking external help. Throughout, I encouraged students to work with discussion, and I also had periodic whole class discussion sessions so that students could share difficulties. As each group had students working on identical individual projects they could always share notes and help each other along. I also asked students to document their progress, explaining that this would give them more credit than the 'right answer'.

Group Component: Each group of 7 students was given a single A4 sheet and asked to divide it into 2 quarters, 3 eighths and 2 sixteenths, as shown Figure 1. Each student takes a piece.

The group was asked to

[i] Show the calculation for the sum of all the pieces being equal to 1.

[ii] Find the ratio of the dimensions of the length l and the breadth b of the sheet of A4 paper.

[iii] Discuss and arrange the following in ascending order: $\frac{l}{2}, \frac{b}{2}, \frac{l}{4}, \frac{b}{4}, \frac{l}{8}, \frac{b}{8}$.

Individual Component: Next, each student in the group was given the following pieces with the accompanying instructions

[i] **Students 1 & 2:** Make a triangle out of the one- quarter A4 sheet. (Procedure: Triangle, Fig. 3)

[ii] **Student 3:** Make a triangle out of the one-eighth A4 sheet. (Procedure: Triangle, Fig. 3)

[iii] **Students 5 & 6:** Make triangles out of the one-sixteenth A4 sheet. (Procedure: Triangle, Fig. 3)

[iv] **Student 4:** Make a square out of the one-eighth A4 sheet. (Procedure: Square, Fig. 4

[v] **Student 7:** Make a parallelogram out of the one-eighth A4 sheet. (Procedure: Parallelogram, Fig. 5)

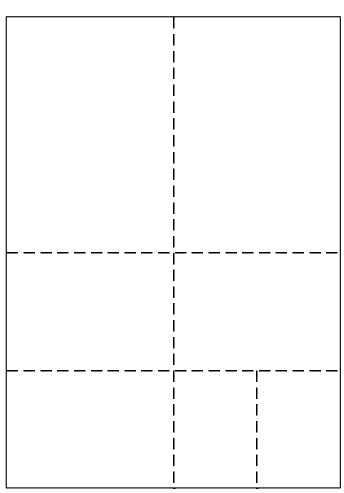


Figure 1. Cut the A4 sheet along the dotted lines to 7 parts - 2 equal large, 3 equal medium and 2 small equal rectangles

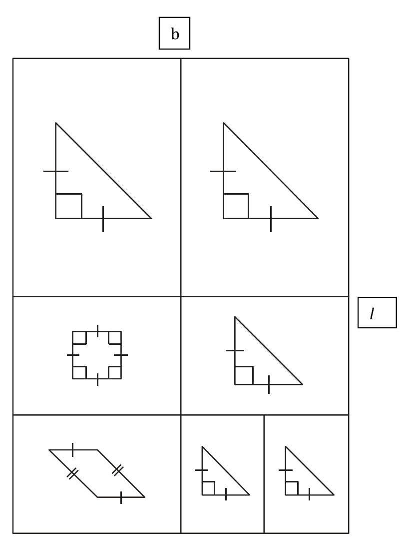


Figure 2.

General questions to be attempted by each student in the group after receiving his/her piece of paper. [The answers are in the accompanying teacher notes.]

1. If the original A4 sheet had length l and breadth b, what are the dimensions of the piece of paper assigned to you? Give your answer in terms of l and b and indicate which of them is smaller and which one is larger.
2. What is the area of the piece of paper you got?
3. What is the ratio of the area of your piece of paper to the area of the original A4 sheet?
4. Instructions for folding most of the shapes require you to start by folding off a square. What are the dimensions of the largest possible square for your piece of paper in terms of l and b?
5. Creating this square requires you to fold off a small rectangular extension – what are the dimensions of this rectangle in terms of l and b?

Procedure Triangle:

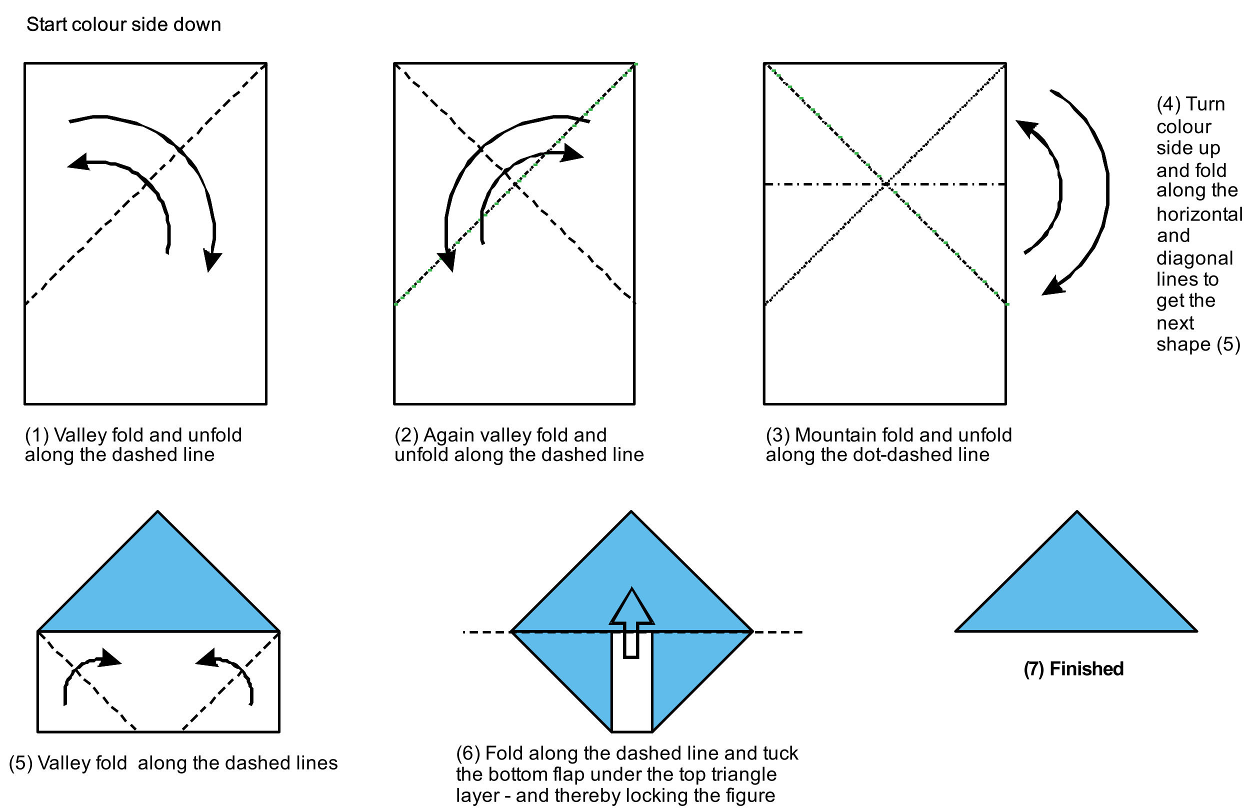


Figure 3

Specific questions to be attempted by the students making the triangle; the answers are in the accompanying teacher notes.

1. Obtain the area of the square that you mark off in terms of l and b in two different ways. Show your calculations.
2. Mountain folds are made along two creases in step 3. Why do these folds result in a triangle (surmounting a rectangle)?
3. Once the extra paper is tucked in, what kind of triangle do you get?
4. What are the angles of this triangle?
5. What are the lengths of the sides of the triangle? (Hint: You will need to use Pythagoras' theorem for this)
6. Find the area of the triangle in two different ways.

Procedure Square:

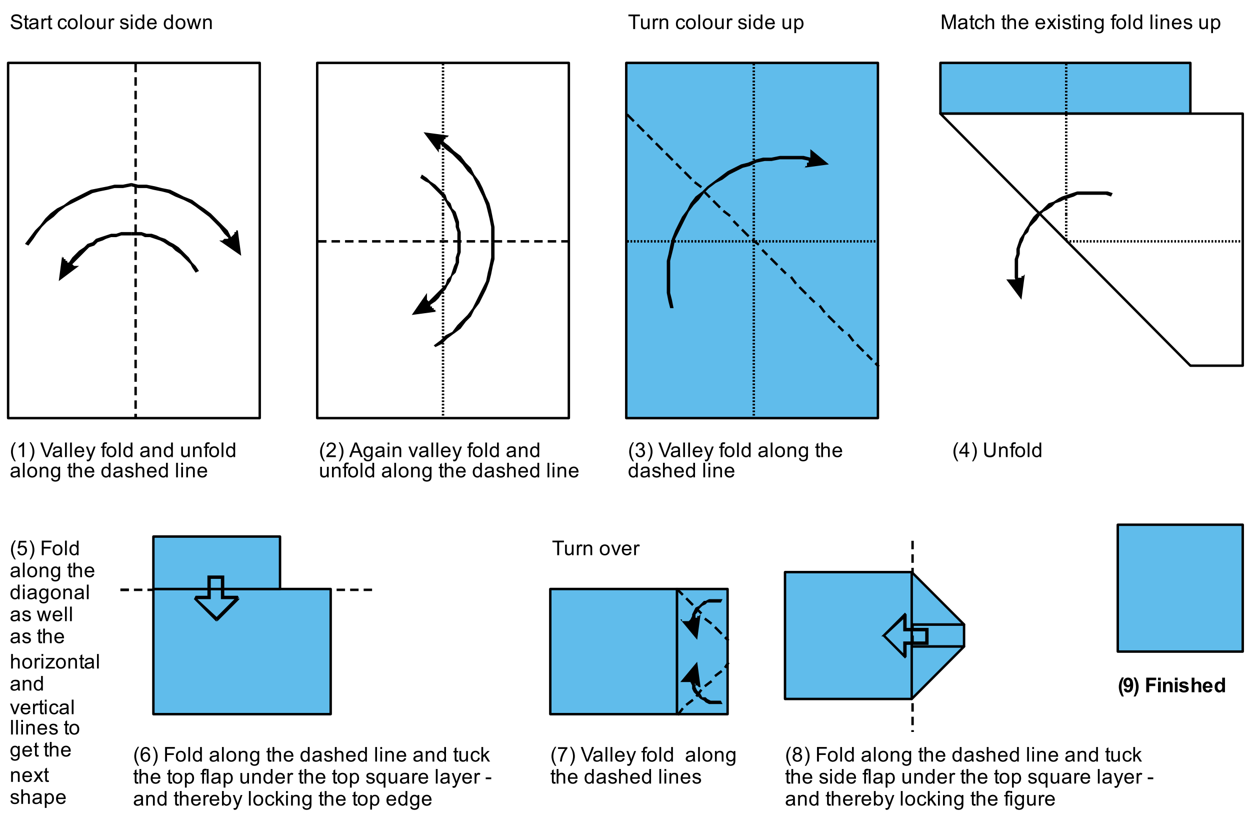


Figure 4

Specific questions to be attempted by the students making the square; the answers are in the accompanying teacher notes.

1. The three creases shown all pass through one point. What is this point?
2. The procedure for making the square involves marking off two rectangles on either side of a central quadrilateral. What are the dimensions of these two rectangles?
3. What angle does the valley fold in step 3 make with the vertical and the horizontal?
4. How do these angles enable the valley fold to create the fourth and fifth steps of Fig. 4?
5. The valley fold marks off a quadrilateral with two rectangles on either side in steps 5-9 of Fig. 4. What are the sides of this quadrilateral?
6. What type of quadrilateral is this? Give reasons for your answer.
7. What is its area?

Procedure Parallelogram:

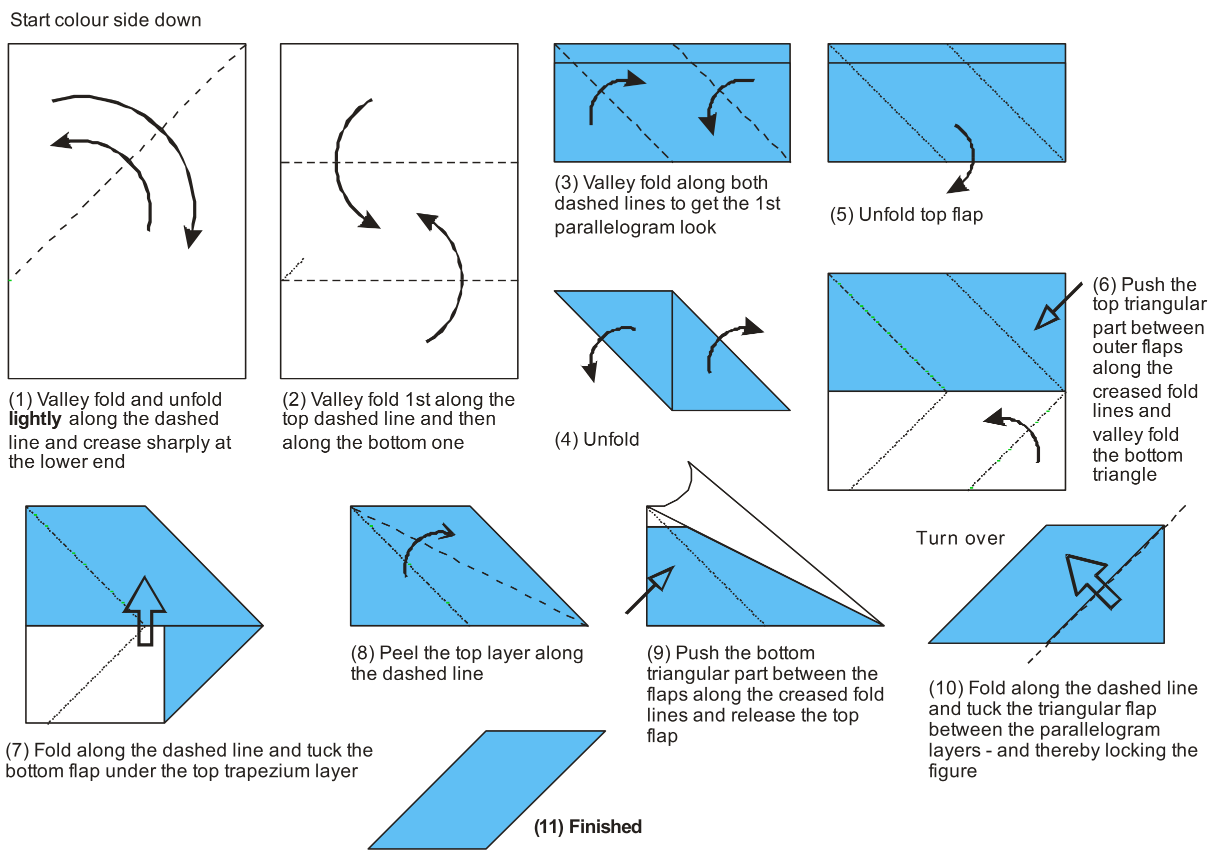


Figure 5

Specific questions to be attempted by the students making the parallelogram.

1. When the paper is folded into three, what are the dimensions of the larger rectangle that is obtained at the bottom (in step 3)?
2. Obtain the area of this rectangle in terms of *l* and *b* in two different ways. Show your calculations.
3. What is the ratio of the length to the breadth of this rectangle?
4. How does folding the two triangles give a figure whose opposite sides are equal and parallel?
5. What are the different kinds of polygons that emerge during this folding? Sketch or photograph them.

Assessment

When the students are being graded on their individual projects, I plan to focus on process rather than product. I will encourage them to put in as much detail as possible. I also plan to be open to their submitting assignments as Power Point presentations or other media where photographs or sketches are part of the work. The planned descriptors for the rubric are as follows:

1. Can the student follow the instructions given for the individual project?
2. Does the student seek help from peers or teachers to follow the instructions? (Note: This is not a negative marking, I encourage students to try on their own and to indicate in their submission how much they were able to manage on their own and what help they took from others.)
3. In answering the individual questions, has the student been able to sketch or describe the reasoning for the answer? Is the answer correct?
4. Has the student been able to obtain the answer by more than one method?

Answers to the general questions to be attempted by each student in the group after receiving his/her piece of paper.

1. **If the original A4 sheet had length l and breadth b, what are the dimensions of the piece of paper assigned to you? Give your answer in terms of l and b and indicate which of them is smaller and which one is larger.**

   Students 1 and 2: $l/2$ and $b/2$ of which $b/2$ is smaller.

   Students 3, 4 and 7: $l/4$ and $b/2$ of which $l/4$ is smaller.

   Students 5 & 6: $l/4$ and $b/4$ of which $b/4$ is smaller.

2. **What is the area of the piece of paper you got?**

   Students 1 and 2: $lb/4$

   Students 3 , 4 and 7: $lb/8$

   Students 5 & 6: $lb/16$

3. **What is the ratio of the area of your piece of paper to the area of the original A4 sheet?**

   Students 1 and 2: 1:4.

   Students 3, 4 and 7: 1:8.

   Students 5 & 6: 1: 16.

4. **Instructions for folding most of the shapes require you to start by folding off a square. What are the dimensions of the largest possible square for your piece of paper in terms of l and b?**

   Students 1 and 2: $\frac{b}{2} \times \frac{b}{2}$.

   Students 3, 4 and 7: $\frac{l}{4}$ and $\frac{l}{4}$

   Students 5 & 6: $\frac{b}{4}$ and $\frac{b}{4}$

5. **Creating this square requires you to fold off a small rectangular extension – what are the dimensions of this rectangle in terms of l and b?**

   Students 1 and 2: $\frac{l-b}{2}$ and $\frac{b}{2}$.

   Students 3 , 4 and 7: $\frac{l}{4}$ and $\frac{b}{2} - \frac{l}{4}$.

   Students 5 & 6: $\frac{b}{4}$ and $\frac{l-b}{4}$.

Answers to the specific questions to be attempted by the students making the triangle.:

1. **Obtain the area of the square that you mark off in terms of l and b in two different ways. Show your calculations.**

   Students 1 and 2 : $\frac{b}{2} \times \frac{b}{2} = \frac{b^2}{4}$

   Student 3 : $\frac{l}{4} \times \frac{l}{4} = \frac{l^2}{16}$

   Students 5 & 6 : $\frac{b}{4} \times \frac{b}{4} = \frac{b^2}{16}$

   The second method is to subtract the area of the rectangular extension from the area of the original piece. Students should be able to show that the answer is the same by both methods.

2. **Mountain folds are made along two creases. Why do these folds result in a triangle (surmounting a rectangle)?**

The mountain folds are along the diagonals of the square which divide the square into four congruent triangles. This allows two of the triangles to be tucked in and the remaining two to be superimposed exactly.

3. **Once the extra paper is tucked in, what kind of triangle do you get?**
   The triangle is isosceles and right-angled.
4. **What are the angles of this triangle? 45°, 45°, 90°**
5. **What are the lengths of the sides of the triangle? (Hint: You will need to use Pythagoras' theorem for this)**

   Students 1 and 2: $\frac{b}{2}, \frac{\sqrt{2}}{4}b, \frac{\sqrt{2}}{4}b$

   Student 3: $\frac{l}{4}, \frac{\sqrt{2}}{8}l, \frac{\sqrt{2}}{8}l$

   Students 5 & 6: $\frac{b}{4}, \frac{\sqrt{2}}{8}b, \frac{\sqrt{2}}{8}b$

6. **Find the area of the triangle in two different ways. The formula 'half base times height' gives the area of the triangles as**

   Students 1 and 2: $\frac{1}{2}\times\frac{\sqrt{2}}{4}b\times\frac{\sqrt{2}}{4}b=\frac{b^2}{16}$

   Student 3: $\frac{1}{2}\times\frac{\sqrt{2}}{8}l\times\frac{\sqrt{2}}{8}l=\frac{l^2}{64}$

   Students 5 & 6: $\frac{1}{2}\times\frac{\sqrt{2}}{8}b\times\frac{\sqrt{2}}{8}b=\frac{b^2}{64}$

Each triangle is $\frac{1}{4}$ the area of the square and the answers can be verified either using this or by using 'half base times height' with the longest side as the base and the height being the height of the rectangle tucked in.

Answers to the specific questions to be attempted by the students making the square:

1. The three creases shown all pass through one point. What is this point?
   This point is the centre of the rectangular piece of paper.

2. The procedure for making a square involves marking off two rectangles on either side of a central quadrilateral. What are the dimensions of these two rectangles? Each is $l/4$ and ½ $(b/2 - l/4)$

3. What angle does the valley fold in step 3 make with the vertical and the horizontal? 45° and 135°

4. How do these angles enable the valley fold to create the fourth and fifth steps of Fig. 4?
   The angles on either side of the valley fold are equal since the diagonals are perpendicular to each other. This allows the folding in of a triangle congruent to the external triangle.

5. The valley fold marks off a quadrilateral with two rectangles on either side in steps 5-9 of Fig. 4. What are the sides of this quadrilateral?
   The sides of this quadrilateral are half the smaller side of the original rectangle i.e. $l/8$.

6. What type of quadrilateral is this? Give reasons for your answer.
   A square since its 4 sides are equal (each is $l/8$ and perpendicular (by folding the 45 degree angles are doubled). Also it is made up of 4 pairs of congruent superimposed isosceles right triangles.

