Azim Premji University

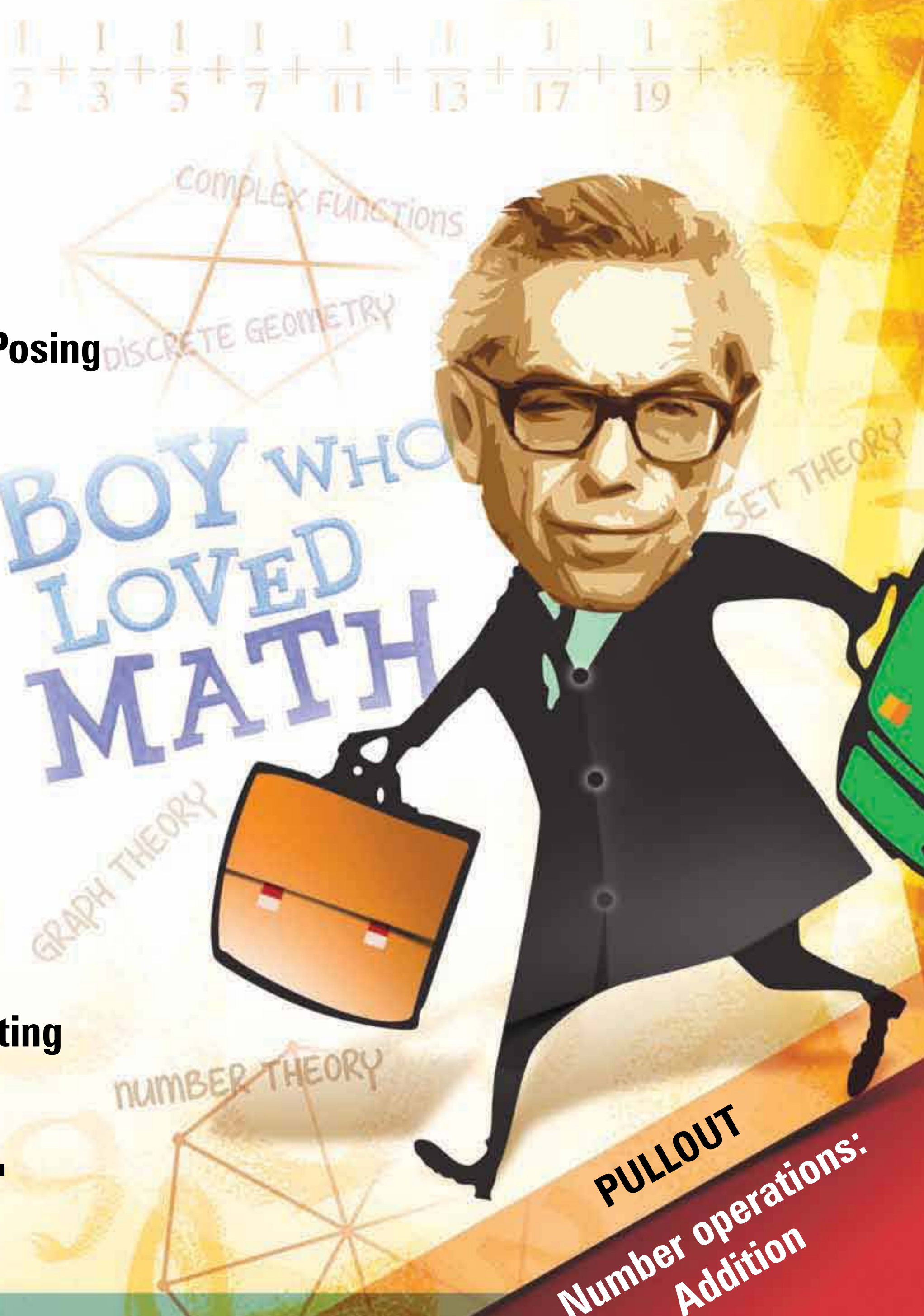
A publication of Azim Premji University together with Community Mathematics Centre, Rishi Valley

COMMUNITY MATH CENTRE · RISHI VALLEY SCHOOL (KFI)