7. What is its area?
   Its area is which is the product of the sides Its area can also be obtained by taking ¼ (area of the original rectangle minus the area of the rectangular extensions).

   i.e. $\frac{1}{4}(lb/8 - 2(l/4 \times \frac{1}{2}(b/2 - l/4))$
   $= lb/32 \; lb/32 + l^2/64 = l^2/64$

Note: Only the most able eighth standard students can attempt this level of algebra which is why only one method of calculating area is asked for.

Answers to the specific questions to be attempted by the students making the parallelogram:

1. When the paper is folded into three, what are the dimensions of the larger rectangle that is obtained at the bottom (in step 3)?

   $\frac{l}{4}$ and $\frac{1}{2}\left(\frac{l}{4}\right)$.

2. Obtain the area of the rectangle that you get in terms of l and b in two different ways (if possible). Show your calculations. Multiplying the above, we get . Else, we can find ½ (area of original rectangle area of first fold)

   $=\frac{1}{2}\left(\frac{lb}{8}-\frac{l}{4}\times\left(\frac{b}{2}-\frac{l}{4}\right)\right)=\frac{l^2}{32}.$

3. What is the ratio of the length to the breadth of this rectangle? 2:1

4. How does folding the two triangles give a figure whose opposite sides are equal and parallel? The rectangle consists of two congruent squares (since the length is twice the breadth). The triangles are folded along the diagonals of these adjacent squares. The diagonals are equal in length and inclined at to the base of the rectangle.

5. What different kinds of polygons emerge during this folding? Sketch or photograph them. A heptagon, a pentagon, two adjacent squares and the parallelogram

Putting the pieces together

Once the individual projects were submitted and graded, the groups of 7 students can come together for more traditional tangram projects. Useful ideas for these may be obtained from [2], [3] and [4] and many more are available on the Internet. The groups work on these with the individual pieces created by them. The outcome does depend on each piece so at this point a shoddily constructed block will affect the whole. Students may choose to redo their individual projects at this stage. Of course, the planned descriptors for the rubric will again focus on process rather than product but the group will gain points for cooperative work and for demonstrating group responsibility and individual responsibility.

References:

[1] Swati Sircar: Sr. Lecturer, Azim Premji University and Chang Wen Wu: Origamist extraordinaire

[2] Hands on Math for Class 6: Jonaki B Ghosh, Haneet Gandhi, Tandeep Kaur

[3] Hands on Math for Class 7: Jonaki B Ghosh, Haneet Gandhi, Tandeep Kaur

[4] Math Masti Booklet 2: math4all (www.math4all.in)

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UN-FOLDING

"Mathematics in/from anything and everything" was the theme of the Association of Math Teachers of India (AMTI) conference at Kochi in January 2014. Swati Sircar, mathematics resource person at Azim Premji University delivered this talk in which she folded the cloth to match the math.

Swati Sircar

In case you are wondering what is about to unfold, let me recap the theme of the conference: **mathematics in/from anything and everything.** What do we mean by anything and everything? Let me take this opportunity to focus on a mundane task that most people, women in particular, do almost every day - folding clothes. Let us see what mathematics is hiding within the folds and where folds can lead us, mathematically of course!

Let me give you a few examples of how girls and women have an edge over the male of the species regarding mathematics! Let's begin with a sari.

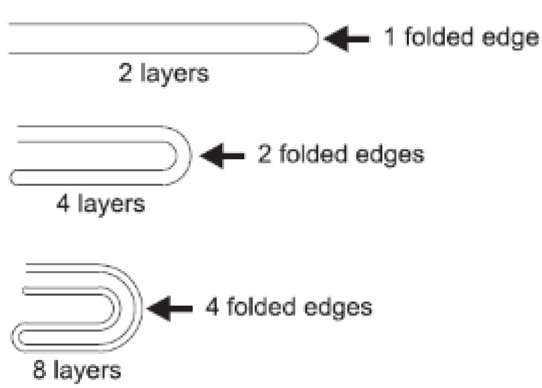


Figure 1.

Keywords: *mathematization, folding, exponents*

The sari is a long piece of cloth which symbolizes female Indian attire. When you fold a sari, you need to be careful while holding the folds, else you will miss some edges! If you count the number of folded edges after each fold, you get the powers of 2, viz. 1, 2, 4, 8 (typically, no one folds a sari beyond that). Why? Well, folding is equivalent to folding in half, so after the first fold you get 1 folded edge. This doubles at the second fold giving 2 folds, which doubles at the third fold giving 4 folds, and so on (see Figure 1). Table 1 displays the count of folded edges after each fold.

Fold number	1	2	3	4	5
Number of folded edges	1	2	4	8	16
Thickness of each fold	1	1/2	1/4	1/8	1/16

Table 1.

You can generalize that n folds will generate edges 2^{n-1}. This can be a good starter for teaching exponents. It is important to draw attention to what are we halving, and whether that is getting 'compensated' elsewhere. Essentially with each fold we halve the length of the (folded) sari. The resulting length is compensated by the number of layers. This is important as we are not cutting and throwing away something but only folding, i.e., the whole is intact. So the resulting length and the number of layers always maintain a reciprocal relation.

Simple halving folds can be used to initiate the study of Geometric Progressions (GP) and the sum of a GP. Take 2 handkerchiefs of the same size but of different colour and place them one over the other. Fold the top one in half. Each 'sheet' now represents ½. Fold the top again in half. Now the top represents,

$$\left(\frac{1}{2}\right)^2 = \frac{1}{4} \quad ,$$

while the bottom represents

$$\frac{1}{2} + \frac{1}{4} = \frac{3}{4} = 1 - \left(\frac{1}{2}\right)^2.$$

If you keep going, after the nth halving, the top represents

$$\left(\frac{1}{2}\right)^n$$

and the bottom represents

$$\frac{1}{2} + \frac{1}{4} + \cdots + \left(\frac{1}{2}\right)^n.$$

One can see from this that

$$\frac{1}{2} + \frac{1}{4} + \cdots + \left(\frac{1}{2}\right)^n = 1 - \left(\frac{1}{2}\right)^n.$$

This idea can be explored further to derive the formula for calculating sum of the first n terms of a GP. Note that you can also do the halving along the diagonals (Figure 2).

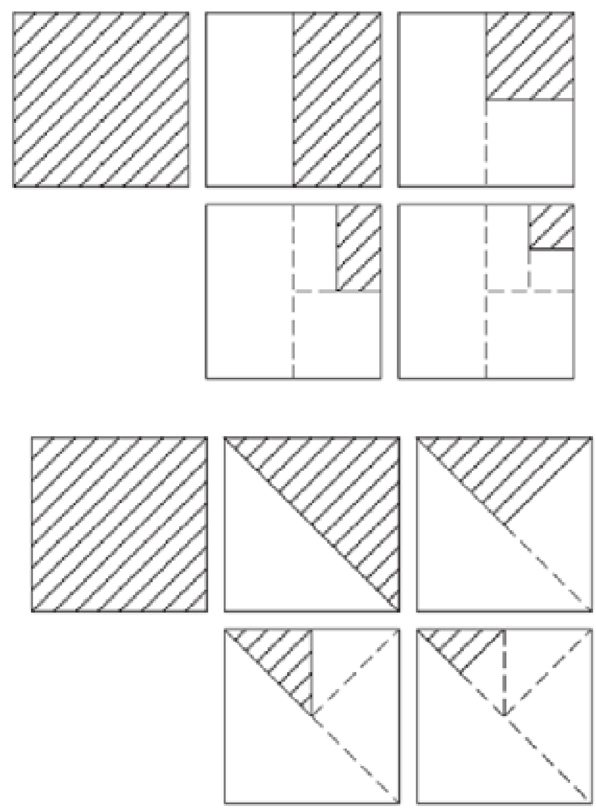


Figure 2.

The pedagogic possibilities in folding are not limited to concepts related to halving (and doubling). Here is another situation where girls and women score over boys (and men). If you wish to fold a bed sheet or a handkerchief, you can start with any edge. Here we are assuming that we want a "nice" way of folding where the edges match up at each fold. Now, take a skirt or a petticoat. Can you start folding along any edge? No. You must start at the top edge. Otherwise the side edges will not match up. Why? The reason lies in the shape.

A bed sheet or handkerchief is rectangular. Hence both pairs of opposite sides are parallel to each other. But a skirt or a petticoat is like an isosceles trapezium (at best), and their vertical sides are not parallel to each other. When we fold a cloth (or paper), a particular edge gets folded in a way that the 2 parts of the edge match, the fold line is perpendicular to that edge, because we are halving the straight angle, i.e.,180° and getting 2 right angles (90°) on either side of the fold. The fold line therefore is the angle bisector of the straight angle represented by the edge (Figure 3).

Since opposite sides of a rectangle are parallel, any line perpendicular to one edge will be perpendicular to the opposite edge as well. So we can start with any edge and fold, and the opposite side will naturally match up. But if we fold either of the vertical sides of a skirt or a petticoat, the fold line perpendicular to that edge will not be perpendicular to the opposite edge, as the vertical edges are not parallel (Figure 4).

However such a situation does arise with men's attire too: bell-bottom pants, after the first fold brings the two trouser legs together. The eternal popularity of the sari ensures that women will always encounter such folds whereas men will have to wait for the vagaries of fashion to experience this aspect of mathematics!

This simple folding technique can be used to test if two lines are parallel or not. Fold along both lines. Now fold a perpendicular to one line. Check if the two parts of the other line have coincided with each other. If they have, then the fold line is also perpendicular to the second line and therefore the two lines are parallel to each other (as both are perpendicular to the fold line). If not, the two lines are not parallel to each other (Figure 5).

If we study a folded petticoat or skirt, more geometry unfolds. The first vertical fold halves the cloth and the 2 parts exactly match. That makes the fold line special. It is the line of symmetry of the skirt or petticoat. Given the isosceles trapezoid shape, there is just one such line. Naturally, we started with that line. Whenever you fold and cut, and then unfold to see the resulting pattern, you cannot but see line symmetry. This can be used with multiple folds to generate the following:

(a) Rotational symmetry (by using folds passing through the same point), or :

(b) Translation symmetry (by using folds parallel to each other).

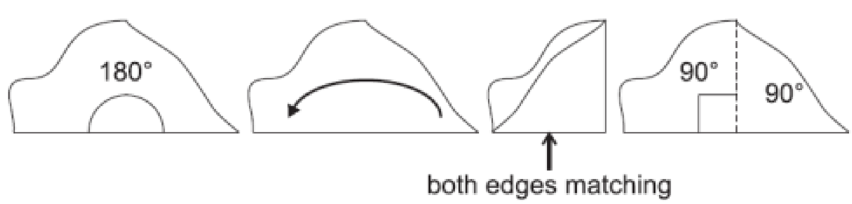


Figure 3.

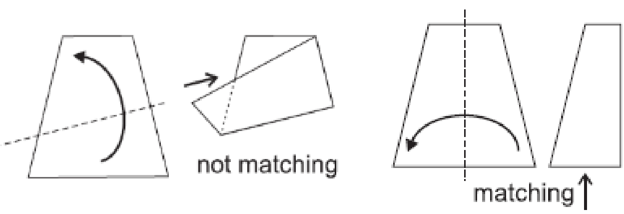


Figure 4.

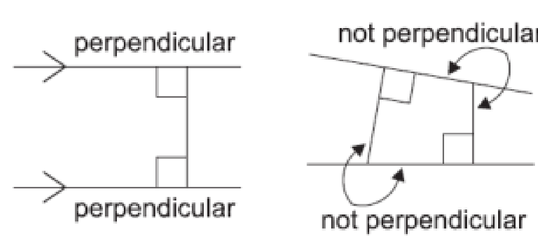


Figure 5.

Beautiful patterns can then be generated (Figure 6).

This helps explain how double reflection on intersecting lines creates rotation, the intersection point being the center of rotation. The angle of rotation is double the acute angle generated by the intersecting lines (Figure 7).

Similarly double reflection on parallel lines creates translation. The distance translated is double the distance between the parallel lines (Figure 8).

So folds are closely linked to line symmetry and reflection and can be used to show many geometric properties of triangles and quadrilaterals, especially those involved with congruence.

Just to give an example, suppose you want to compare the sides and angles in a triangle. There are 2 theorems related to these – side relations implying angles relations and their reverses – for scalene as well as for isosceles triangles. Let us take a closer look at them through the lens of paper folding:

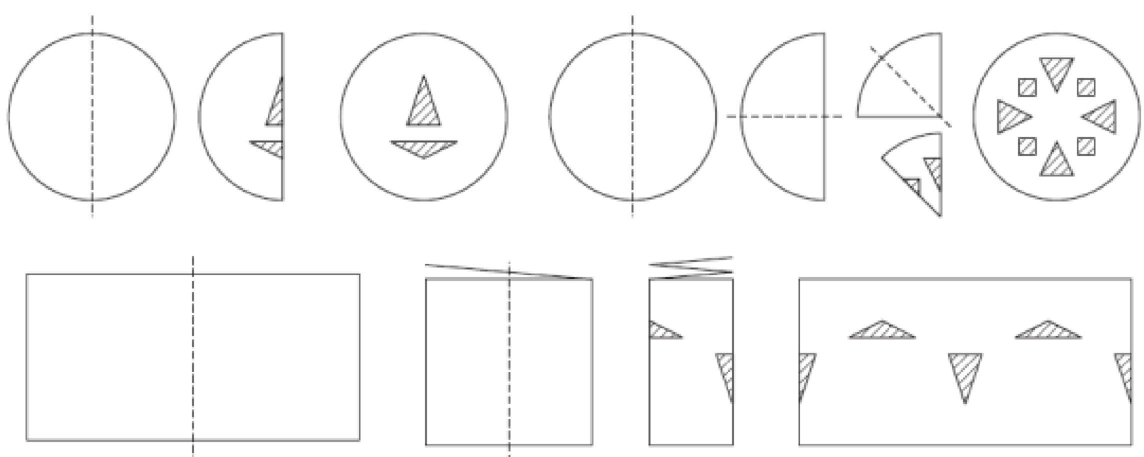


Figure 6.

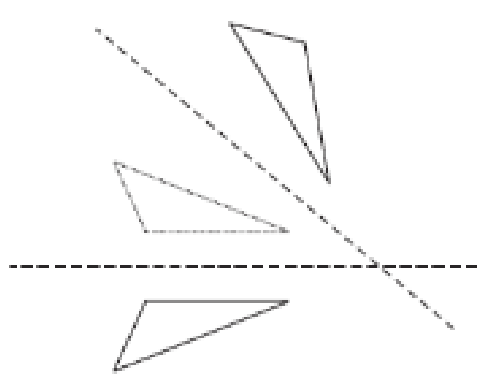


Figure 7.

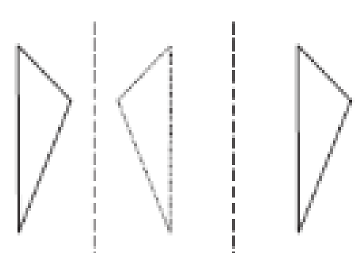


Figure 8.

Part A: angle relations implying side relations

Paper Folding Theorem

To compare any 2 sides of any triangle fold one on the other starting from the common vertex. The fold is actually the bisector of the angle between these 2 sides (Figure 9).

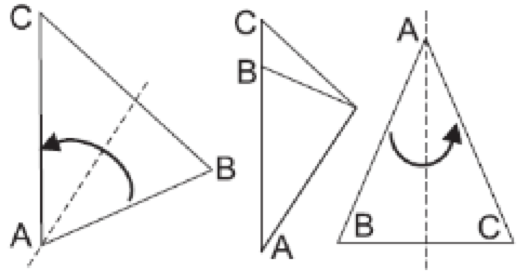


Figure 9.

As you can see from the right, the diagrams match what you get through folding. The construction in each case is exactly the corresponding fold line.

Now let us look at two theorems:

Theorem 1: In ΔABC, ∠C < ∠B ⇒ AB < AC

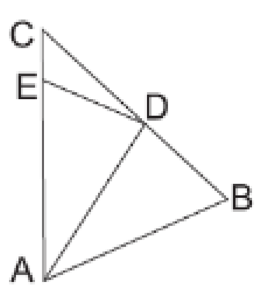


Figure 10.

Draw the angle bisector of ∠A that meets BC at D (Figure 10).

∴ ∠DAB = ∠DAC = ½∠A

∠ADC = ∠ABD + ∠BAD = ∠B + ½∠A >

∠C + ½∠A = ∠ ACD + ∠DAC = ∠ADB

∴ We can cut off an angle equal to ∠ADB from ∠ADC.

Let E be a point on AC such that ∠ADE = ∠ADB. Then ΔABD ≅ ΔAED, by ASA.

∴ AB = AE < AC.

Theorem 2: In ΔABC, ∠B = ∠C ⇒ AB = AC

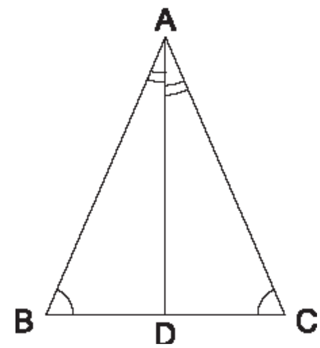


Figure 11.

Draw the angle bisector of ∠A that meets BC at D (Figure 11).

∴ ∠DAB = ∠DAC = ½∠A

∴ ΔABD ≅ ΔACD by AAS

∴ AC = AB

Part B: side relations implying angle relations

Paper Folding

Similarly to compare any 2 angles, one can halve their common side, i.e., fold the perpendicular bisector of their common side (Figure 12).

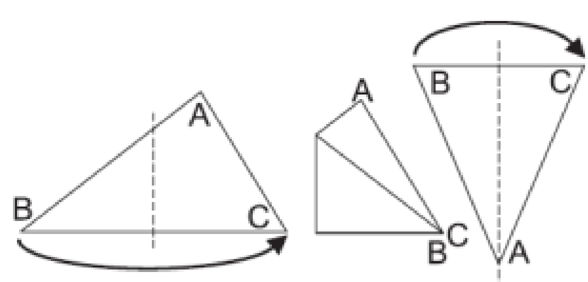


Figure 12.

Observe how the folded figure for scalene (or unequal angles) overlapped with the unfolded triangle generates the diagram on the right (Figure 13).

Here the corresponding theorems are as follows:

Theorem 3: In ΔABC, AB < AC ⇒ ∠C < ∠B

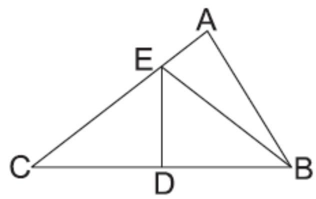


Figure 13.

Draw the perpendicular bisector of BC that meets AC at E*, while D is the midpoint of BC

∴ CD = BD and ∠EDC = ∠EDB

And since ED is common side, by SAS, ΔEDC ≅ ΔEDB

∴ ∠C = ∠ECD = ∠EBD < ∠ABD = ∠B

* For an explanation of why E is always between A and C see ** below

For isosceles triangles, i.e., AB = AC ⇒ ∠C = ∠B, the fold is the perpendicular bisector of BC (which one can observe goes through A). The proof uses the perpendicular from A to BC. Therefore though the lines are all same, their meanings are a bit different.

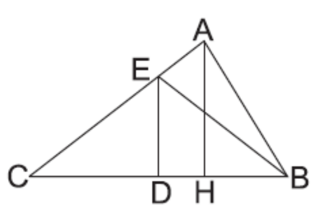


Figure 14.

**It always used to bother me why the perpendicular bisector of BC will intersect the larger side AC as opposed to the shorter one AB. On reflection the following turns out: Let AH ⊥ BC (Figure 14), ∴ by Pythagoras $AC^2 = AH^2 + CH^2$ and $AB^2 = AH^2 + BH^2$, AH is common, AB < AC ⇒ BH < CH ∴ the midpoint D of BC falls within CH ∴ the perpendicular bisector of BC cuts side AC and not side AB.

Note how this involves Pythagoras which comes much later in the syllabus. But the similar logic in "angle to side" i.e., ∠C < ∠B ⇒ AB < AC is simpler. However, textbooks usually include the proof of "side to angle" i.e., AB < AC ⇒ ∠C < ∠B without mentioning the above. Then "angle to side" is proved by contradiction.

The reader can explore which other properties of triangles and quadrilaterals (and angles) can be demonstrated through folding.

One figure stands out as an exception to the above, as we cannot use folding to check its properties. Any guesses? It's the parallelogram. Why? Recall that folds correspond precisely to line symmetry. Incidentally, the parallelogram is the only quadrilateral that has rotational but not line symmetry. Every other quadrilateral with any kind of symmetry has a line of symmetry. The only property of the parallelogram that can be demonstrated with folds is that the diagonals bisect each other. I will leave it to the reader to figure out how to do so. You can refer to the annexure for the basic folds. Interestingly these basic folds have a 1-1 onto mapping with the basic constructions!

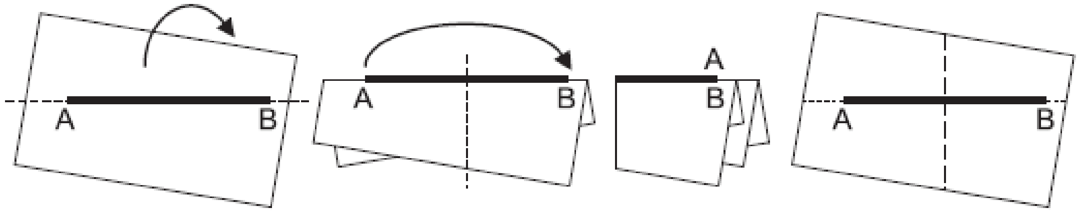
But before going more deeply into comparing folds with constructions (with compass and straight edge), let me get back to the original theme. This symmetry aspect of folding is crucially used in one profession whose benefits we all enjoy. Can you name this profession? It is tailoring. We human beings externally have bilateral symmetry on a gross scale. Naturally, our clothing imitates that. And the tailor smartly uses this symmetry by folding the cloth before drawing and cutting.

Let me wrap it up (or fold it) with a treat that folding enables but Euclidean straight-edge and compass construction does not: trisecting an angle. It is possible to trisect any angle by folding, but we know that we cannot do the same with a compass and straight edge. If you are curious about this, please refer to *At Right Angles*, Volume 1, No. 2, "Axioms of Paper Folding" (page 16) by Shiv Gaur.

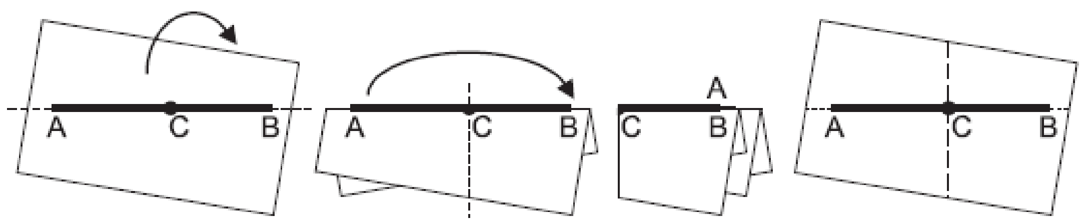
Annexure

Basic Folds

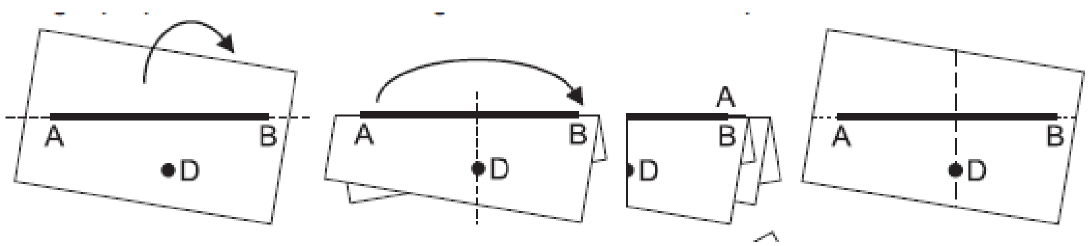
1. bisecting a line segment



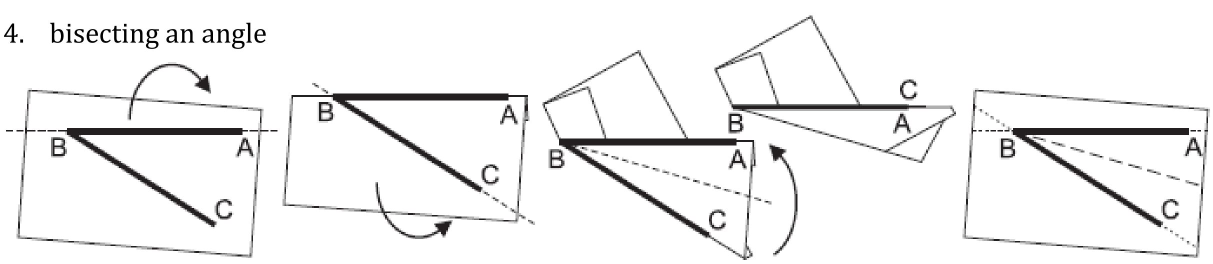
2. folding a perpendicular to a line segment from an internal point



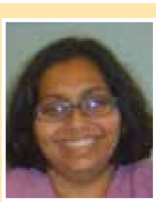
3. folding a perpendicular to a line segment from an external point



4. bisecting an angle



*Easier option: cut along the sides of the angle and then fold



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PENTOMINOES

Gautham Dayal

Pentomino puzzles were invented (or discovered) in the early 1900s by Henry Dudeney, an English inventor of puzzles (who is unfortunately not as well known as he should be). They then appeared sporadically in recreational mathematical magazines in the 1930s and 1940s. Interest in them was revived when Solomon Golomb wrote about them in the 1950s. They were popularized by Martin Gardener in his column MATHEMATICAL GAMES that appeared in the Scientific American as well as in his books on recreational mathematics. While they are a valuable educational resource in their avatar as puzzles, they can also be used effectively to build spatial intuition.

So what are pentominoes?

Take five identical squares. Now place them one at a time so that (apart from the first square) each square touches at least one of the squares already placed along a complete edge. The various shapes that can result are known as pentominoes ('penta' meaning five and 'mino' to suggest that they are related to dominos).

Keywords: *Pentomino, puzzle, spatial, congruent, transformation, translation, rotation, reflection*

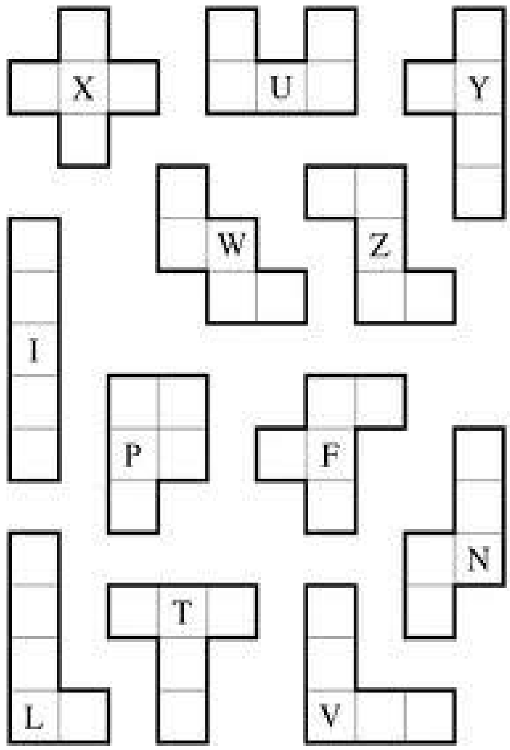

Figure 1: The twelve pentominoes

It turns out that it is possible to make *twelve* such shapes using five squares. These twelve shapes make up a set of pentominoes. In order to talk about the various pieces, it helps to name them. They are usually denoted by alphabets as in Figure 1.

Pentominoes are often used in the same way that *tangrams* are. So, a pentomino puzzle is a shape drawn on a sheet of paper. Solving the puzzle requires one to place the 12 (or sometimes fewer) pentomino pieces on a plane surface to form a shape similar to the one given on the sheet. A Google search gives a large number of sites devoted to such puzzles; the more useful of these sites grades the puzzles in order of difficulty.

In this article, we are not as concerned with *difficult* pentomino constructions (which is a natural and worthwhile place to aim to go once we are familiar with these pieces), as with ways in which pentominoes can be used in a classroom with a group of students to develop geometric intuition. However we give some examples of these construction puzzles at the end of the article, including some references.

Finding pentominoes – an activity

A worthwhile question to investigate with a group of students is that of finding all possible pentominoes and we sketch here one way to do this. While this can be done with paper and pencil, it sometimes helps to have a number of squares cut out of card (or other suitable material).

One begins with the definition of pentominoes and what it can mean for two squares to touch correctly. We observe that there is only one monomino (the square). We see different ways to add a square to this monomino and see that all the possibilities end with the same result (a domino).

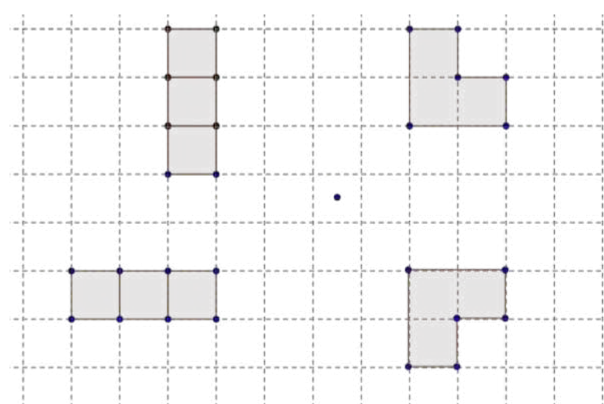

Figure 2: From domino to tromino

To move to trominoes, one sees the various ways to join a square onto a domino. While there are many ways of doing this, there are only two distinct ones ('I' and 'L'; see Figure 2). This is a good time to use words such as 'congruent', 'rotate' and 'reflect'.

Now add a square to the two possible trominoes to get tetrominoes. At this stage, it may be noticed that some pairs of shapes are not identical if only translations and rotations are allowed, but can be superposed if reflection is permitted (that is, flipping over). We find that there are five tetrominoes; see Figure 3.

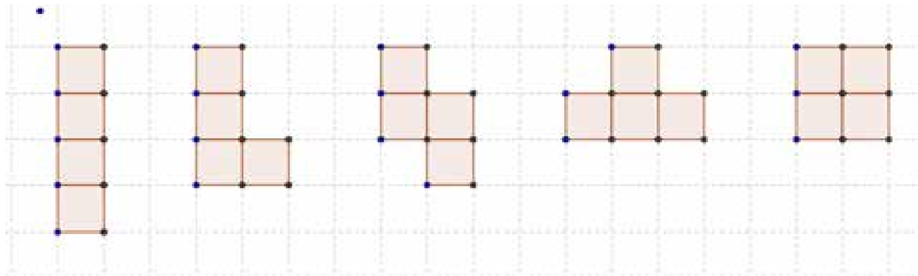

Figure 3: The five tetrominoes

Add another square to obtain pentominoes. Students will need to keep track of the different ways in which a square can be added and the various congruences which occur to find the full list of possible pentominoes. Using the symmetries of each tetromino also makes this enumeration more efficient.

Getting to know the pentominoes

There are a couple of ways to develop a sense of familiarity with the pieces.

- One way which works well with younger students is to give them a set of pentominoes and let them construct any figure they like (this strategy also works with other dissection puzzles like the tangram): figures of animals, houses, vehicles, etc. They could be given sheets of squared paper to copy out the silhouette of the figures they have built. They can also be asked to mark out the positions of the pieces on their drawings. Often this gives rise to arrangements that can later be used as puzzles for other children.
- While it is commonly thought that a pentomino puzzle must be made using all the twelve pieces, this is not necessary. One could start off with puzzles using just two pieces and build from there. For examples of such puzzles, please look at the CIMT website from which the Space-filling problems in Figure 4 have been taken.

While these are particularly suited to develop spatial abilities in children, they are also suitable to train children to think methodically and develop reasons for eliminating possible options.

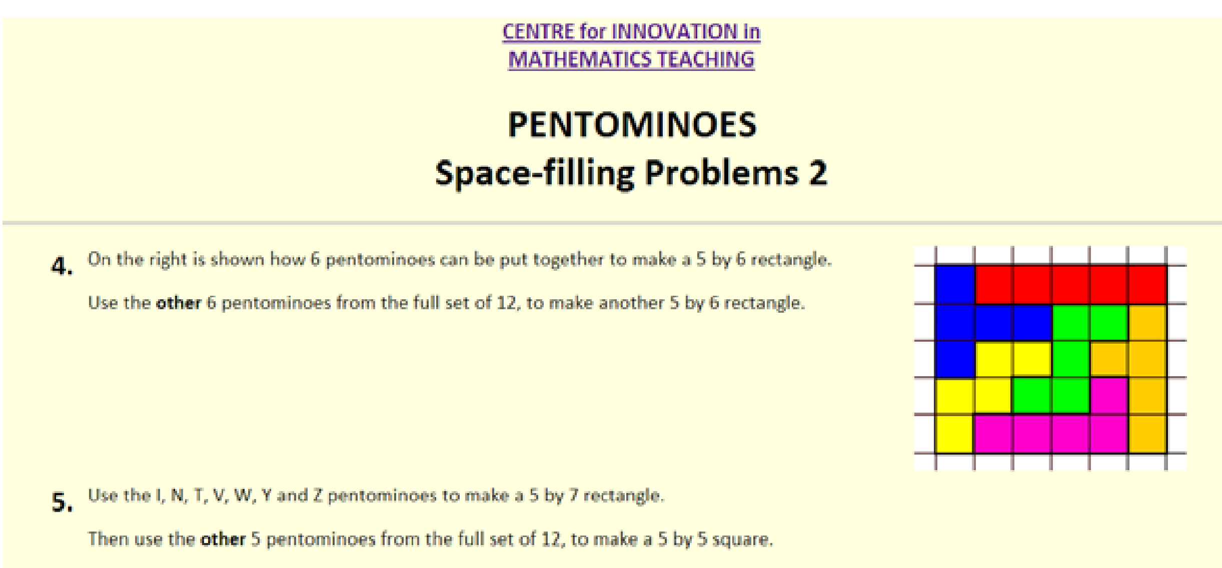

Figure 4: Pentomino puzzles from the CIMT website (http://www.cimt.plymouth.ac.uk/)

A board game

While constructing shapes is usually the key feature while using pentominoes, there is at least one board game using the pentomino pieces, which can be played by two or three players.

Mark out an eight by eight grid (or use a chess board whose squares have the same size as the squares making up your set of pentominoes). The first player chooses a piece and places it on the board in such a way that it lies entirely on the board and follows the grid. The next player has to lay a piece in a similar manner, with the additional requirement that it does not lie over a previously placed piece. Players take turns laying pieces until one of them is no longer able to put down a piece. The last player who can lay down a piece is the winner.

The idea behind this game is related to that of pentomino exclusion, an idea that is explored in Chapter 3 of Golomb's book (referred to below).

Making pentominoes

While sets of pentominoes are commercially available, one would recommend making them by hand, especially with students. In this section we suggest a few ways in which a set of pentominoes can be made. Since much of what is involved in this activity can be used to make other puzzles and manipulatives, this opens up many opportunities. You could choose depending of the time, resources and material available.

- Graph paper and card – draw out the twelve pentominoes on graph paper so that each square has a side of 20mm or 25mm. It works better if the pieces are drawn out with no shapes touching. Stick the graph paper on a sheet of heavy KG card. Cut the shapes out using scissors. If this is being done by older students or teachers, use a blade and ruler to cut out the inside corners to get more precise cuts.
- Wood – most plywood shops sell wooden beading which is 3/4th inch or an inch thick. Choose pieces that are neatly cut. Using a tri-square, mark out pieces of the sizes you need and cut them out using a saw with fine teeth (even a hacksaw will work). Don't cut them all into squares to be more efficient (for example, cut out the I piece as a single piece, five units long, the X as one piece three units long and two squares). Using a wood glue (Fevicol SH, for example) stick these together to get the shapes you need.

  You need to keep in mind that wood sticks best 'along the grain' while orienting your pieces. Also, getting precise cuts might take some practice and patience.

  If one is able to find a workshop that cuts beading, it is worthwhile trying to get some lengths of wood that are square in cross-section. Cutting these out into cubes allows one to make pentominoes suitable for three-dimensional puzzles as well as experiment with making other puzzles like the 'soma cube'.
- Acrylic or other plastics – this is the most "high-tech" method. One can draw out the pentomino pieces using sketching software (such as Inkscape or SketchUp, both available on the net without a fee). The files can then be given to a laser cutter to be cut out on a variety of materials. Tracking down a laser cutter involves enquiring at shops that make rubber stamps or trophies. Often operators of these machines will also be able to sketch out the pieces on appropriate software for those reluctant to use Inkscape / SketchUp (though learning to use these software tools is a worthwhile skill to learn). The advantage of this method is that it is possible to make a number of sets fairly quickly, and access to a laser-cutting workshop could lead you to explore and make other *dissection* puzzles.

Resources

The book "Polyominoes: Puzzles, Patterns, Problems, and Packings" (revised and expanded second edition) by Solomon Golomb (pub: Princeton University Press, 1994) has a number of excursions into the geometry of pentominoes. While not directly accessible to most school students, it is certainly possible for teachers to adapt some of the material in it for use with students.

The internet has a large number of sites devoted to pentominoes for students of all ages, and a particularly good one is the CIMT website http://www.cimt.plymouth.ac.uk/resources/puzzles/pentoes/pentoint.htm which has pentomino problems accessible to school children, many of which are inspired by Golomb's book mentioned above.

Appendix: Some pentomino puzzle sites

There are many good sites for such puzzles, for example:

- http://puzzler.sourceforge.net/docs/pentominoes.html
- http://isomerdesign.com/Pentomino/
- http://gp.home.xs4all.nl/PolyominoSolver/Polyomino.html

Here are some typical pentomino puzzles which we have taken from the site http://isomerdesign.com/Pentomino/. (In Figure 5, the regions shown shaded represent 'holes'.)

EIGHT BY EIGHT SQUARES WITH 'HOLES' IN FOUR DESIGNATED SQUARES

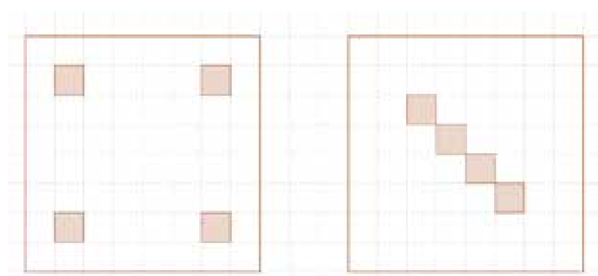

Figure 5: Two pentomino puzzles requiring making a eight by eight shape

TRIPLICATION: USING NINE PENTOMINOES TO CREATE THE FOLLOWING SHAPES

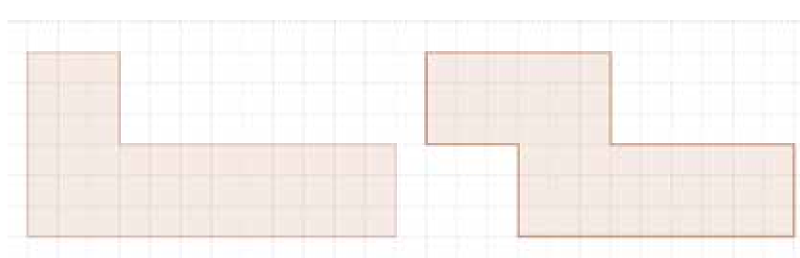

Figure 6: Using nine pentominoes to create a triplicate 'L' and a triplicate 'N'

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Gautham's interests are in the ways that Making and Play can be used as ways to learn math and physics. He has more recently been exploring the opportunities that technology provides in teaching/learning. In Gautham's words: ' I enjoy working with materials,particularly wood. While I do use this interest to make puzzles and toys, I find that I am not very good at solving these puzzles.' Gautham may be contacted at gautham.vanya@gmail.com

How To Prove It

This continues the 'Proof' column begun earlier. In this 'episode' we study some problems concerning the prime numbers, and a theorem from triangle geometry.

Shailesh Shirali

A property of prime numbers

The following is a striking property of the primes:

If p is a prime number exceeding 3, then $p^2 - 1$ *is a multiple of 24.*

In general, statements about prime numbers are daunting to prove — in part because the primes are so highly irregular in their distribution. Indeed, we do not have any formula to generate the primes. So how might we go about proving the above statement?

Let's check it first. The primes exceeding 3 are: 5, 7, 11, 13, 17, 19, 23, ... Squaring them and subtracting 1, we get the numbers 24, 48, 120, 168, 288, 360, 528, ... It is easily checked that each of these numbers is a multiple of 24. Indeed, their greatest common factor or GCD is 24. (Another term for GCD is HCF: 'highest common factor'. But GCD is currently the accepted term in higher mathematics.)

A strategy for proving the result. Here is an approach to finding a proof: *Suppose that the claim is true. What does it lead to, what does it imply? By studying these implications, can we uncover a proof?* Let's do just this. An obvious implication of the given statement, which holds because $24 = 3 \times 8$, is the following: *If p is a prime number exceeding 3, then* $p^2 - 1$ *is a multiple of both 3 and 8.*

Keywords: *Prime number, divisibility, least common multiple, direct proof, indirect proof, proof by contradiction, SAS congruence, Euclid*

Now an idea strikes us. If we show that a number K is a multiple of both 3 and 8, would it follow that K is a multiple of 24? Yes. The reason for this is seen by listing the multiples of 3 (namely: 3, 6, 9, 12, 15, 18, 21, 24, 27, 30, ...) and the multiples of 8 (namely: 8, 16, 24, 32, 40, ...). On examining these lists we find that the numbers common to them are 24, 48, 72, ...; they are all multiples of the least number in the list, which is 24 (that's precisely why 24 is called the 'least common multiple' or LCM of 3 and 8).

So we have found a strategy for solving the problem: *Prove that if* $p > 3$ *is prime, then* $p^2 - 1$ *is divisible by both* 3 *and by* 8.

But this is easy!

Divisibility by 8 Since p is a prime number exceeding 3, it is odd. But we know from what we proved in the an earlier (*How To Prove It*, November 2013) that if n is odd, then $n^2 - 1$ is a multiple of 8. Hence it must be true that if $p > 3$ is a prime number, $p^2 - 1$ is a multiple of 8.