At Right Angles

A RESOURCE FOR SCHOOL MATHEMATICS

Volume 2, No. 2
July 2013

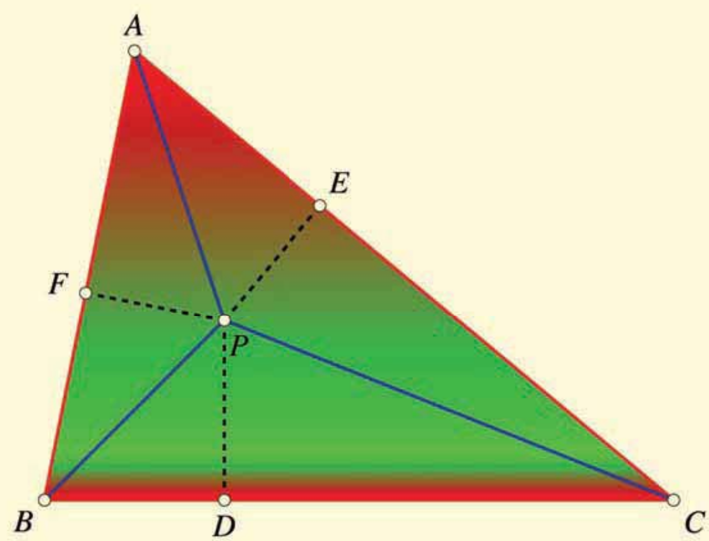


Notes on the Cover Image

Paul Erdős

Paul Erdős was a problem poser par excellence. It is said that he had the curious ability of knowing the right level of problem with which to challenge those around him!

Here is a geometric inequality posed by him as a problem,which is now regarded as a classic – the Erdős-Mordell inequality: If Pis a point within a triangle ABC, and D,E,F are the feet of the perpendiculars from P to the sides, then PA+PB+PC≥2(PD+PE+PF). Equality holds just when triangle ABC is equilateral and P is its centroid.



$$PA + PB + PC \geq 2(PD + PE + PF)$$

A proof was provided in 1937 (the 'Mordell' in the name of the theorem comes from the proof). Subsequently some generalizations have been discovered, and elementary geometrical proofs found. An analogue of this result is known for tetrahedra too.

From The Chief Editor's Desk . . .

It is most appropriate that we feature in this issue one of the great characters of twentieth century mathematics: Paul Erdős, the itinerant mathematician, who lived out of a suitcase, collaborated with mathematicians everywhere he went, and wrote more papers than anyone else in the history of the subject; for this is the centenary of his birth. R Ramanujam writes about him in the context of problem-posing. We hope this article will herald a regular 'Erdős column' in At Right Angles, devoted to Erdős-style problems.

It is always nice to see derivations based on a minimum of higher concepts. Sadagopan Rajesh shows us how Brahmagupta's elegant formula for the area of a cyclic quadrilateral can be derived using Heron's formula and the geometry of similar triangles. B Sury takes up the second instalment of his article on the Principle of Inclusion and Exclusion. This is followed by Part 2 of my own article on Harmonic Triples. Devang Ram Mohan writes on a subject that may not immediately be recognized as being part of mathematics: the art of making matches optimally! In fact, optimization in matchings is a well known topic in operations research, and it was the topic of last year's Nobel Prize in Economics. So it is very appropriate that you read about it here.

In 'Classroom' we start with an essay on Angles which explores some pitfalls that can waylay the young student, concerning measurement of angles. Sneha Titus talks about two contrasting styles of teaching mathematics — the 'path smoothing' way and the 'enabling' way. This matter has considerable relevance for us in India, for path smoothing is so rooted in our culture. Ajit Athle then leads us through two interesting problems in geometry and draws out more lessons in the art of problem solving in geometry. Tanuj Shah follows by showing how the combinatorics of Braille makes for a rich Math Club topic, and Prithwijit De narrates an allegorical tale based on an elegant combinatorial property of perfect squares.

In 'Tech Space' Thomas Lingefjärd points to some interesting GeoGebra explorations that can be made in the fertile territory common to number theory and geometry. Then in 'Reviews', Dheeraj Kulkarni talks about a fascinating set of math videos freely available on the web, devoted to the topic of Dimension; would that more such videos become available! Padmapriya Shirali has the closing piece, a Pullout on the teaching of Number Operations; this continues the piece which appeared in the previous issue, on Place Value.

One of the aims of starting this magazine was to facilitate a networking of mathematics teachers across the country, and to help bring about a culture of dialogue amongst us. We have begun in small ways to do this, through math teacher workshops. We hope to reach out in a much greater way through an online presence. We are working on this - do stand by for further announcements! You could also contact us if you would like to arrange a workshop in your city.