(For those who missed that issue, here is a quick proof. Let n be odd. Then $n = 2k + 1$ for some integer k. This yields: $n^2 - 1 = (2k + 1)^2 - 1 = 4k^2 + 4k = 4k(k + 1)$. Since k and $k + 1$ are a pair of consecutive integers, one of them is even, hence $k(k + 1)$ is even; and this implies that $4k(k + 1)$ is a multiple of 8. Hence $n^2 - 1$ is a multiple of 8.)

Divisibility by 3 Since $p > 3$ is prime, on division by 3 it leaves a remainder of 1 or 2. So $p = 3k + 1$ or $3k + 2$ for some integer k. Now we need to check the divisibility of $p^2 - 1$ by 3 for these two forms.

- If $p = 3k + 1$, then $p^2 - 1 = (3k + 1)^2 - 1 = 9k^2 + 6k$, which is a multiple of 3.
- If $p = 3k + 2$, then $p^2 - 1 = (3k + 2)^2 - 1 = 9k^2 + 12k + 3$, which too is a multiple of 3.

Either way, $p^2 - 1$ is a multiple of 3.

Since $p^2 - 1$ is a multiple of both 3 and 8, it follows that $p^2 - 1$ is a multiple of 24.

Remark on the strategy followed You may wonder why we selected the numbers 3 and 8. Because $3 \times 8 = 24$? Not quite. Instead of 3 and 8, what if we select 4 and 6? It is easy to show that if $p > 3$ is prime, $p^2 - 1$ is a multiple of both 4 and 6. But since the LCM of 4 and 6 is 12, this would only prove that $p^2 - 1$ is a multiple of 12. It would *not* prove that $p^2 - 1$ is a multiple of 24.

Here are two more such results. In both we consider the effect of division by 120.

1. *If* p *is a prime number exceeding* 5*, the remainder when* p^2 *is divided by* 120 *is either* 1 *or* 49*.*
2. *If* p *is a prime number exceeding* 5*, then* $p^4 - 1$ *is a multiple of* 120*.*

For example, take the primes 17 and 19. We have:

$$17^2 = 289 = (120 \times 2) + 49,$$
$$19^2 = 361 = (120 \times 3) + 1,$$

and:

$$17^4 - 1 = 83520 = 120 \times 696,$$
$$19^4 - 1 = 130320 = 120 \times 1086.$$

We ask you to find the proofs of these statements. *Hint*. $120 = 3 \times 5 \times 8$. Hence you must consider the effect of dividing p^2 by 3, 5 and 8 respectively.

Direct and Indirect Proof

Proofs do not all follow the same approach; they come in different flavours and different colours. For example, proofs can be *direct* or *indirect*, and this is a crucial distinction. We now elaborate on this matter. Say we are given two 'propositions' or assertions, P and Q, and we are required to show: "If P is true, then Q is true" (more briefly: "If P, then Q", or "$P \implies Q$"). A "direct proof" is one where we start with P and travel 'directly' to Q, along a linear chain of deductions. In an 'indirect proof' the starting point may not be P. Instead we may ask: Could it be that Q is *not* true? What might be the consequences of assuming that Q is not true? What would it tell us about P? Thus we consider various alternatives to Q and then eliminate them, one by one, forcing us to 'accept' Q.

Direct proof We give two examples of direct proof. Note how they start with the given premise and proceed in a linear way to the desired conclusion.

Example 1. Prove: “For any integer n, the remainder in the division $n^2 \div 4$ is 0 or 1.”

Proof: Suppose that n is even. Then $n = 2k$ where k is some integer. This implies that $n^2 = 4k^2$, so n^2 is a multiple of 4.

Next, suppose that n is odd. Then $n = 2k + 1$ where k is some integer. This implies that $n^2 = 4k^2 + 4k + 1 = 4k(k + 1) + 1$, and we see that n^2 is 1 more than a multiple of 4 and hence leaves a remainder of 1 under division by 4.

Example 2. Prove: “If n is a positive integer such that the number $x := 2^n - 1$ is prime, then the number $\frac{1}{2}x(x + 1)$ is perfect.” (A ‘perfect number’ is one for which the sum of the proper divisors equals the number itself. Example: 6 is perfect, since $1 + 2 + 3 = 6$. In the rule stated, if we take $n = 3$, we get $x = 7$, which is prime, and this gives us the perfect number $\frac{1}{2}(7 \times 8) = 28$. This general rule was first mentioned by Euclid in *The Elements*.)

Proof: We must show that $\frac{1}{2}x(x + 1) = 2^{n-1} \cdot x$ is perfect. The number has two distinct prime divisors (2 and x). This fact enables us to enumerate its full list of divisors:

$$\begin{cases} 1, & 2, & 2^2, & 2^3, & \ldots, & 2^{n-1}, \\ x, & 2x, & 2^2 \cdot x, & 2^3 \cdot x, & \ldots, & 2^{n-1} \cdot x. \end{cases}$$

Of these, all are proper factors except the very last one, $2^{n-1} \cdot x$, which is the number itself. We must now find the sum of all the proper factors. For this we use an often-used identity: *The sum of the first several powers of* 2, *starting with* 1, *is* 1 *less than the next higher power of* 2. Thus, $1 + 2 = 2^2 - 1$, $1 + 2 + 2^2 = 2^3 - 1$, $1 + 2 + 2^2 + 2^3 = 2^4 - 1$, and so on. Using the identity we find the sum of the proper factors of $2^{n-1} \cdot x$:

$$(1 + 2 + 2^2 + \cdots + 2^{n-1}) + (1 + 2 + 2^2 + \cdots + 2^{n-2})\,x$$
$$= (2^n - 1) + (2^{n-1} - 1)\,x = x + (2^{n-1} - 1)\,x$$
$$= 2^{n-1} \cdot x.$$

So the sum of the proper factors of $2^{n-1} \cdot x$ equals the original number, $2^{n-1} \cdot x$, just as we wished to prove.

Indirect proof Direct proof may seem the most natural kind of proof. But there are situations where a direct proof does not seem possible, or is too difficult. In such cases, it may be simpler to look for an indirect proof. Here, the significance of the word ‘indirect’ is that the proof proceeds by elimination of the alternatives other than the one we wish to prove. Occasionally we come across situations where the indirect route is more natural than the direct one; it may even be aesthetically more pleasing. A few examples will serve to illustrate these comments.

Example 3. Prove: “If $n > 1$ is an integer such that $2^n - 1$ is prime, then n is prime.”

Proof: How do we show that a number (known to exceed 1) is prime? Here are two ways: either we show that it has no proper divisors; or we show that it cannot be composite. The latter is the indirect way, and it is what we adopt here.
We have been told that $2^n - 1$ is prime.
Since n is either prime or composite, there are two possible situations which can occur:

(A) $2^n - 1$ is prime and n is prime.

(B) $2^n - 1$ is prime and n is composite.

These two possibilities are contrary to each other (they cannot both occur). Also, there are no possibilities other than these. (So (A) and (B) form a mutually exclusive list.) We wish to show that it is (A) that occurs, and an obvious strategy for doing so is to show that (B) *cannot* occur. This is what we now do.

Suppose that $n > 1$ is composite; then $n = ab$ for some two integers $a > 1$ and $b > 1$, and $2^n - 1 = 2^{ab} - 1$. Let $k = 2^a$. Then:

$$2^n - 1 = (2^a)^b - 1 = k^b - 1. \qquad (1)$$

The number $k^b - 1$ has a factorization which is easy to anticipate:

$$k^b - 1 = (k - 1)\left(k^{b-1} + k^{b-2} + k^{b-3} + \cdots + k + 1\right). \qquad (2)$$

(This comes from observing that $k^2 - 1 = (k - 1)(k + 1)$, $k^3 - 1 = (k - 1)\left(k^2 + k + 1\right)$, etc.)

Both factors in the factorization (2) exceed 1; for, the smaller of the two factors is $k - 1$, and $k - 1 = 2^a - 1$ which exceeds 1 since a exceeds 1. Hence $k^b - 1$ is not prime, i.e., $2^n - 1$ is not prime. Note what has happened: by supposing that n is composite, it has turned out that $2^n - 1$ is composite as well. But this means that possibility (B) has been falsified; it cannot occur. Hence it is possibility (A) which must occur. Therefore, if $2^n - 1$ is prime, it must be that n itself is prime.

Do you see why this proof is called ‘indirect’?

Example 5. Prove: "If a triangle has two equal angles, then the sides opposite to the equal angles are equal." Stated otherwise: "In $\triangle ABC$, if $\angle B = \angle C$, then $AB = AC$."

Proof: First, some background. This problem is Theorem I.6 in *The Elements*; it comes just after I.5: "In an isosceles triangle, the angles opposite the equal sides are equal." (Or: "In $\triangle ABC$, if $AB = AC$, then $\angle B = \angle C$.") At this point in the text, the only congruence result available is "SAS congruence" (I.4): "If two sides of one triangle are equal, respectively, to two sides of another triangle, and the angles included by the two pairs of sides are equal, then the two triangles are congruent to each other." (The fact that the angle is 'included'between the two sides is crucial.) Euclid uses it to prove I.5 as shown in Figure 1.

If we attempt to prove Theorem I.6 the same way, we run into a difficulty. Try it out for yourself! You will find that no matter where we locate D on BC (possibilities: foot of internal bisector of angle BAC; midpoint of BC; foot of perpendicular from A to BC), we are unable, using SAS congruence, to show that $\triangle ABD \cong \triangle ACD$. In each case, we find that 'SAS' fails to apply; either the sides are wrong, or the angle itself is wrong.

Given: $AB = AC$. Draw AD to bisect $\angle BAC$. Now compare $\triangle ABD$ and $\triangle ACD$.

SAS congruence applies: $AB = AC$, AD is a shared side, and $\angle BAD = \angle CAD$. Hence $\triangle ABD \cong \triangle ACD$, and $\angle B = \angle C$.

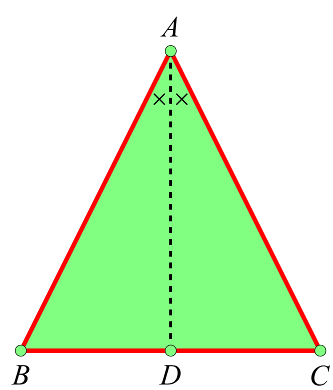

FIGURE 1. Theorem I.5: In Δ *ABC*, if *AB*= *AC*, then $\angle$B =$\angle$C

Given: $\angle ABC = \angle ACB$. Suppose $AB > AC$. Locate D on AB such that $DB = AC$. Now compare $\triangle DBC$ and $\triangle ACB$.

SAS congruence applies: $DB = AC$, $BC = CB$, and $\angle DBC = \angle ACB$. Hence $\triangle DBC \cong \triangle ACB$. But this is absurd!

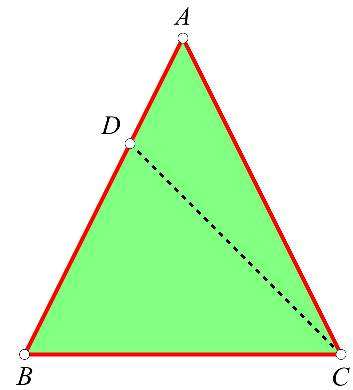

FIGURE 2. Theorem I.6: In Δ *ABC*, if $\angle B$ = $\angle C$, then *AB* =$\angle$*AC*

So Euclid uses a different strategy, and it is very ingenious. He asks, "Suppose that what is to be proved is *not* true (i.e., the sides are not equal). What happens then?" Now the possibility that $AB \neq AC$ can be subdivided into two possibilities: $AB > AC$, $AC > AB$. If we can show that both these are not possible (or "absurd" to use Euclid's words), then the desired conclusion would follow ($AB = AC$). To carry out this aim, Euclid assumes that $AB > AC$, argues as in Figure 2, and arrives at the conclusion that $\triangle DBC \cong \triangle ACB$. But this is absurd, since $\triangle DBC$ is contained within $\triangle ACB$, and the part cannot be equal to the whole. The absurd conclusion came about because of what we had assumed: $AB > AC$. If we had assumed instead that $AC > AB$, a similar absurdity would follow. So neither of these assumptions can be made. But then the only possibility left is $AB = AC$. And this is just what we wanted to show.

Note the indirectness of the strategy. The direct approach was found to be infeasible, so Euclid adopts the indirect route. His proof is an example of *proof by contradiction*.

Closing remark. When you see indirect proof for the first time in a mathematics class, you may get the impression that it is a form of reasoning peculiar to mathematics. But in fact we employ this kind of reasoning routinely in daily life, without realizing it. When you read a crime thriller and encounter the word 'alibi', you are dealing with just this form of reasoning! Here's how this happens. Say a crime has occurred in some house, and the police have pinpointed the time of the crime: it happened at 11 pm. The chief suspect for the crime is Mr. X. But the hopes of the police to lay the blame on Mr. X are dashed when he produces an alibi: he can show that at 11 pm that night he was in some other city. The police case that X is the culprit now crumbles, as follows. *Claim.* X is not guilty. *Proof.* Suppose not; i.e., suppose that X committed the crime. But then he must have been at the scene of the crime at 11 pm. On the other hand, he was in some other city at exactly that time; that's what his alibi is all about! So we reach a contradictory state of affairs. (We assume that X does not belong to the league of 'X Men'and has not yet mastered the art of being in two places at the same time.) Consequently we must give up the assumption made at the beginning, about X being guilty. Hence, Mr. X is not guilty!

Note the laborious way in which we wrote out the argument. In actuality, such reasoning happens in a flash, and we are not even aware that we have thought it out in this way.

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A date-of-birth computation

Say you were born on day d of month m in year y. Here d is a number between 1 and 31, m is a number between 1 and 12, and y is a number between 0 and 99 (inclusive in each case).

For example, if the date of birth is 15 August 1947, then $d = 15$, $m = 8$, $y = 47$. We now do some arithmetical operations on d, m, y as described below.

1. **Write down d.**
2. **Multiply by 4. Add 13. Multiply by 25.**
3. **Subtract 200. Add m.**
4. **Multiply by 2. Subtract 40. Multiply by 50.**
5. **Add y.**
6. **Subtract 10,500.**

The result should be a number giving your birth day, month and last two digits of the year in which you were born.

Example

Suppose your birthdate happens to be 15 August 1947, or 15-08-47. Here is how the computations go, starting with $d = 15$:

- **15 → 15 x 4 = 60 → 60 + 13 = 73 → 73 x 25 = 1825**
- **1825 → 1825 – 200 = 1625 → 1625 + 08 = 1633**
- **1633 → 1633 x 2 = 3266 → 3266 – 40 = 3226 → 3226 x 50 = 161300**
- **161300 → 161300 + 47 = 161347 → 161347 – 10500 = 150847**

The number obtained at the end is 150847, or 15-08-47.

Why does this work? Find an explanation!

Guessing the formula for

Sum of an Arithmetic Progression

in the classroom

C⊗MαC

This short note is based on a note written by K. R. S. Sastry (see [1]) in which he puts into practice the constructive pedagogy of George Pólya: "First guess, then prove".

The context used is that of finding a formula for the sum S_n of the first n terms of the arithmetic progression ('AP') with first term a and common difference d:

$$a,\ a+d,\ a+2d,\ a+3d,\ a+4d, \ldots.$$

The textbooks typically give the following formula,

$$S_n = \frac{n\left[2a+(n-1)d\right]}{2},$$

and prove it using Gauss's technique of reversing the terms. As a result, students are rarely if ever presented with the challenge of *finding* a formula (which is clearly not the same as being *given* the formula and then being asked to prove it).

Keywords: *Sequence, consecutive number, generalization, triangular number, pictorial*

But it is comparatively easy to lead students to an answer. We ask them to construct a table of partial sums:

n	Unsimplified sum of first n terms	Simplified sum
1	a	a
2	$a+(a+d)$	$2a+d$
3	$a+(a+d)+(a+2d)$	$3a+3d$
4	$a+(a+d)+(a+2d)+(a+3d)$	$4a+6d$
5	$a+(a+d)+(a+2d)+(a+3d)+(a+4d)$	$5a+10d$

We thus get a sequence of partial sums:

$$a,\ 2a+d,\ 3a+3d,\ 4a+6d,\ 5a+10d,\ 6a+15d,\ 7a+21d, \dots.$$

Now we must guess a formula for these expressions. The pattern in the sums is easy to see. Each sum is (naturally) a multiple of a added to a multiple of d. The coefficient of a is always equal to the number of terms (again, naturally so). What about the coefficient of d, the numbers $0, 1, 3, 6, 10, \dots$? These numbers should be familiar to students if they have studied the triangular numbers (which surely is a must-study topic at the middle school level), and they may know (or should know!) that the formula $n(n+1)/2$ generates the numbers. In this case, the sequence is displaced by one unit (it starts with 0 rather than 1, the second term is 1 rather than the first term, and so on), hence the formula that applies is $(n-1)n/2$, obtained by replacing n in the previous formula by $n-1$. So it appears by an examination of the expressions that the sum S_n to n terms is given by:

$$S_n = na + \frac{(n-1)n}{2}d.$$

Once the formula has been empirically found, it is easy to see that it must be correct: we get it by adding the 'a' terms and the 'd' terms separately. And it is easy to transform the formula to the usual forms (where 'last term' means 'n^{th} term'):

$$\begin{aligned} S_n = na + \frac{(n-1)n}{2}d &= \frac{2na+(n-1)nd}{2} \\ &= \frac{n\left(2a+(n-1)d\right)}{2} = \frac{n\left[a+\left(a+(n-1)d\right)\right]}{2} \\ &= \text{number of terms} \times \frac{\text{first term} + \text{last term}}{2} \\ &= \text{number of terms} \times \text{average of first term and last term.} \end{aligned}$$

References

i. K. R. S. Sastry, "First guess, then prove", The Mathematics Teacher (pub: National Council of Teachers of Mathematics), Vol. 73, No. 4 (April 1980), pp. 247

The COMMUNITY MATHEMATICS CENTRE (CoMaC) is an outreach arm of Rishi Valley Education Centre (AP) and Sahyadri School (KFI). It holds workshops in the teaching of mathematics and undertakes preparation of teaching materials for State Governments and NGOs. CoMaC may be contacted at comm.math.centre@gmail.com or shailesh.shirali@gmail.com

Graphing with Desmos – An online graphing calculator

"At Desmos, we imagine a world of universal math literacy, where no student thinks that math is too hard or too dull to pursue. We believe the key is learning by doing. When learning becomes a journey of exploration and discovery, anyone can understand – and enjoy! – math"- ***Desmos Team (www.desmos.com/about)***

Sangeeta Gulati

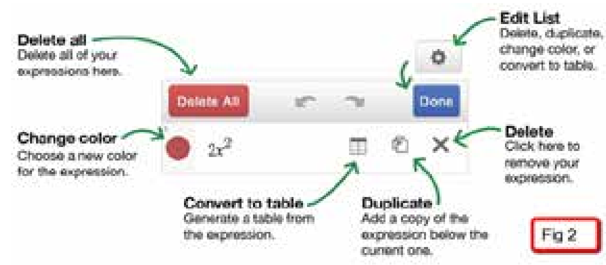
The appropriate use of open source technology can enliven the mathematics classroom and open up many learning opportunities. In this article we will describe how Desmos, an online graphing calculator, can enable the visualization of concepts and lead to meaningful explorations by students. Having used Desmos for more than a year, I truly believe in the philosophy and vision of the Desmos team. This online calculator can instantly plot any equation, be it lines, parabolas, derivatives of functions or Fourier series. Data tables can be easily generated and these open up opportunities for curve fitting and modeling activities. Sliders make it a breeze to demonstrate function transformations. As Desmos runs on browser-based html5 technology, it works on any computer or tablet without requiring any downloads. It is intuitive, beautiful math. And best of all: it's completely free.

In this article, we will take a tour of the features of Desmos and explore the possibilities it opens for a teacher and a student.

Keywords: *graphing calculator, freeware, dynamic, parameters, slider*

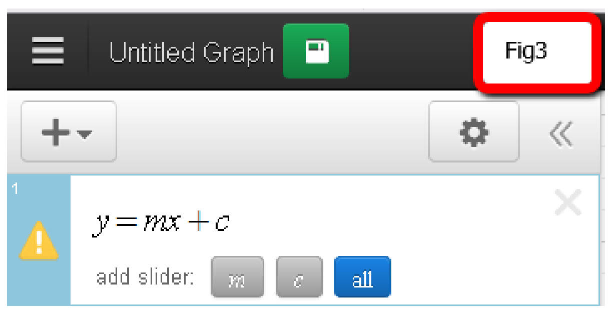
Getting started with Desmos

Desmos may be accessed from www.desmos.com. You can create an account or sign in with your Google account. The "Launch Calculator" option may be used without an account but signing in gives you the option of saving the output for future reference.

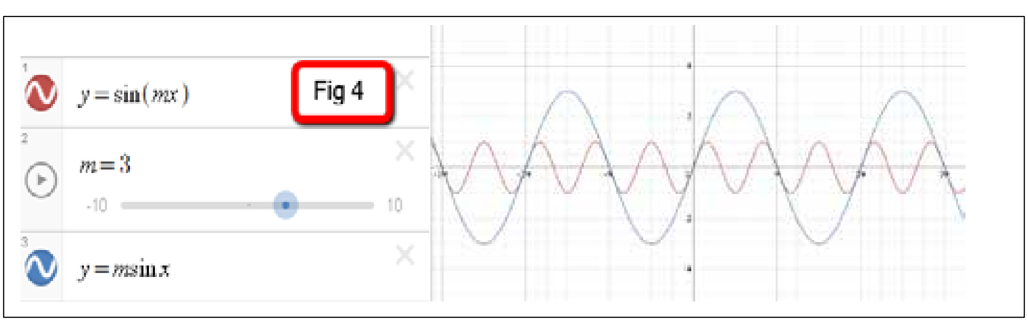


To create a new graph, just type your expression in the expression list bar. As you are typing your expression, the calculator will immediately start drawing your graph on the graph paper (indeed, even before you finish typing!). Once you are done with that task, you can edit your function, hide the function, change the colour or delete the function.

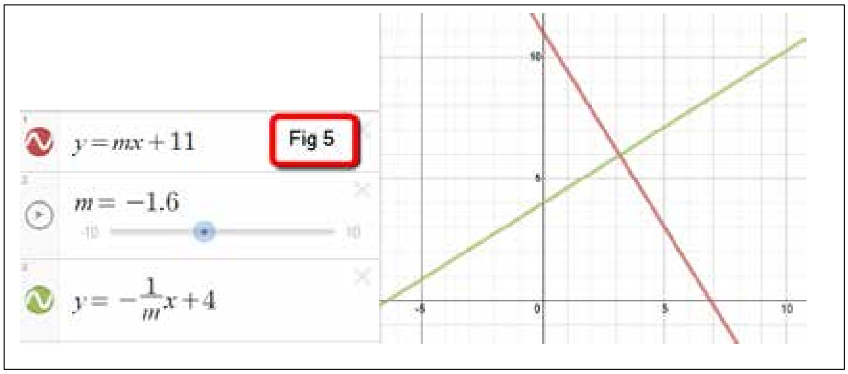
To graph a single line, enter a linear expression like $y = 2x + 3$. To make a dynamic graph, use parameters in place of constants. Typing $y = mx + c$ gives you a prompt to add sliders (Fig 3), for m and c, clicking on 'all' brings up a ready-to-use dynamic graph. Drag the sliders to create 'live' graphs on the screen!



You can use the same variables in different expressions to plot curves that change together. For example, Fig 4 shows the effect of varying m in the two expressions $y = \sin mx$ and $y = m \sin x$. This allows the teacher and student to explore transformations and visually understand the effect of changing a parameter.

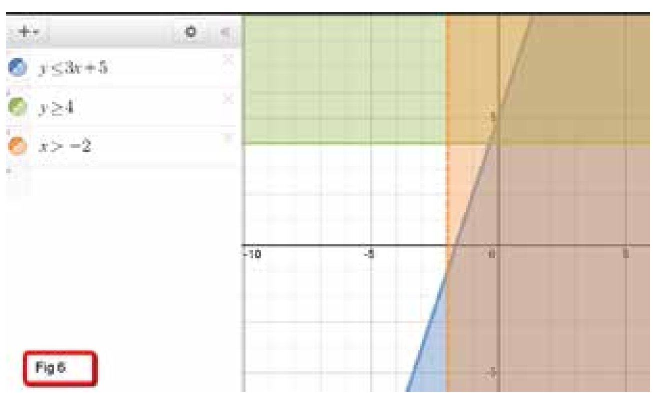


There is no better way but to 'see' (Fig 5) two lines perpendicular to each other when their slopes are negative reciprocals of each other! Desmos brings up many such 'aha' moments.



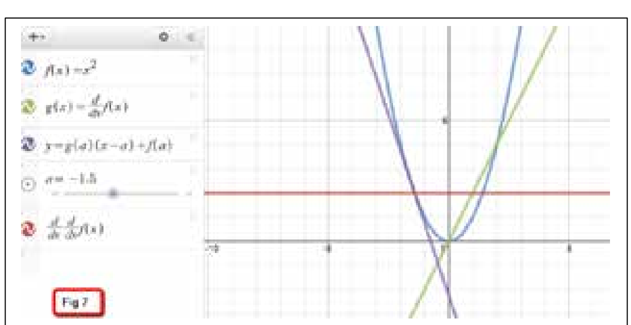
Is any special syntax needed for the input?

Typing expressions into the expression bar does not require the user to know any special syntax; one simply types in the function using a natural syntax (examples: sqrt (x) gives $\sqrt{x}$, abs (x) gives the modulus function $|x|$, pi gives π, and so on). Alternatively, we may use the 'Functions' key in the Desmos keyboard to obtain the required functions.



Graphing inequalities

Graphing inequalities (Fig 6) with Desmos is particularly easy. Try typing in $y > x$ or $y > 2x + 3$ or $y > x^2 + 1$ and see what happens. Or check the output from $x^2 + x + 3 > y > x^2 + 1$. The effect will surely come as a surprise! Desmos gives us great freedom to play with inequalities, enabling us to check the effects of making incremental changes in the defining constraints. We are spared much of the tedium of plotting by hand.



Graphing functions and their derivatives

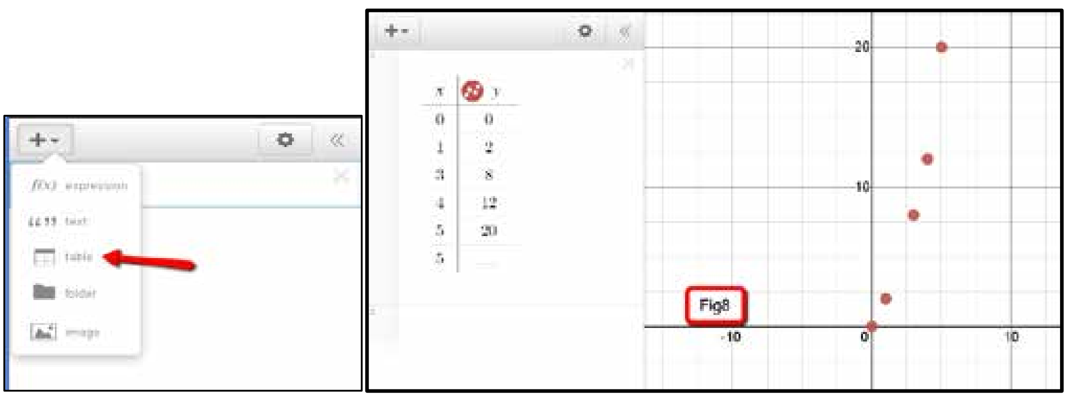
Finding the derivative (Fig 7) of a function is as easy as typing *d/dx f* (*x*), or *d/dx d/dx f* (*x*) for the second derivative, and you can build a tangent line accordingly using the point-slope form. This makes for an excellent demonstration of the relationship between a function and its derivative.

Graphing functions defined in a piecewise manner

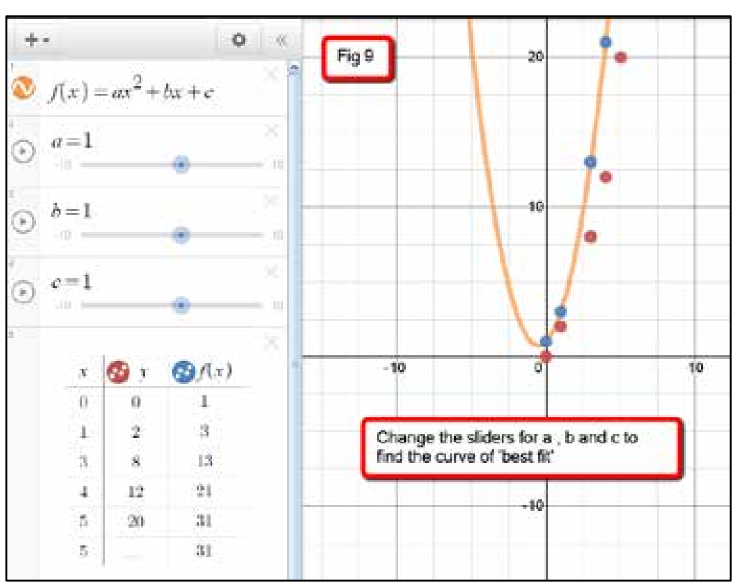
Plotting functions defined in piecewise manner can be handled in a single step. To limit the domain or range (*x* or *y* values of a graph), we simply add the restriction to the end of the equation in curly brackets, {}. For example, $y = 2x$ $\{x > 0\}$ would graph the line $y = 2x$ for x greater than 0.

Using the 'Table' feature of Desmos

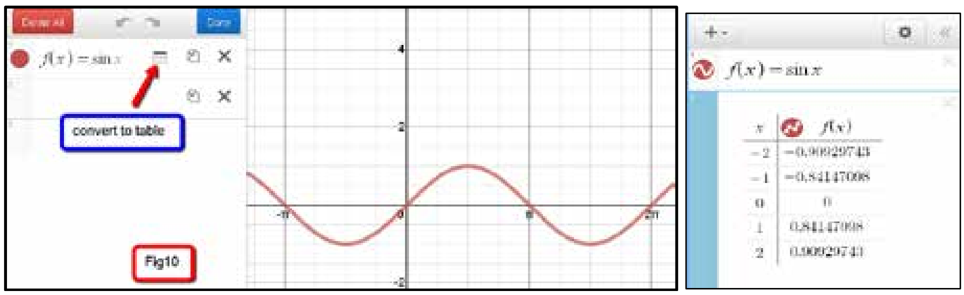
A significant feature of Desmos is the Table (Fig 8); it is excellent for creating a table of data just as one would do with paper and pen. As one enters the values in each row, the corresponding point gets plotted on the graph paper.

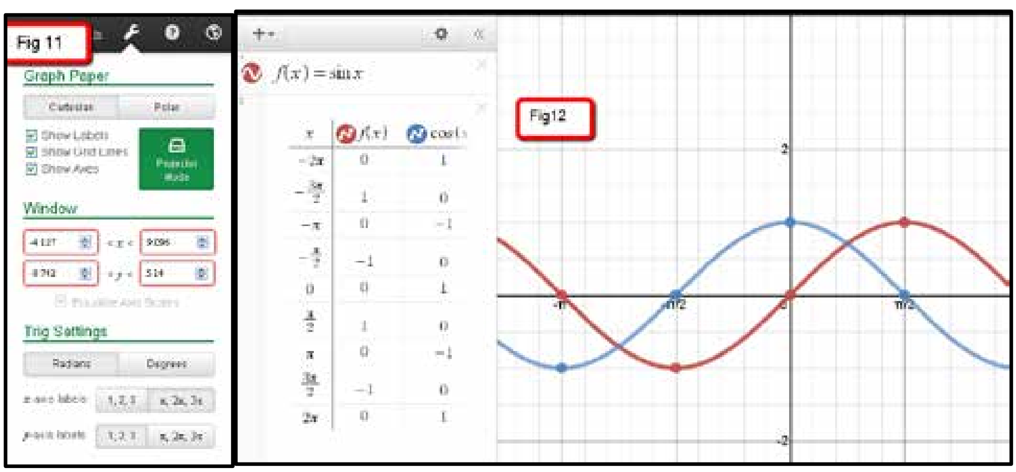


Using 'expressions' (+ add item), you can input a function which you think will best fit the curve and add a column in the table with the header *f* (*x*) which will automatically fill in the predicted values. This is most effective when instead of typing in one specific function we take a general function (Fig 9) and use sliders to find the curve of best fit.



Desmos also allows us to convert a function into a table of values (Fig 10)! And the fun doesn't stop here; if the table so generated does not make sense, as in the case of trigonometric functions, we would like to have values of x expressed in terms of π, we can change each entry by typing in 'pi', 'pi/2' or '-2pi' and the corresponding points will get highlighted on the graph. It is also useful to know that for trigonometric functions, you can change the settings so that the scale on the x - axis is in radians (Fig 11). We can also add a column (Fig 12) for, say, cos (x) to do a comparison between the two functions. The possibilities are amazing!

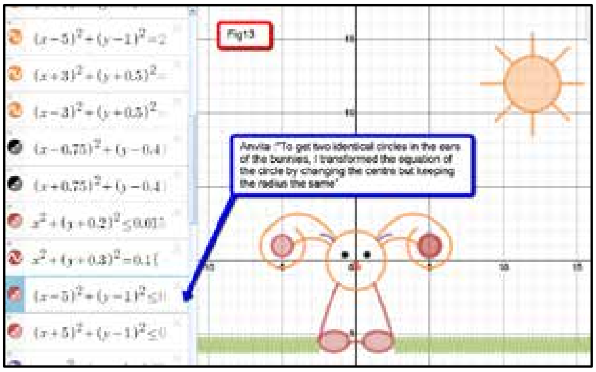


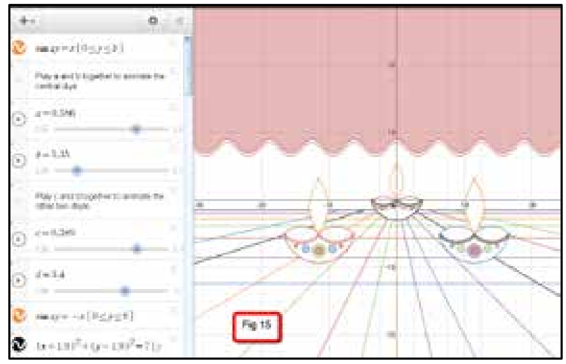


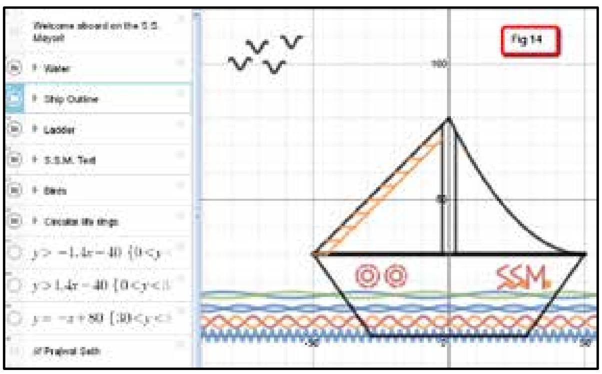
Samples of student work

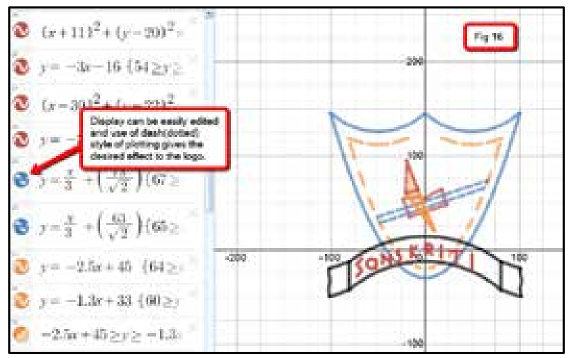
Technology, if used appropriately, can enable teachers to create meaningful learning opportunities for students. The remaining part of the article will describe the explorations done by students of grade 11 on piecewise functions using Desmos. The task assigned to students required them to create an interesting picture, of their own choice, using the elementary functions and their properties. They had to suitably restrict the domains of the functions to obtain the desired output. During this process they developed many new insights on properties of functions. It is known that technology enables educators to help students unlock their potential, and through this exercise Desmos enabled me to witness this happening at first hand with my own students; the results far exceeded my expectations. The students threw themselves into the task with great enthusiasm. They learned about restricting domains of functions and transformations, they explored conics – a topic not discussed till then in class – and came up with beautiful art work (Figs 13, 14, 15, 16: work of Anvita, Prajwal and Narayani of Sanskriti School).

As they presented their work before the class, I could see the high level of understanding they had developed for the functions. I was amazed. What I could not achieve after doing numerous problems on Domain and Range of functions, the students had achieved on their own. They used sliders to create animated graphs which made their work a piece of art.









I hope this tour of Desmos has inspired you to try this beautiful tool for yourself, and explore the many possibilities it opens for all of us.

I am thankful to my students especially Prajwal, Anvita, Harshita and Narayani who put in precious time and effort to create such beautiful work; they reconfirmed my belief that learning and teaching of Mathematics can be fun!

Web links

[1] www.desmos.com/
[2] http://support.desmos.com/home
[3] https://www.youtube.com/user/desmosinc
[4] http://dynamath.wikispaces.com/Maths+and+Art (to see more samples of students' work)

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Adventures in problem solving

First and last digits of perfect squares

$\mathcal{C}\otimes\mathcal{M}\alpha\mathcal{C}$

problem corner

First digits

In this section we solve the following problem:

Find the smallest perfect square N whose digits start with 1234567.

It is assumed that we are working in base ten. Note that we do not know how many digits N has. It may seem that such a problem can be solved only using trial and error, by playing with a calculator, but we shall show otherwise. (We do use a calculator, but it is only for computation of two square roots.)

The number of digits in N is either odd or even. If it is the former, let the number of digits be denoted by $2k+1$; else let it be denoted by $2k+2$. Since a number with n digits lies between 10^{n-1} and $10^n - 1$ (inclusive at both ends), the following can be said:

- If N has $2k+1$ digits, then
$$N = \underbrace{1234567\ldots}_{2k+1 \text{ digits}} = 1.234567\ldots \times 10^{2k}.$$
- If N has $2k+2$ digits, then
$$N = \underbrace{1234567\ldots}_{2k+2 \text{ digits}} = 1.234567\ldots \times 10^{2k+1} = 12.34567\ldots \times 10^{2k}.$$

(The dots indicate digits of N which we do not know as yet.) We consider both possibilities and see which one gives us a smaller perfect square.

Suppose that N has an odd number of digits In this case $N = 1.234567\ldots \times 10^{2k}$, so:

$$1.234567 \times 10^{2k} \leq N < 1.234568 \times 10^{2k}.$$

(Note the strict inequality sign on the right side. Note also that the dots have fallen away now.) Taking square roots (this is where we need the calculator!) we get:

$$1.111110705 \times 10^{k} \leq \sqrt{N} < 1.111111155 \times 10^{k}.$$

Observing the decimal expansions carefully, we write this as:

$$1111110.705 \times 10^{k'} \leq \sqrt{N} < 1111111.155 \times 10^{k'}$$

where $k' = k - 6$. From this it is clear that the least possible value of $\sqrt{N}$ is 1111111 (obtained by taking $k' = 0$, i.e., $10^{k'} = 1$). And indeed we find that

$$1111111^2 = 1234567654321,$$

a number with thirteen digits. *This is the least square of the required type, i.e., whose digits start* 1234567 …, *given that it has an* ***odd*** *number of digits.*

There are of course infinitely many squares of the stated kind having an odd number of digits. For we also have:

$$11111107.05 \times 10^{k''} \leq \sqrt{N} < 11111111.55 \times 10^{k''}$$

where $k'' = k - 7$. This implies that the next smallest such square (after 1111111^2) is 11111108^2 which is a number with fifteen digits: $11111108^2 = 123456720987664$. And after this comes $11111109^2 = 123456743209881$.

Suppose that N has an even number of digits In this case $N = 12.34567\ldots \times 10^{2k}$, so:

$$12.34567 \times 10^{2k} \leq N < 12.34568 \times 10^{2k}.$$

(As earlier, note the strict inequality sign on the right side, and the fact that the dots have fallen away.) Taking square roots we get:

$$3.513640562 \times 10^{k} \leq \sqrt{N} < 3.513641985 \times 10^{k}.$$

(Now you will see why we rewrote $1.234567\ldots \times 10^{2k+1}$ as $12.34567\ldots \times 10^{2k}$.) Hence we have:

$$3513640.562 \times 10^{k'} \leq \sqrt{N} < 3513641.985 \times 10^{k'},$$

where $k' = k - 6$. From this it is clear that the least possible value of $\sqrt{N}$ is 3513641. And indeed we find that

$$3513641^2 = 12345673076881,$$

a number with fourteen digits. *This is the least square of the required type, i.e., whose digits start* 1234567 …, *given that it has an* ***even*** *number of digits.*

As earlier we can say that there are infinitely many squares of the stated kind having an even number of digits. For we also have:

$$35136405.62 \times 10^{k''} \leq \sqrt{N} < 35136419.85 \times 10^{k''},$$

where $k'' = k - 7$.

This implies that the next smallest such square is 35136406^2, a number with sixteen digits: $35136406^2 = 1234567026596836$. Other squares which also have sixteen digits and start with 1234567 are 35136407^2 which equals 1234567096869649; 35136408^2 which equals 1234567167142464; ...; and 35136419^2 which equals 1234567940143561.

So the answer to the stated problem is:

- The least such square is a number with thirteen digits, 1234567654321.
- The next such square is a number with fourteen digits, 12345673076881.

We invite you to find the smallest perfect cube whose digits start with 1111111.

Last digits

The analysis carried out in the previous section shows, in effect, that the initial digits of a perfect square can be any string whatever: specify any finite string of digits, and we can find a perfect square with those as the initial digits.

What about the digits that come "at the opposite end"of a perfect square? Can any corresponding statement be made? This question brings up some interesting mathematics and also some surprises.