- Shailesh Shirali

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At Right Angles is a publication of Azim Premji University together with Community Mathematics Centre, Rishi Valley School and Sahyadri School (KFI). It aims to reach out to teachers, teacher educators, students & those who are passionate about mathematics. It provides a platform for the expression of varied opinions & perspectives and encourages new and informed positions, thought-provoking points of view and stories of innovation. The approach is a balance between being an 'academic' and 'practitioner' oriented magazine.

At Right Angles
A Resource for School Mathematics

Azim Premji University
A Publication of Azim Premji University together with Community Mathematics Centre, Rishi Valley and Sahyadri School, Pune

Contents

Features

This section has articles dealing with mathematical content, in pure and applied mathematics. The scope is wide: a look at a topic through history; the life-story of some mathematician; a fresh approach to some topic; application of a topic in some area of science, engineering or medicine; an unsuspected connection between topics; a new way of solving a known problem; and so on. Paper folding is a theme we will frequently feature, for its many mathematical, aesthetic and hands-on aspects. Written by practising mathematicians, the common thread is the joy of sharing discoveries and the investigative approaches leading to them.

In the Classroom

This section gives you a 'fly on the wall' classroom experience. With articles that deal with issues of pedagogy, teaching methodology and classroom teaching, it takes you to the hot seat of mathematics education. 'In The Classroom' is meant for practising teachers and teacher educators. Articles are sometimes anecdotal; or about how to teach a topic or concept in a different way. They often take a new look at assessment or at projects; discuss how to anchor a math club or math expo; offer insights into remedial teaching etc.

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Contents contd.

Tech Space

'Tech Space' is generally the habitat of students, and teachers tend to enter it with trepidation. This section has articles dealing with math software and its use in mathematics teaching: how such software may be used for mathematical exploration, visualization and analysis, and how it may be incorporated into classroom transactions. It features software for computer algebra, dynamic geometry, spreadsheets, and so on. It will also include short reviews of new and emerging software.

Problem Corner

Reviews

Letter

Pullout

feature

Paul Erdős

The Artist of Problem-Posing

Notes from a small suitcase

Paul Erdős has been described as one of the most universally adored mathematicians of all time. No mathematician prior to him or since has had quite the lifestyle he adopted: the peripatetic traveller living out of a suitcase, moving from one friend's house to another for decades at a stretch, and all the while collaboratively generating papers; no one has had quite the social impact he has had, within the community of mathematicians. This article offers a glimpse of his life and work.

R. Ramanujam

A theorem on Facebook

It's very likely that you have a Facebook account, and of course, you have many friends on Facebook. If *X* is your friend on Facebook, then you are *X*'s friend on Facebook too. But it's possible that *Y* is *X*'s friend and not yours. Still, I am sure you and *X* have many common friends, forming a *trio* of friends. Now, here is a question for you.

What is the smallest number n of people on Facebook such that there is ***definitely*** *a trio among them, either all of whom are friends with each other, or none of whom are friends with each other?*

With three people, say *A, B* and *C,* it is easy that we will not have this property: let *A* and *B* be friends, neither of whom are friends with *C*. What about four people, *A, B, C* and *D*? Again it is easy: make *A* and *B* friends, *C* and *D* friends, and no more. In both these case the desired trio is not to be found.

When we have five people, it is a little more difficult, but a picture can help think about it. Let us have points denoting the 5 persons; draw a red line connecting them to denote that they are friends on Facebook, and a blue line between them to denote that they are not. Now a red pentagon with the persons on vertices and blue lines to 'opposite' vertices (as in Figure 1) should convince you that we can indeed have a situation without the desired property.

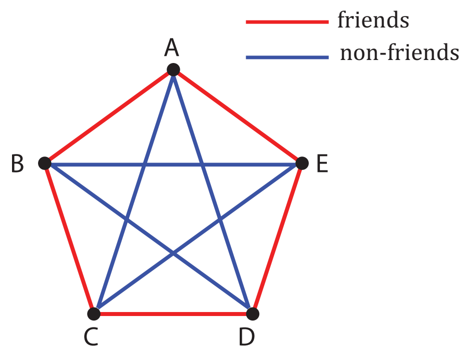

Figure 1: Pentagon with 'red' edges, 'blue' diagonals: no trio of the desired kind

Now to six. Take some time off now to draw pictures like the above. The hexagon with its diagonals does not help; while we get many interesting pictures, we get none that works like the one with five vertices. At this point, we start suspecting that six might indeed be the smallest number we seek. But then we need a proof that among **any six** persons on Facebook, we have a trio, either all of whom are friends, or not-friends.

Call the newcomer F. We first observe that we already know something about F!

Claim. Among the other five persons, there are at least three among them such that F is a friend of all three, or F is friends with none of the three.