Consider the terminating digit ("last digit"or units digit). It is known that the last digit of a perfect square is one of $\{0, 1, 4, 5, 6, 9\}$; there are six possibilities. So if we select a digit at random, the probability that it is a possible last digit of a perfect square is 0.6.

How many possibilities are there for the last two digits (when viewed as a two-digit number)? Example: Since $23^2 = 529$ and $28^2 = 784$, two such numbers are 29 and 84. Once again, we count the possibilities using a computer. We find that there are 22 in all:

00, 01, 04, 09, 16, 21, 24, 25, 29, 36, 41,
44, 49, 56, 61, 64, 69, 76, 81, 84, 89, 96.

Hence, if we select an ordered pair of digits at random (there are $10^2 = 100$ such pairs), the probability that there exists a perfect square with those as the last two digits (in the same order) is 0.22. Note the substantial drop in probability, from 0.6 to 0.22.

We move a step higher. How many possibilities are there for the last three digits? It is much less obvious what the answer is, so we head back to the computer and let it generate the answer. It turns out that there are 159 possibilities for the last three digits:

000, 001, 004, 009, 016, 024, 025, 036, 041, 044, 049, 056, 064, 076, 081, 084,
089, 096, 100, 104, 116, 121, 124, 129, 136, 144, 156, 161, 164, 169, 176, 184,
196, 201, 204, 209, 216, 224, 225, 236, 241, 244, 249, 256, 264, 276, 281, 284,
289, 296, 304, 316, 321, 324, 329, 336, 344, 356, 361, 364, 369, 376, 384, 396,
400, 401, 404, 409, 416, 424, 436, 441, 444, 449, 456, 464, 476, 481, 484, 489,
496, 500, 504, 516, 521, 524, 529, 536, 544, 556, 561, 564, 569, 576, 584, 596,
600, 601, 604, 609, 616, 624, 625, 636, 641, 644, 649, 656, 664, 676, 681, 684,
689, 696, 704, 716, 721, 724, 729, 736, 744, 756, 761, 764, 769, 776, 784, 796,
801, 804, 809, 816, 824, 836, 841, 844, 849, 856, 864, 876, 881, 884, 889, 896,
900, 904, 916, 921, 924, 929, 936, 944, 956, 961, 964, 969, 976, 984, 996.

Therefore, if we select an ordered triple of digits at random (there are $10^3 = 1000$ such triples), the probability that there exists a perfect square with those as the last three digits (in the same order) is 0.159.

We see the makings of a curious sequence here. Let a_k denote the number of possible k-digit 'endings' of a perfect square, so $a_1 = 6$, $a_2 = 22$, $a_3 = 159$. Using a computer we generate more values of the sequence (I used *Mathematica*); here is what we get:

k	1	2	3	4	5	6
a_k	6	22	159	1044	9121	78132

What is the law of formation of the sequence? It is far from obvious!

We shall analyze this very interesting problem on some other occasion. For now we only give the answer. It can be shown (see references [1] and [2] for details) that:

$$a(k) = \begin{cases} \dfrac{5 \cdot 10^k + 40 \cdot 5^k + 7 \cdot 2^k + 56}{72} & \text{if } k \text{ is even,} \\[2ex] \dfrac{5 \cdot 10^k + 50 \cdot 5^k + 11 \cdot 2^k + 110}{72} & \text{if } k \text{ is odd.} \end{cases}$$

For example, the formula gives: $a(2) = (500 + 1000 + 28 + 56)/72 = 1584/72 = 22$, which is correct. The following also can be shown: for all $k \geq 1$,

$$a(k+8) = 130a(k+6) - 3129a(k+4) + 13000a(k+2) - 10000a(k).$$

What a very surprising pair of relations! For now we shall leave you to marvel at them.

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Problems for the Middle School

Problem Editor : R. ATHMARAMAN

problem corner

Problems for Solution

Problem III-2-M.1

What is the least multiple of 9 which has no odd digits?

Problem III-2-M.2

Which number is larger: 31^{11} or 17^{14}?

Problem III-2-M.3

What is the remainder when 2015^{2014} is divided by 2014?

Problem III-2-M.4

Find the least natural number larger than 1 which is simultaneously a perfect square, a perfect cube, a perfect fourth power, a perfect fifth power and a perfect sixth power. How many such numbers are there?

Problem III-2-M.5

A group of ten people (men and women), sit side by side at a long table, all facing the same direction. In this particular group, ladies always tell the Truth while the men always lie. Each of the ten people announces: "There are more men on my left, than on my right." How many men are there in the group? (This problem has been adapted from the Berkeley Math Circle, Monthly Contests.)

SOLUTIONS OF PROBLEMS IN ISSUE-III-1

Solution to problem III-1-M.1

Show that the following number is a perfect square for every positive integer n:

$$\underbrace{111111\ldots111111}_{2n\text{ digits}} - \underbrace{222\ldots222}_{n\text{ digits}}.$$

Let a_n denote the given integer; e.g., $a_1 = 11-2 = 9$, $a_2 = 1111-22 = 1089$. Observe that $a_1 = 3^2$ and $a_2 = 33^2$. That gives us a clue to the solution. Let b_n denote the number with n ones, e.g., $b_4 = 1111$. The proof that a_n is a perfect square for all n is illustrated for the case $n = 4$ (the general case is written the same way):

$$
\begin{aligned}
a_4 &= 11111111 - 2222 \\
&= 11110000 - 1111 = 1111 \times 10000 - 1111 \times 1 \\
&= 1111 \times (10^4 - 1) = 1111 \times 9999 \\
&= 1111 \times 1111 \times 9 = 1111 \times 1111 \times 3 \times 3 \\
&= (3 \times 1111)^2.
\end{aligned}
$$

In general, a_n is the square of the number 333 ... 333 which has n threes.

Solution to problem III-1-M.2 *On a digital clock, the display reads* $6:38$*. What will the clock display twenty-eight digit changes later?*

Let us compute the digit-changes, step by step.

From	To	# digit changes	Cumulative total
6 : 38	6 : 39	1	1
6 : 39	6 : 40	2	3
6 : 40	6 : 49	9	12
6 : 49	6 : 50	2	14
6 : 50	6 : 59	9	23
6 : 59	7 : 00	3	26
7 : 00	7.01	1	27
7.01	7 : 02	1	28

The time is $7:02$ after twenty-eight digit changes are over.

Solution to problem III-1-M.3 *The figure shows a hall $ABCDEF$ with right angles at its corners. Its area is 2520 sq units, and $AB = BC$, $CD = 30$ units, $AF = 60$ units. A point P is located on EF such that line CP divides the hall into two parts with equal area. Find the length EP.*

Let $AB = x$; then $BC = x$. The area of the hall is then $60(30 + x) - 30x = 1800 + 30x$ which equals 2520 (given information; see Figure 1). Hence $x = 720/30 = 24$, which leads to $DE = 60 - 24 = 36$.

Let $PE = y$. Then the area of the trapezium $CDEP$ is $\frac{1}{2} \times 2520 = 1260$. Hence:

$$\frac{1}{2}(30 + y) \times 36 = 1260,$$

$$\therefore\ 30 + y = \frac{2520}{36} = 70, \quad \therefore\ y = 40.$$

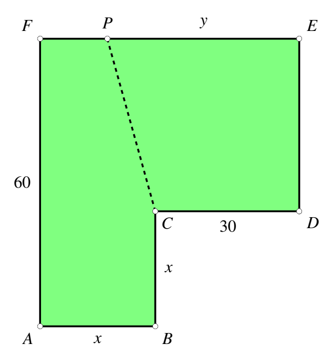


FIGURE 1

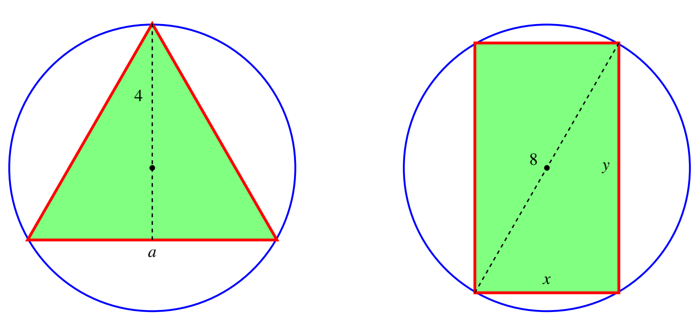


FIGURE 2

Solution to problem III-1-M.4 *In a circle with radius 4 units, a rectangle and an equilateral triangle are inscribed. If their areas are equal, find the dimensions of the rectangle.*

Let the side of the equilateral triangle be a, and let the rectangle have dimensions x, y (see Figure 2). The radius of the circle is 4 units. The height of the equilateral triangle is $a \times \sqrt{3}/2$, and since the radius of the circle is 2/3 of the height, we get

$$4 = \frac{2}{3} \times a \times \frac{\sqrt{3}}{2}, \quad \therefore\ a = 4\sqrt{3}.$$

Hence the area of the triangle is $\frac{\sqrt{3}}{4}a^2 = \frac{\sqrt{3}}{4} \times 48 = 12\sqrt{3}$. This is also the area of the rectangle. Since the diagonal of the rectangle has length 8, we have: $xy = 12\sqrt{3}$ and $x^2 + y^2 = 8^2$. We must solve these equations for x, y. The second equation yields $y^2 = 64 - x^2$. Substituting into the first one and squaring, we get:

$$x^2(64 - x^2) = 432, \quad \therefore\ x^4 - 64x^2 + 432 = 0.$$

Treating this as a quadratic equation in x^2, we get:

$$x^2 = \frac{64 \pm \sqrt{64^2 - 4 \times 432}}{2} = \frac{64 \pm \sqrt{2368}}{2} = 32 \pm 4\sqrt{37}.$$

So the sides of the rectangle are $\sqrt{32 + 4\sqrt{37}}$ and $\sqrt{32 - 4\sqrt{37}}$.

Solution to problem III-1-M.5 *Find the value of*

$$\left\lfloor \frac{2014^3}{2012 \times 2013} \right\rfloor - \left\lfloor \frac{2012^3}{2013 \times 2014} \right\rfloor.$$

Let $a = 2013$. The expression within the first '$\lfloor\ \rfloor$' then equals:

$$\frac{(a+1)^3}{(a-1)a} = \frac{a^3 + 3a^2 + 3a + 1}{a^2 - a}$$
$$= a + 4 + \frac{8}{a-1} - \frac{1}{a},$$
$$\therefore\ \left\lfloor \frac{(a+1)^3}{(a-1)a} \right\rfloor = a + 4,$$

since $1 > \frac{8}{a-1} > \frac{1}{a}$. Similarly, the expression within the second '$\lfloor\ \rfloor$' equals:

$$\frac{(a-1)^3}{a(a+1)} = \frac{a^3 - 3a^2 - 3a + 1}{a^2 + a}$$
$$= a - 4 + \frac{8}{a+1} - \frac{1}{a},$$
$$\therefore\ \left\lfloor \frac{(a-1)^3}{a(a+1)} \right\rfloor = a - 4,$$

since $1 > \frac{8}{a+1} > \frac{1}{a}$. Therefore the given quantity equals $(a + 4) - (a - 4) = 8$.

problem corner

Problem for the Senior School

PRITHWIJIT DE & SHAILESH SHIRALI

Problems for Solution

Problem III-2-S.1
Let n be a positive integer not divisible by 2 nor by 5. Prove that there exists a positive integer k, depending on n, such that the number $111\ldots1$, where the digit 1 is repeated k times, is divisible by n.

Problem III-2-S.2
Let $\mathbb{R}$ be the set of all real numbers, and let b be a real number, $b \neq \pm 1$. Determine a function $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) + bf(-x) = x + b$, for all $x \in \mathbb{R}$.

Problem III-2-S.3
Determine all three-digit numbers N such that: (i) N is divisible by 11, (ii) $N/11$ is equal to the sum of the squares of the digits of N. (This problem appeared in the International Mathematical Olympiad 1960.)

Problem III-2-S.4
You are given a right circular conical vessel of height H. First, it is filled with water to a depth $h_1 < H$ with the apex downwards. Then it is turned upside down and it is observed that water level is at a height h_2 from the base. Prove that

$$h_1^3 + (H - h_2)^3 = H^3.$$

Can h_1, h_2 and H all be positive integers?

Problem III-2-S.5
Let $\{a_n\}_{n=1}^{\infty}$ be a sequence defined as follows: $a_1 = 3, a_2 = 5$ and:

$$a_{n+1} = |a_n - a_{n-1}|, \quad \text{for all } n \geq 2.$$

Prove that $a_k^2 + a_{k+1}^2 = 1$ for infinitely many positive integers k.

Solutions of problems in Issue-III-1

Solution to problem III-1-S.1 *Let $f(x) = ax^2 + bx + c$, where a, b, c are positive integers. Show that there exists an integer m such that $f(m)$ is a composite number.*

We shall prove this by actually exhibiting such an integer m. Let $f(1) = n$; then $n = a + b + c$. Now consider the value of $f(n + 1)$:

$$f(n+1) = a(n+1)^2 + b(n+1) + c$$
$$= an^2 + 2an + bn + (a+b+c)$$
$$= an^2 + 2an + bn + n = n(an + 2a + b + 1).$$

Since a, b, c are positive integers, $n > 1$ and $an + 2a + b + 1 > 1$. So $f(n+1)$ is a product of two integers both of which exceed 1. Therefore $f(n+1)$ is composite.

Solution to problem III-1-S.2 *Show that the arithmetic progression* 1, 5, 9, 13, 17, 21, 25, 29, 33, *…contains infinitely many prime numbers.*

Another way of expressing this is: *Show that there are infinitely many primes of the form* $4k+1$. It so happens that the corresponding problem with $4k-1$ instead of $4k+1$ is easier to solve. This is because of the following property: *The product of numbers all of the form* $4k+1$ *is also of that form.*

From this it follows: *If an odd positive integer n is of the form* $4k-1$, *then it has at least one prime factor of that form*.

Now consider the primes of the form $4k-1$. They are: 3, 7, 11, 19, …. Suppose there is a last such prime, say p. Now construct the following number n:

$$n = 4(3 \times 7 \times 11 \times \cdots \times p) - 1.$$

This is of the form $4k-1$, so it has a prime factor q of this form. The prime q cannot be any of 3, 7, 11, …, p, as n is not divisible by these primes. So we have found a new prime of the form $4k-1$. Hence there cannot be a 'last prime' of this form. Therefore there are infinitely many primes of the form $4k-1$.

This method of proof does not work for primes of the form $4k+1$ because we *cannot* make a statement like this one: 'If n is of the form $4k+1$, then it has at least one prime factor of that form.' (An easy counterexample to this hypothesis is the number $21 = 3 \times 7$.) Some other approach is needed. This 'other approach' is provided by the following at-first-sight-surprising fact which we do not prove here: *The prime factors of a number of the form* $4m^2+1$ *are all of the form* $4k+1$. Examples: $4 \times 4^2 + 1 = 65 = 5 \times 13$, and $4 \times 6^2 + 1 = 145 = 5 \times 29$. Taking this to be a fact, the rest of the proof is easy.

Suppose that there is a last prime p of the form $4k+1$. We now construct the number $n = 4(5 \times 13 \times 17 \times \cdots \times p)^2 + 1$. The prime factors of n are all of the form $4k+1$ and distinct from 5, 13, 17, …, p. Hence p cannot be the last such prime. So there are infinitely many such primes.

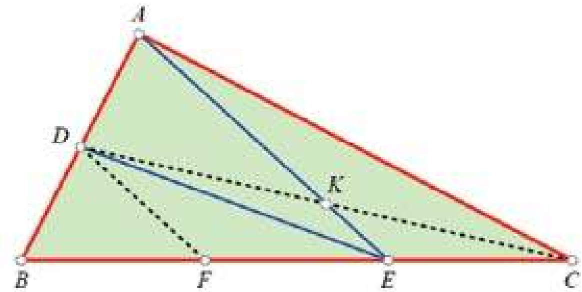


FIGURE 1.

Solution to problem III-1-S.3 *In* $\triangle ABC$, *the midpoint of* AB *is* D, *and* E *is the point of trisection of* BC *closer to* C. *Given that* $\measuredangle ADC = \measuredangle BAE$, *determine the magnitude of* $\measuredangle BAC$.

Let K be the point of intersection of CD and AE. (See Figure 1.) Observe that in triangle AKD, $\measuredangle KAD = \measuredangle KDA$. Hence $DK = AK$. Let F be the midpoint of BE. Note that DF is parallel to AE. In triangle CDF, E is the midpoint of CF, and EK is parallel to DF. Therefore K is the midpoint of CD. Hence in triangle ACD, $CK = DK = AK$. It follows that $\measuredangle CAD = \measuredangle BAC = 90°$.

Solution to problem III-1-S.4 *Given a* $\triangle ABC$, *does there necessarily exist a point* D *on side* BC *such that* $\triangle ABD$ *and* $\triangle ACD$ *have equal perimeter? If such a point* D *exists, then we can similarly obtain points* E *and* F *on* AC *and* AB, *respectively, such that* BE *and* CF *bisect the perimeter of* ABC. *Are the lines* AD, BE, CF *concurrent?*

Let $BC = a$, $CA = b$ and $AB = c$. Let D be a point on BC, between B and C, such that $BD = x$ and $CD = y$. AD bisects the perimeter of triangle ABC if and only if $x + c = y + b = s$, where $2s = a + b + c$. Thus $x = s - c$ and $y = s - b$. Since $s - c$ and $s - b$ are positive quantities whose sum is a, it is possible to find a point D on BC such that $BD = x$ and $CD = y$. See Figure 2. (More precisely, D is the point where the ex-circle opposite vertex A touches BC.)

For the second part, concurrency of the three line segments AD, BE, CF follows from the converse of Ceva's theorem. (For we have, in the same way: $CE = s - a$, $EA = s - c$, $AF = s - b$, $FB = s - a$.

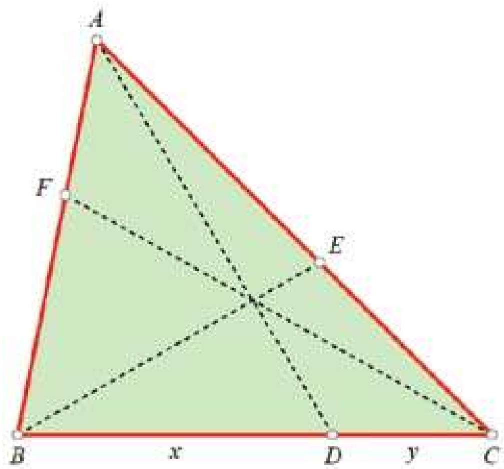


FIGURE 2.

- $BC = a, CA = b, AB = c$
- $c + x = b + y = s$
- $x = s - c$
- $y = s - b$

EA = s − c, AF = s − b, FB = s − a.

Hence

$$\frac{BD}{DC} \times \frac{CE}{EA} \times \frac{AF}{FB} = 1.$$

The converse of Ceva's theorem states that if D, E, F are points on BC, CA, AB such that the above equality holds, then AD, BE, CF concur. Hence the claim.)

Solution to problem III-1-S.5 *Let $A = 5^{2013}$ and $B = 4^{2013}$. Is $4^A + 5^B$ a prime number?*

Observe that 4 divides $A - 1$. Let $A - 1 = 4m$. Now

$$4^A + 5^B = 4(4^m)^4 + (5^{4^{2012}})^4$$

is of the form $4x^4 + y^4$. But:

$$4x^4 + y^4 = \left[(y - x)^2 + x^2\right]\left[(y + x)^2 + x^2\right].$$

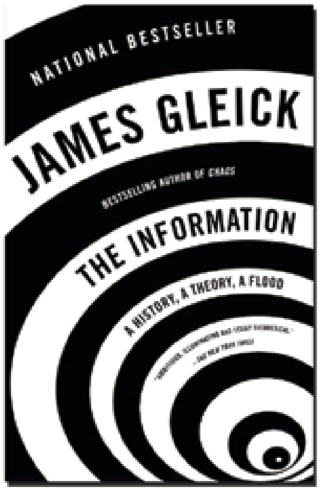
Put $x = 4^m$ and $y = 5^{4^{2012}}$. Since both factors exceed 1, $4^A + 5^B$ is composite.

A Natural History of Information

A review of 'The Information: A History, a Theory, a Flood' by James Gleick

review

KESHAV MUKUNDA



In 1948, a paper was submitted to a Bell Labs technical journal, in which the author proposed a new theoretical framework to analyse problems in communication. Suggesting that it be rejected for publication, the reviewer of the paper said it was "poorly motivated and excessively abstract", and went on to say, "it is unclear for what practical problem it might be relevant ... the author mentions computing machines – I guess one could connect such machines, but a recent IBM memo stated that a dozen or so such machines will be sufficient for all the computing that we'll ever need in the foreseeable future, so there won't be a whole lot of connecting going on" [1]. While this clearly mistaken reviewer remains anonymous, the author of the submitted paper entitled "A Mathematical Theory of Communication", a relatively young mathematician and engineer named Claude Shannon, went on to become one of the founding pioneers of the new field called *information theory*. In fact, Shannon's 1948 paper – published around the same time that the transistor was invented – essentially created the field of information theory, which has deeply influenced the development of engineering and computer science. Among

Keywords: *information, bit, binary, digit, communication*

numerous other ideas, Shannon is credited with conceptualizing the digital computer and circuit design theory, as well as being the first to use the word *bit* as a contraction of the phrase 'binary digit', in the same 1948 paper.