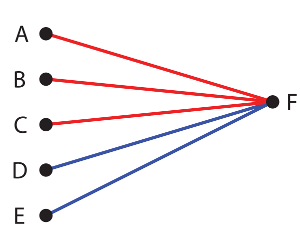

Figure 2: Whom is F friends with?
This figure shows one of many possibilities.

Why? Suppose not. Let us argue with reference to a picture like the one we drew earlier; see Figure 2. We focus on the 'lines' coming out from F. Each line is red or blue. Since there are five such lines, one colour occurs three times or more. Whichever that colour is, we get the three persons we want. (If this colour is red, then the three persons the lines connect to are friends of F, else they are non-friends.)

Nice. Suppose that the three persons identified in the previous step are A, B, C. (It could be any three; we have renamed them as A, B, C.) Their relationship with F is the same: all friends or all non-friends. Suppose they are all friends of F. Now if any two of A, B, C are friends with each other, these two together with F form a trio of friends. And if no two among A, B, C are friends with each other, then A, B, C form a trio of non-friends! Either way we get the trio we need.

Please check that all possible cases can be disposed of in a similar way. So we have proved a **Facebook Theorem** that is valid for any six of the millions of members who use that site, knowing nothing at all about them!

The picture that we drew was a *graph*, with *edges* connecting pairs of *vertices*. We used two kinds of edges, red and blue. We can call this an *edge-colouring* of the graph with two colours. When every pair of vertices has an edge between them, we call it a *complete* graph. A complete graph on n vertices is denoted by K_n. (So K_2 is just an edge, K_3 is a triangle, and K_4 is a quadrilateral with its two diagonals.)

In this language, what we showed was the following: if each edge of K_5 is coloured red or blue, then a monochromatic K_3 may not get created, but if each edge of K_6 is coloured red or blue, then a monochromatic K_3 necessarily does get created. ('Monochromatic' means that all edges have the same colour.)

The critical number 6 is an example of a **Ramsey number** (named after the mathematician and logician Frank Plumpton Ramsey) of a graph, the minimum number of vertices needed to force a monochromatic subgraph inside it. More rigorously, given any two numbers s and t, the

Ramsey number $R_2(s, t)$ is the smallest integer m satisfying the property that if the edges of K_m are coloured red or blue, then no matter which way it is done there is either a subgraph K_s with all red edges, or a subgraph K_t with all blue edges. With k colours, we can similarly speak of $R_k(s, t)$. What we showed above was: $R_2(3, 3) = 6$.

Why should anyone care about Ramsey numbers? For one reason, finding them is extremely hard! Only a handful are known, and Table 1 lists all the known Ramsey numbers of the form $R_2(s, t)$. You will find it a nice challenge to show that $R_2(3, 4) = 9$.

In the absence of any practical algorithm for computing exact values of Ramsey numbers, a great deal of research effort has been concentrated on obtaining bounds instead. For *diagonal* Ramsey numbers, i.e., Ramsey numbers of the form $R_2(s, s)$, some bounds are known. For instance, it can be shown without too much difficulty that $R_2(s, s) \le 4^s$, an upper bound. Getting lower bounds is much harder.

s	3	3	3	3	3	3	4	4
t	4	5	6	7	8	9	4	5
$R_2(s, t)$	9	14	18	23	28	36	18	25

Table 1: All the known Ramsey numbers

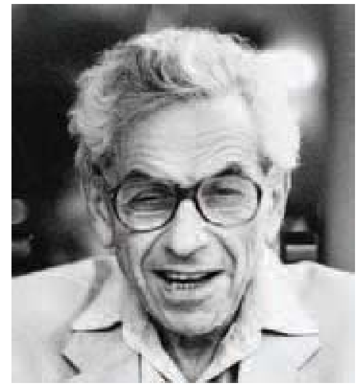

Figure 3: Paul Erdős having a chuckle.
Source: http://24.media.tumblr.com/tumblr_maobc7dXYQ1qipuzxo1_1280.jpg

In 1947, the mathematician **Paul Erdős** (pronounced Air-dsh) proved this remarkable theorem

Theorem 1. *Let k, n be positive integers such that* $2\binom{n}{k} < 2^{\binom{k}{2}}$. *Then* $R_2(k, k)$ *is greater than n.*

In order to show that $R_2(k, k) > n$, it suffices to show that there exists at least one colouring of the edges of K_n which results in no monochromatic K_k. Erdős showed this probabilistically! The details are given in Figure 4.