The story of Shannon's work, as well as a wider look at the evolution of the idea of information, appears in a recent book called *The Information: A History, a Theory, a Flood.* The book is by the well-known science writer James Gleick, who has written several bestsellers. Published in 1987, his first book *Chaos: Making a New Science* described the development of chaos theory, the mathematical study of sensitive dynamical systems, and was important in helping popularize chaos theory and fractals. Among other things, the book spread greater awareness of the phrase "the butterfly effect", making it a cultural meme that has since appeared in movies and pop culture generally. *Chaos* won a Pulitzer Prize in 1988, and has sold millions of copies since then. Among Gleick's other widely acclaimed books are two biographies, *Genius: The Life and Science of Richard Feynman,* and *Isaac Newton,* both finalists for the Pulitzer Prize as well.

Unlike a more straightforward historical biography of a person, *The Information* describes the evolution of an idea, bringing together strands of history and culture to show how a crucial new construct emerged in our understanding of the world. This new construct was defined, measured and articulated as 'information'. It now appears to be everywhere we look, and this ubiquity is perhaps the reason why the idea of information was traditionally overlooked. After all, anything that is considered self-evident and 'obvious' usually hides a deeper and richer understanding of the world; as the mathematician E. T. Bell put it, *"obvious* is the most dangerous word in mathematics" [2].

On the face of it, you might wonder what a book describing the history of the concept of information has to do with mathematics. In fact, the theoretical foundations that help to describe and measure information are mathematical in nature. When Shannon thought about transmitting information, he did not consider the meaning or sense of the message; as he put it, "semantic aspects of communication are irrelevant" to the engineering problem, although he conceded that "frequently the messages have *meaning*" [3]. Instead, he formulated information as a mathematical measure of the number of possible states that a message could take, using symbols from a finite underlying alphabet. In this sense information is directly related to the idea of entropy, and can be measured in a similar way. Many of the other concepts described in this book also have a solid mathematical basis, and the whole history of the conceptualization of information shows us the ways in which mathematics can be applied to all sorts of questions about the world. Gleick is at ease writing and describing these mathematical ideas, and as a textured background to these ideas he provides a narrative that takes in a wide sweep of history, linking the technical to the cultural.

The book begins with the description of the talking drums in sub-Saharan Africa that are used to transmit complex messages across long distances, being sent from village to village by relay. The language of the drum involves sound combinations that have a few different dimensions – tones as well as vowels and consonants - which are used to encode detailed messages. This sets the stage for the investigation of the 'amount' of a message that can be reliably transmitted using relatively simple alphabets; in fact, the example shows that the 'size' of a message is not always directly correlated with the amount of information it carries. Gleick then moves on to investigate the early attempts at creating long distance telegraph systems, and then telephones, bringing out curious stories of the many people and inventions that flourished in the early years of each invention. There are the Chappe brothers in 18th century France, for example, who devised an early form of telegraph using a network of tall towers with men communicating between them using flags. Even the famous mathematician C.F. Gauss, together with the physicist Weber, came up with a scheme involving electric currents that travelled through wires to deflect small needles left or right. We clearly see two separate and interconnected problems emerging: creating a useful language or alphabet of the message, and inventing effective communication technology itself.

Gleick then discusses how language, when moving from the oral to the written, brought with it questions of representation and standardization. How exactly were the letters in the alphabet to be arranged to spell a given word: for example, would it be *wordes* or *words, colume* or *column*? This leads to the story of how the Oxford English Dictionary was originally compiled, and to its subsequent development. Moving from the written to the published word to transmitting them through wires, the book then considers how language was encoded in electronic switches, bringing logic and language together in the early computers. Again, through this journey we meet several people involved in these ideas through history: Charles Babbage, Ada Byron, and their early mechanical computer called the 'analytical engine'; Augustus De Morgan, George Boole and their symbolic 'algebra of logic'; and of course Alan Turing, with his formalization of the meanings of ideas such as 'algorithm' or 'computation'.

From here the journey to Shannon's 1948 paper was not self-evident. Gleick describes Shannon's early ideas, and the people he worked with. Some were more responsive to his ideas than others. For example, Turing and Shannon often had lunch together and discussed their work, and in an interview given in 1982, Shannon said that Turing "didn't always believe these . . . my ideas . . . he didn't believe they were in the right direction" [4]. Once the idea of information emerged, it spread quickly to various disciplines to different levels of success. In the 1950s, as the structure of DNA was discovered, the biologist Francis Crick described the copying of a sequence of nucleic acids as a transfer of information. At the time, this was meant as a metaphoric description, but soon biologists and geneticists would talk of information, alphabets, and the transcription of codes in a literal sense. Information theory permeated economics, philosophy and physics, while it also remained significant and useful in the growing computer industry.

Gleick eventually argues that the idea of information is more universally fundamental than we might think. In fact, some theoretical physicists now suggest that space and time are themselves simply constructed by the exchange of discrete bits of information. In this view, information is the essence out of which everything else in the physical universe arises; or, as the physicist John Archibald Wheeler put it, "all things physical are information-theoretic in origin" [5].

The book doesn't move linearly through history, but instead weaves between different times and different discoveries to tease out the threads of the various insights that led to the concept of information. In hindsight, it might seem obvious to us now that the idea of information would emerge in certain historical contexts, and we can now easily see and name these ideas in those contexts; but it would have taken a great leap of understanding at the time to see how all the pieces fit together. By giving us this non-linear narrative, Gleick delightfully shows not just that our human scientific understanding of the world meanders in several different directions with no evident direction of 'progress', but also that each human idea does not constructively build on the ones that came before. We almost get the sense that there are several plot lines evolving in what is a large detective story, and Gleick brings them together in a satisfying way.

In all of the discussion on information, however, Gleick sidesteps issues of the control of access to information, steering clear of any political analysis or discussion of how information and state power are closely related. Any history of information would surely have to acknowledge these relations, and it would have been interesting if the book considered this. Still, he does mention Wikipedia and the ways in which entries can be silenced or censored by vandals and deletionists, as the community of editors struggle to reach an elusive compromise. In fact, this struggle is not just between opposing points of view, but also with our management of the sheer quantity of information available to us now. To Gleick, the bold Wikipedia project is one attempt to deal constructively with the new flood of information we are continually exposed to. One factor in this flood is the curious fact that it apparently takes more energy to actually delete electronic information than to simply store it, in an entropic sense. Why delete or forget anything, in that case? However, as we electronically preserve more information now than ever before, this 'information overload' makes it difficult for us to decide on the value of any given piece of knowledge. The more information we have access to, the harder it is to filter out irrelevant noise to find what we want, and then understand what it means. The challenge of making sense of it all is more relevant than ever, and Gleick is optimistic about our collective ability to manage the challenges and even to create meaning in what could become a bewildering jumble.

Although many of the concepts in the book can be quite complicated, Gleick gives us a very readable account, going into details only as much as is necessary for us to get a sense of the mathematics and engineering involved. Even these are not presented in an abstract way, but are woven into the historical account and help to move the narrative forward. High-school students might find some of the book challenging, but it will probably help them to see the world in a new way, making connections that they had not known before. It can show them mathematics in a new light, being applied to very practical problems at the centre of modern communications and technology. Teachers of mathematics would hopefully find the book fascinating and would be able to appreciate it at a deeper level. There are a great many mathematical ideas that you might be surprised to find in a book about information: randomness and normality of numbers, Gödel and incompleteness, quantum mechanics and uncertainty – but the links that Gleick fashions between them is intellectually satisfying. In addition, a reader who has already heard a little bit about some of the people in this book – Babbage, Morse, Turing, and others – would find that this adds to the value of the unusual perspective that Gleick brings. *The Information*, perhaps the first natural history of information ever attempted, lays out for us the long course we've followed to get to where we are today.

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Appendix: Bits and Bytes and Shannon's concept of entropy

Consider an alphabet of symbols, each of which may be used to send a message, or transmission. Each symbol *i* used in the transmission could be selected with probability p_i and this probability could depend both on the symbol selected and its location within the transmission. Shannon proposed that the amount of information carried by a transmission is given by

$$H = -K \sum_i p_i \log p_i.$$

Here *K* is a positive constant and the summation is across the symbols of the alphabet. Why did he decide to use this measure? To create a way to measure the amount of information a transmission contains, Shannon set out three reasonable conditions that any such measure would have to satisfy. (See reference [1].) He then proved mathematically that there is only one possible way to define the measure, namely the one shown above.

Choosing different bases for the logarithm naturally gives us different choices of units to measure information. In his 1948 paper, Shannon suggested that using a base of 2 would be convenient for electronic devices, and that the units in this case could be called *binary digits*, or simply *bits*. He noted that this name was suggested by the mathematician J. W. Tukey, a colleague at Bell Labs. The name certainly has stuck! Note that as an electronic switch has two stable positions, ON or OFF, it carries 1 bit of information.

How are the probabilities p_i for the symbols in the alphabet found? For human languages, their values can be empirically estimated. Shannon gave the following example: the English language can be thought to contain an alphabet of 27 symbols: the usual 26 letters, plus a space. In everyday written English communications, not every symbol is equally probable, and their successive choices are not independent either. If each symbol is selected randomly with probability $\frac{1}{27}$ and each choice is made independently, then a transmission might look like this:

XFOML RXKHRJFFJUJ ZLPWCFWKCYJ FFJEYVKCQSGHYD QPAAMK ZAACIBZLHJQD.

Instead, if we were to use the naturally occurring frequencies of the letters in the English language, and also select each letter with a probability that depends on the previous two letters (using the naturally occurring frequencies of the various three-letter combinations) then a transmission would look like this:

IN NO IST LAT WHEY CRATICT FROURE BIRS GROCID PONDENOME OF DEMONSTURES OF THE REPTAGIN IS REGOACTIONA OF CRE.

It's clear that the resemblance to a 'meaningful' English sentence has increased, though it is still gibberish!

(Some readers may be reminded of the following lines which occur in Lewis Caroll's poem *Jabberwocky* which is part of his 'nonsense verse' work:

'Twas brillig, and the slithy toves
Did gyre and gimble in the wabe;
All mimsy were the borogoves,
And the mome raths outgrabe.

The lines seem to be telling us something, though the words do not belong to any English dictionary!)

At the time of Shannon's 1948 paper, the formula for the measure of information H was already well-known in the field of statistical mechanics. In this context, the formula describes the "entropy of the system". Roughly speaking, entropy is a measure of the 'level of disorder' in a thermodynamic system, a way of measuring how far away the system is from equilibrium. If a thermodynamic system can have several microstates, each occurring with a possibly different probability p_i then the entropy of the system is defined to be

$$s = -k_B \sum_i p_i \log p_i$$

Here k_B is called the Boltzmann constant, and the summation is across all the microstates. So Shannon had created a measure of information which is an extension of the thermodynamic concept of entropy. In this sense, information can be thought of as a form of entropy.

For interested readers, [3] is the original landmark paper where Shannon introduces these notions.

References

[1] http://en.wikipedia.org/wiki/Entropy_(information_theory)#Characterization

[2] http://en.wikipedia.org/wiki/A_Mathematical_Theory_of_Communication

[3] http://cm.bell-labs.com/cm/ms/what/shannonday/shannon1948.pdf

Azim Premji University

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Rishi Valley

DIVISION

PADMAPRIYA SHIRALI

A VISUAL APPROACH

At Right Angles
A Resource for School Mathematics

INTRODUCTION

DIVISION

It is well known that division and subtraction are generally found to be more difficult to learn by children, as compared with addition and multiplication. Particularly in the case of division, children have difficulty both in identifying the situations requiring division and the complex formal procedure involved in the long division process. There is also the role played by zero as a place value holder in the quotient. Given that these are the three main difficulties, teachers need to slow down and exercise care while teaching the division concept. I have often seen teachers not making use of the place values of numbers while explaining division procedure. Unless the place value is emphasized, the logic of the placement of quotient in the right place cannot possibly be understood.

Do plenty of warm-up activities involving these two division contexts before introducing division:

a) Division as equal distribution;

b) Division as repeated subtraction.

(Note: Division as a rate or reducing scale factor is taught only at the upper primary level.)

Keywords: Division, repeated subtraction, equal distribution, equal groups, sharing, multiplication, dividend, quotient, remainder, divisor

ACTIVITY ***ONE*** | Distribution into equal groups and internalising division contexts

- Draw two line segments parallel to each other. Ask a certain number of children to distribute themselves equally along the two segments (*but first ensure that the number of children is even*). Each opposing pair can shake hands to verify the one to one correspondence.
- Draw 3 large circles on the ground. Ask a certain number of children (*but first ensure that the number of the children is a multiple of* 3) to distribute themselves equally between the three circles. Let them count out their number for verification.
- Draw a large square on the ground. Ask a certain number of children (*but first ensure that the number of the children is a multiple of 4*) to distribute themselves equally on the four sides of the square.

It is important for the teacher to let the children figure out how to do the distribution on their own. They may blunder at the start, but they will eventually come out with effective ways of solving the problem.

ACTIVITY ***TWO*** | Distributing objects into equal groups and internalising division contexts

Materials required: Square pieces, straws and rubber bands, coloured buttons. Peg board and pegs or graph board and seeds

Let four children share the seeds or marbles amongst themselves equally.

The seeds or marbles can also be placed in paper plates or bowls.

Let another group of children work with straws and share out straws equally amongst themselves.

Let yet another group of children arrange square pieces in the required number of rows with the same number of pieces in each row.

Activities 1 and 2 should be done over several days with varied materials and in different contexts to provide children with a firm grounding in the division experience.

ACTIVITY *THREE*

Division as repeated subtraction

Materials required: Beads, straws

Give a child, say, 20 beads. Ask him to remove 2 beads at a time from the pile. Ask him to record the number of times he removes 2 beads. At the end let him record the statement as follows.

"20 seeds, 2 seeds removed 10 times."

Repeat this activity with other numbers, making sure that there will be no remainder left in the initial stage.

Often teachers introduce the symbol for division or the formal way of writing division too soon. While the child is still struggling to understand the division concept and internalise it, he or she is confronted with new symbols and complex recording procedures. It is best to give children exposure to activities involving division for at least 10 days before we introduce the symbol.

ACTIVITY *FOUR*

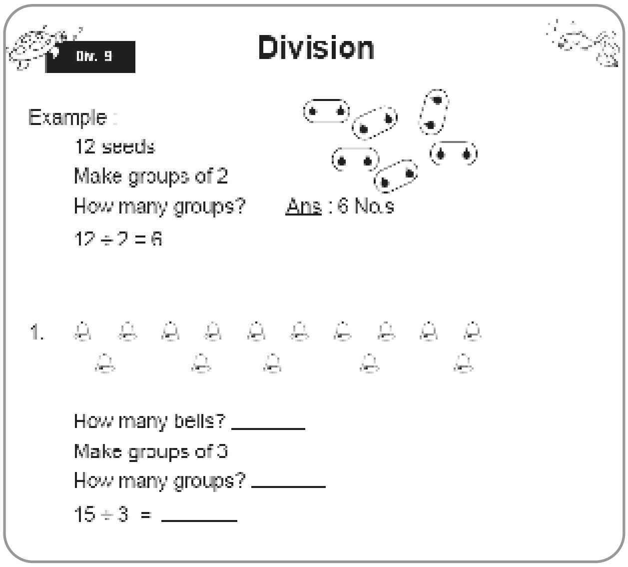
Introduction to the division symbol

Materials required: Seeds or square pieces

Let the children use pairing and count the number of pairs and record the information as shown in the picture.

Let them do repeated subtraction and count the number of times subtraction has taken place, and let them then record the information.

It is important that they record the result in both forms, as a grouping and as a division fact, till they internalize the relationship between grouping and division. The same holds if one is doing it through repeated subtraction.



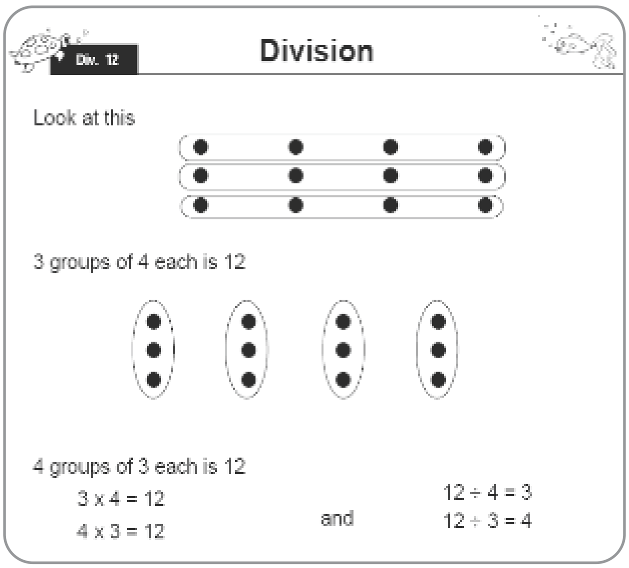
ACTIVITY *FIVE*

Helping children to see the connection between multiplication and division

Materials required: Square pieces, seeds. Peg boards with pegs

It is quite possible that children may intuitively use multiplication facts to arrive at the answers for division problems. In fact, children who have internalised multiplication concepts quite thoroughly may straight away use multiplication facts by converting the division problem into complementary multiplication problem. Example: 12 ÷ 4 may be converted to: "4 times which number equals 12?"

However, not all children may see the connection. Hence it becomes necessary for the teacher to lead the children into this discovery by asking directed questions.



How many square pieces did you have at the start? 12. Into how many rows are you going to distribute them? 4. How many pieces have you placed in each row? 3 pieces. How do we state this as a division fact?

$$12 \div 4 = 3.$$

Can you describe this arrangement (as shown in the picture) as a multiplication situation?

$$3 \times 4 = 12.$$

ACTIVITY *SIX*

To show that every division fact gives rise to another division fact

Materials required: Square pieces

Arrange square pieces in an array form as shown in the picture for activity 5.

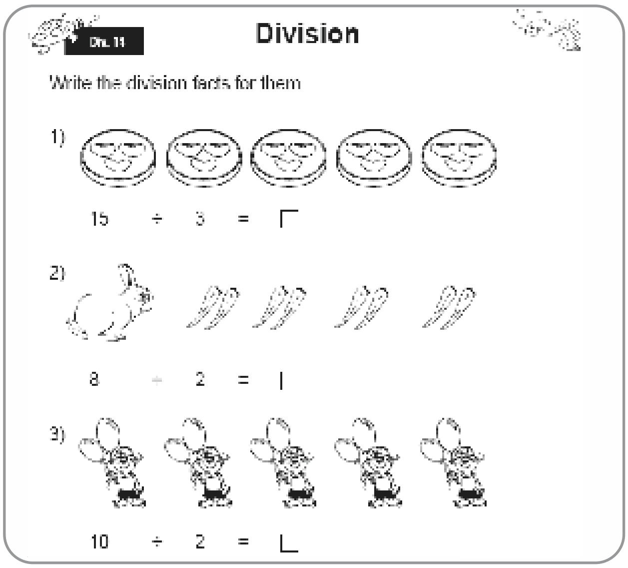
Help children to record the division fact by looking at it one way; that is, as $12 \div 4 = 3$.

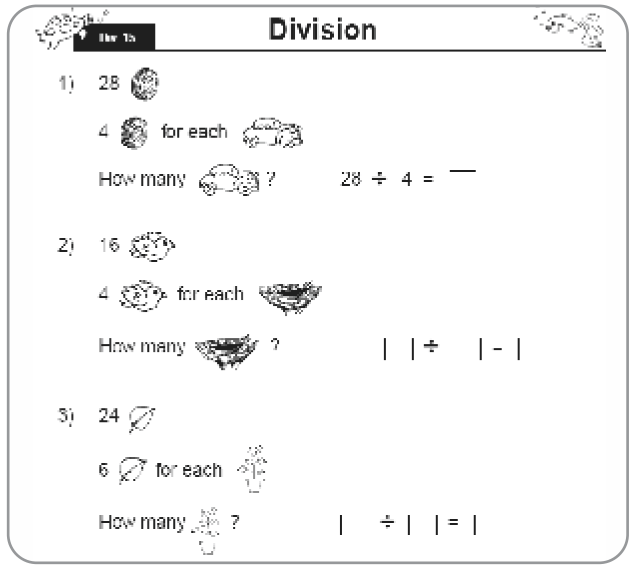
Now turn the array the other way round to record the other division fact arising from the same situation; that is, $12 \div 3 = 4$.

ACTIVITY *SEVEN*

To show division using plenty of visuals and creating context situations

Materials required: Square pieces





Create many visual word problems or give plenty of context situations which help children understand the division concept.

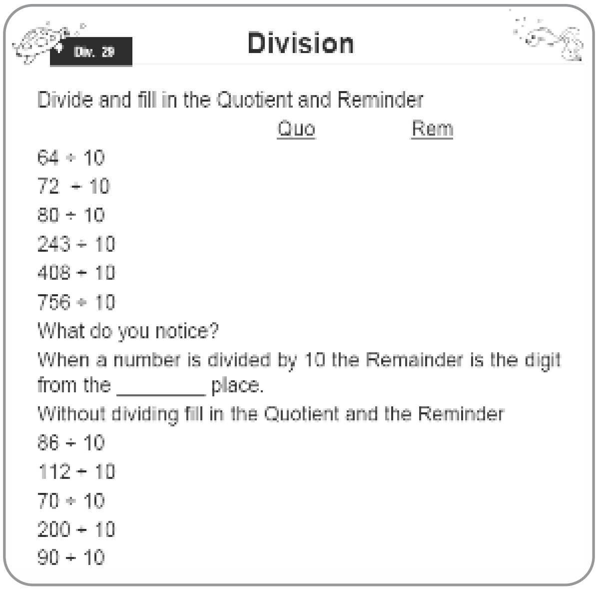
ACTIVITY *EIGHT*

To show the division procedure for one step division problems

Materials required: Square pieces

Take any problem, say: 16 ÷ 8. Take 16 square pieces and distribute equally to 8 people. Let children see that each person gets 2 pieces. Now simultaneously show the recording procedure, emphasizing the fact that the total number being shared (namely, 16) is written in the centre, the number of people among whom it is shared is written on the left, and the number each one gets is written on top. Use place value headings on the dividend. It is best to get children into the practice of writing the quotient on top, synchronizing it with the correct place value. This reduces the scope for errors, and zero as a place value holder becomes evident.

ACTIVITY NINE | Division by ten



Div. 29

Division

Divide and fill in the Quotient and Reminder

	Quo	Rem
64 ÷ 10		
72 ÷ 10		
80 ÷ 10		
243 ÷ 10		
408 ÷ 10		
756 ÷ 10		

What do you notice?

When a number is divided by 10 the Remainder is the digit from the ________ place.

Without dividing fill in the Quotient and the Reminder

86 ÷ 10

112 ÷ 10

70 ÷ 10

200 ÷ 10

90 ÷ 10

Show several problems with division by 10. Make a table as shown, showing the division facts along with quotient and remainder. Let children notice that in division by 10, the number in the units place becomes the remainder, and the 'rest' of the number becomes the quotient.

ACTIVITY TEN | Division of zero by a number, and division by zero

$$12 \div 0 = ?$$

This is best explained through repeated subtraction.

Ask the question: "How many times can zero be subtracted from 12 to get zero at the end"?

No matter how many times we subtract zero from 12, we will never get a zero as the answer! It goes on indefinitely. So the division cannot be done at all.

Division of zero by a number:

$$0 \div 8 = ?$$

While teaching multiplication facts through the flow technique, we had established that any number multiplied by 0 yields 0. We could now use this fact to show that the answer would be 0.

Note on terminology: It is not necessary to introduce all the words – '*dividend*', '*divisor*', '*quotient*', '*remainder*' – at the lower primary level. 'Remainder' alone will make sense, as it is a commonly used word in English. Instead, we can refer to these numbers as: How many (objects or sweets) need to be distributed? To how many people? How many will each get?

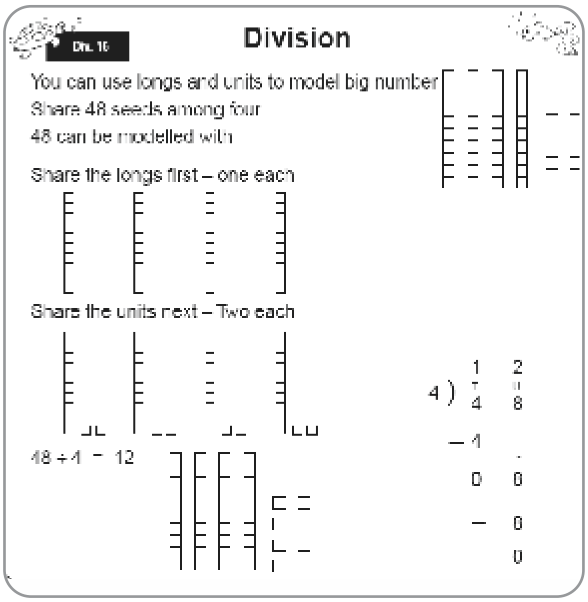
ACTIVITY ***ELEVEN***

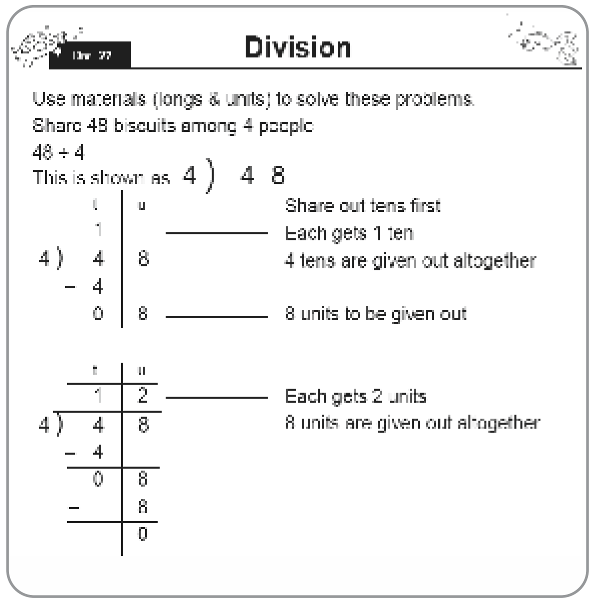
Division of a double digit number by a single digit number (without any exchange and remainder)

Materials required: Place value Kit.

Introduction to division of a double digit number by a single digit number is best done through place value material. In the first stage we introduce problems which do not require exchange from tens to units.

Example: "Share 48 rupees amongst 4 people."





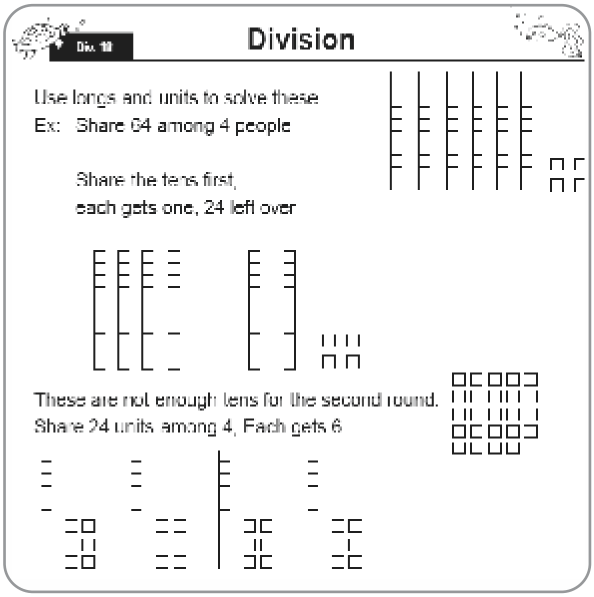
As shown in the picture, we use tens and units material to show 48. We start with **tens** (this is an important point to note as in all the other arithmetical operations we start from right to left, and it is in the division operation alone that we move from left to right) and ask the question how many tens (at each point, read the number with its place value to draw the child's attention to it) can we share out equally amongst 4 people? Each one gets 1 **ten** (*emphasize again the place value*). This is recorded in the division problem as 1 ten in the tens place over the 4. It is important to emphasise the place value all the way through. In my school days we used to write the quotient on the right hand side of the dividend. But this way one does not see the correspondence between the digits of the quotient and the dividend. Placement of zeroes as place holders is also not clear in this form of writing.

Now as we subtract the 4 tens given away, we move to the second step. Many children take time to learn a two-step division problem; hence we must go very slowly, articulating every action. I usually like to use a downward arrow to indicate "bringing down the next number." This focuses the child's attention on it, makes him understand what is happening, and serves as a visual aid. We now take down 8 units and each gets 2 units which is then recorded on top of 8 as a quotient. After subtraction there are no units left.

ACTIVITY *TWELVE*

Division of a double digit number by a single digit number (with exchange) and with or without remainder

Materials required: Place value Kit.



Here again division of a double digit number by a single digit number involving exchange is best done through place value material.

Example: Share 64 rupees amongst 4 people.

As shown in the picture, we use tens and units material to show 64 and make 4 groups. Since we need to share 6 tens among 4 people we ask the question: "How many tens can we share out equally among 4 people?" So we first distribute 4 of the 6 tens to the 4 people. (Sometimes children write a lower multiple than what is possible under the dividend and end up with a remainder which is larger than the divisor. Point out to them that when that happens they could have taken a higher multiple). Each gets 1 ten. This is recorded in the division problem as 1 ten in the tens place over the 6. Now when we subtract what is given away to the 4 people, we are left with 2 tens. We now take down 4 units. We convert the 2 tens into 20 units. The number now reads as 24 units. 24 units shared amongst 4 people is 6 units for each person. So we record 6 units over the units place.

Extension: Problems involving remainder should also be taken up and explained using materials in a similar way.

ACTIVITY ***THIRTEEN***

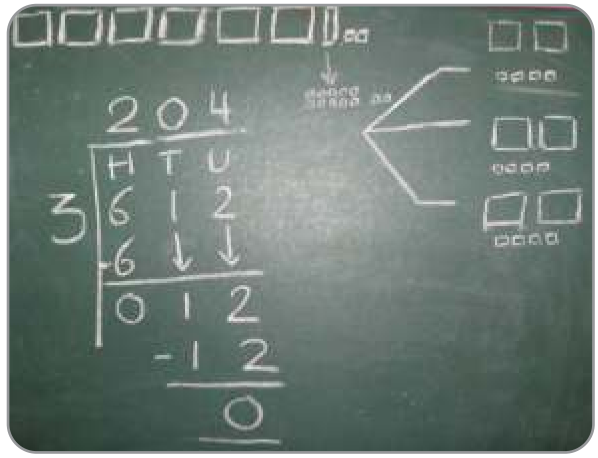
Division of a three digit number by a single digit number (with zero in the quotient)

Materials required: Place value Kit.

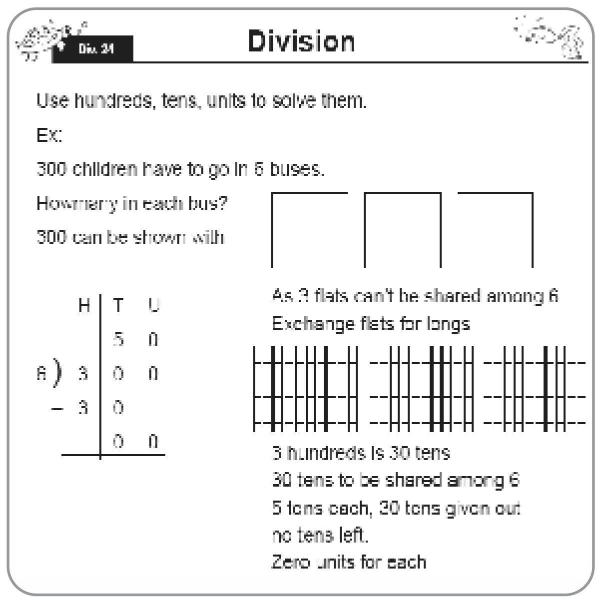
Share 612 rupees amongst 3 people:

$$612 \div 3$$

Let us use place value materials to demonstrate this as shown in the picture.



As shown in the picture we use hundreds, tens and units material, or fake money, to show 612. Since we need to share 6 hundreds among 3 people we ask the question: "How many hundreds can we share out equally among 3 people?" So we first distribute 6 hundreds to the 3 people, and each one gets 2 hundreds. This is recorded in the division problem as 2 hundreds in the hundreds place over the 6. Now as we subtract what is given away to the 3 people we are left with 0 hundreds. We now take down 1 ten. We cannot share 1 ten as it is (without exchanging) amongst 3 people. Since no tens are being given out to the three people, we write 0 tens in the tens place of the quotient. Now we bring down the 2 units. We convert the 1 ten into 10 units. The number now reads as 12 units. 12 units shared amongst 3 people is 4 units for each person. Now we record 4 units over the units place.



Extension: Different problems which give rise to zero in the quotient, for example: $408 \div 4$, $400 \div 5$, $600 \div 5$ can be explained using the same kind of reasoning.

While discussing problems which give rise to a remainder we need to draw the child's attention to the fact that the remainder will always be less than the divisor.

Verification: Show the children the method for verifying that their answer is right by multiplying the quotient with the divisor and adding the remainder to the product so obtained. It must match with the dividend.

ACTIVITY FOURTEEN | Division of a double digit number by a double digit number

It is best to postpone teaching this to class 5. Also by this point the child does not (indeed, should not) need to take recourse to the use of concrete materials. Terminology like 'dividend', 'divisor', 'quotient' and 'remainder' can be introduced at this level.

Children face a lot of difficulty in understanding the procedure of division by a double digit number, and many errors happen in this area. Working with multiples of a double digit number is difficult as children do not know the multiplication facts of these numbers. In the initial stages let them construct the multiplication table for the required number and use the facts in solving the problem. At a later point one can help them to use estimation methods to figure out the possible quotient. Estimation would involve looking at the numerals in the highest place value both of the quotient and the divisor. For example, 785 ÷ 24 requires the child to look at 7 hundred and 2 tens which will result in 3 tens (7 hundred divided by 2 tens). Also, let children see that estimation does not always give the exact quotient but it helps in getting close to the possible quotient; it could be at times 1 less than that or 1 more than that.

The games suggested here involve the usage of all four operations. They can be modified to reinforce divisions alone or multiplications and divisions.

GAME 1 | CLIMB THE LADDER

Aim: To give practice in using 4 operations with small numbers for two players.

Equipment: Two dice, game board with two ladders (numbers 1 to 10 written on each rung), two counters.

The first player rolls the two dice (let us say 5 and 6 arise), using those numbers and any operation in any combination, he must try to make an answer of 1.

He might arrange them as $6 - 5 = 1$. The player can then place his counter on the first rung of the ladder. When a player cannot make the required number, he loses his turn and stays where he is on the ladder.

The second player now rolls the dice in turn to move his counter on the ladder.

Extension: The game can be extended to 20 rungs on the ladder. We can also have 3 dice to bring in 2 operations at a time.

GAME 2 EQUATIONS

Aim: To give practice in using 4 operations with slightly larger numbers.

Equipment: 52 number cards

Numbers 1 to 9 (3 of each), 10 to 18 (2 of each), 19 to 25 (1 of each)

Also make 30 sign cards (+, –, ×, ÷)

No. of players: Two to four

Shuffle the number cards and deal five cards to each player. The object of the game is to arrange these five cards, together with an equals to sign and three sign cards of their choice, to form an equation. Children should be told that × and ÷ get priority over + and –.

A player who manages to make an equation with the original cards scores 10 points. If a player cannot make an equation with his set of 5 number cards and exchanges 1 number card for a new one, then he loses 2 points for every card exchanged.

GAME 3 DOMINOES

Aim: To provide practice in simple multiplication and division at speed.

Equipment: Make a set of 32 dominoes. Each domino is divided by a line into a 'left side' and a 'right side'. The first domino has "Start" written on the left side and a simple problem on the right side (ex. 9 × 3, 24 ÷ 6). On each of the remaining dominoes, the answer for the previous domino's problem is written on the left side, and another problem is given on the right side (the answer for which is on the left side of the next domino, and so on). The last domino has "End" written on the right side.

This game is self-correcting in nature. If a single child plays it alone, he will have dominoes left over if he makes a mistake. If several are playing together, others can point out the mistakes.

Padmapriya Shirali

Padmapriya Shirali is part of the Community Math Centre based in Sahyadri School (Pune) and Rishi Valley (AP), where she has worked since 1983, teaching a variety of subjects – mathematics, computer applications, geography, economics, environmental studies and Telugu. For the past few years she has been involved in teacher outreach work. At present she is working with the SCERT (AP) on curricular reform and primary level math textbooks. In the 1990s, she worked closely with the late Shri P K Srinivasan, famed mathematics educator from Chennai. She was part of the team that created the multigrade elementary learning programme of the Rishi Valley Rural Centre, known as 'School in a Box'. Padmapriya may be contacted at padmapriya.shirali@gmail.com

The Closing Bracket . . .

One wonders why a field as basic as Education – a field at once crucial and vitally important in every possible way — receives such little education in the public domain. In name of course it does. New schemes get proposed, new critiques are offered, and the cycle continues. Yet there is barely a dent made in the hard shell of the problem. Are we as a society at all interested in doing something about it? It would appear not. One comes across any number of armchair critics who will tell you exactly what is wrong with the education system, and who will not do a thing about it.

A decade the former Government initiated a series of actions that culminated in the National Curriculum Framework 2005. Included in it was a superbly crafted (and freely available) document referred to widely as the "Position Paper in Teaching of Mathematics". It contains a wealth of wisdom on the teaching of mathematics, in matters ranging from curriculum and teaching style to assessment and examinations. I wonder why we have allowed so important and valuable a document to simply go waste and gather dust on shelves. (Note: This metaphor will now have to be adapted to modern times! So I should say "gather e-dust on e-shelves hidden in some cloud!") Are we now going to take steps to produce version 2.0 of this document? It would be a tragic waste of good human energy of we do so. In fact, such a step is only a strategy for postponing action. So let us pray that this will not happen.

One wonders why we cannot simply start at our own individual levels — teaching and learning beautiful mathematics which is available in such abundance everywhere, especially in the modern world. Is it that we are all waiting for big organizational initiatives? And if so, why? The following quote from J Krishnamurti is of relevance. "In a world of vast organizations, vast mobilizations of people, mass movements, we are afraid to act on a small scale; we are afraid to be little people clearing up our own patch. We say to ourselves, 'What can I personally do? I must join a mass movement in order to reform.' On the contrary, real revolution takes place not through mass movement but through the inward revolution of relationship—that alone is real reformation, a radical, continuous revolution. We are afraid to begin on a small scale. Because the problem is so vast, we think we must meet it with large numbers of people, with a great organization, with mass movements. Surely, we must begin to tackle the problem on a small scale, and the small scale is the 'me' and the 'you'."

— Shailesh Shirali

Specific Guidelines for Authors

Prospective authors are asked to observe the following guidelines.

1. Use a readable and inviting style of writing which attempts to capture the reader's attention at the start. The first paragraph of the article should convey clearly what the article is about. For example, the opening paragraph could be a surprising conclusion, a challenge, figure with an interesting question or a relevant anecdote. Importantly, it should carry an invitation to continue reading.
2. Title the article with an appropriate and catchy phrase that captures the spirit and substance of the article.
3. Avoid a 'theorem-proof' format. Instead, integrate proofs into the article in an informal way.
4. Refrain from displaying long calculations. Strike a balance between providing too many details and making sudden jumps which depend on hidden calculations.
5. Avoid specialized jargon and notation — terms that will be familiar only to specialists. If technical terms are needed, please define them.
6. Where possible, provide a diagram or a photograph that captures the essence of a mathematical idea. Never omit a diagram if it can help clarify a concept.
7. Provide a compact list of references, with short recommendations.
8. Make available a few exercises, and some questions to ponder either in the beginning or at the end of the article.
9. Cite sources and references in their order of occurrence, at the end of the article. Avoid footnotes. If footnotes are needed, number and place them separately.
10. Explain all abbreviations and acronyms the first time they occur in an article. Make a glossary of all such terms and place it at the end of the article.
11. Number all diagrams, photos and figures included in the article. Attach them separately with the e-mail, with clear directions. (Please note, the minimum resolution for photos or scanned images should be 300dpi).
12. Refer to diagrams, photos, and figures by their numbers and avoid using references like 'here' or 'there' or 'above' or 'below'.
13. Include a high resolution photograph (author photo) and a brief bio (not more than 50 words) that gives readers an idea of your experience and areas of expertise.
14. Adhere to British spellings – organise, not organize; colour not color, neighbour not neighbor, etc.
15. Submit articles in MS Word format or in LaTeX.

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At Right Angles welcomes articles from math teachers, educators, practitioners, parents and students. If you have always been on the lookout for a platform to express your mathematical thoughts, then don't hesitate to get in touch with us.

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Articles involving all aspects of mathematics are welcome. An article could feature: a new look at some topic; an interesting problem; an interesting piece of mathematics; a connection between topics or across subjects; a historical perspective, giving the background of a topic or some individuals; problem solving in general; teaching strategies; an interesting classroom experience; a project done by a student; an aspect of classroom pedagogy; a discussion on why students find certain topics difficult; a discussion on misconceptions in mathematics; a discussion on why mathematics among all subjects provokes so much fear; an applet written to illustrate a theme in mathematics; an application of mathematics in science, medicine or engineering; an algorithm based on a mathematical idea; etc.

Also welcome are short pieces featuring: reviews of books or math software or a YouTube clip about some theme in mathematics; proofs without words; mathematical paradoxes; 'false proofs'; poetry, cartoons or photographs with a mathematical theme; anecdotes about a mathematician; 'math from the movies'.

Articles may be sent to :
AtRiA.editor@apu.edu.in

Please refer to specific editorial policies and guidelines below.

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The magazine has zero tolerance for plagiarism. By submitting an article for publishing, the author is assumed to declare it to be original and not under any legal restriction for publication (e.g. previous copyright ownership). Wherever appropriate, relevant references and sources will be clearly indicated in the article.

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