Erdős's probabilistic proof of Theorem 1.

Consider an edge blue/red colouring of K_n in which the colour for each edge is assigned *randomly* and *independently*, with probability $^1/_2$ for each.

How many copies are there of K_k in this configuration? Clearly as many as there are subsets of size k in the set $\{1, 2, 3, \dots, n\}$, i.e., $\binom{n}{k}$. What is the probability that any particular copy is monochromatic? Each of the $\binom{k}{2}$ edges in the chosen K_k gets a particular colour with probability $^1/_2$, and there are two colours to choose from, so the probability is equal to

$$2 \cdot \frac{1}{2^{\binom{k}{2}}} = 2^{1-\binom{k}{2}}.$$

Hence the probability that there exists a monochromatic K_k is at most

$$\binom{n}{k} \cdot 2^{1-\binom{k}{2}}.$$

(For, the probability of a union of several events is at most the sum of the probabilities of the individual events.)

This quantity is less than 1 by the assumption of the theorem, hence the probability that there exists a colouring with **no** monochromatic K_k is greater than 0. Therefore, there exists a colouring with no monochromatic K_k, and we are done.

Figure 4: A random proof!

The master who could count by tossing coins

Erdős was 33 years old when he proved Theorem 1. This way of proving the existence of something by showing that the probability that it exists is positive is typically Erdős's. He loved the **probabilistic method** and used it to great advantage to solve problems in number theory, combinatorics and graph theory. He would set up some mechanism for counting the number of ways of doing something, compute the probability of an event, and show that some mathematical object exists. Joel Spencer, who did a lot of work with Erdős refers to this as 'Erdős magic'.

Erdős could ask questions on counting pretty much anything. Consider k points and t lines on the plane. We might ask many questions concerning them, but here is one of Erdős's questions: What is the maximum number $f(k, t)$ of incidences between the points and the lines? He conjectured that the points of a square grid and a certain set of lines give the optimal order of magnitude. This was confirmed only decades later by Szemerédi and Trotter, in 1983.

Here is Erdős proposing a question for the student journal *Quantum*. Let $f(n)$ be the largest integer for which there is a set of n distinct points $x_1, x_2, \ldots x_n$ in the plane such that for every x_i there are at least $f(n)$ points x_j which are equidistant from x_i. Determine $f(n)$ as accurately as possible. Is it true that $f(n)$ is approximately n^{ϵ} for every $\epsilon > 0$? Erdős offered $500 for a proof and 'much less' for a counterexample. The question was settled in 1990.

This was also typical of the Erdős style; he posed thousands of problems, and offered prize money for solving many of them.

Very early on, Erdős was attracted to number theory, but there too he turned to counting orders of magnitude. In 1934, when he was 21 years old, Erdős heard of Simon Sidon's work on sequences of integers with pairwise different sums. In 1938 he asked: what is $f(n)$, the maximum number of positive integers $a_i \leq n$ such that the pairwise products $a_i a_j$ are all distinct? He answered the question by reducing it to a question in graph theory.

Through his questions, Erdős led us in many directions that we could not have imagined to exist. Here is an example. In 1927 van der Waerden published a celebrated theorem, which states that if the positive integers are partitioned into finitely many classes, then at least one of these classes contains arbitrarily long arithmetic progressions. In 1936, Erdős and Turàn realised that it ought to be possible to find arithmetic progressions of length k in any 'sufficiently dense' set of integers, which would show that the partitioning in van der Waerden's theorem was, in a sense, a distraction. The conjecture was proved by Szemerèdi in 1974. Not only is it a very difficult proof, but the *regularity lemma* that he used in the proof has become a central tool in graph theory and theoretical computer science. (Szemerèdi was awarded the prestigious Abel prize last year.)

> **A question both deep and profound**
> **Is whether a circle is round.**
> **In a paper of Erdős**
> **Written in Kurdish**
> **A counter example is found.**

Erdős made another related conjecture, far more famous and still open. *Let X be any set of positive integers such that the series* $\sum_{x \in X} \frac{1}{x}$ *diverges. Then X contains arbitrarily long arithmetic progressions.* Note that the set of primes is an example of such a set. The general question is open (as noted), but Green and Tao showed in 2004 that the primes contain arbitrarily long arithmetic progressions. (Terence Tao was awarded the Fields prize in 2006.)

All this is very deep mathematics, but what about the fun part? Often, it was recreational mathematics that led Erdős to the deep end. Here is an example. A distinct pair of numbers (m, n) is said to be *amicable* if the sum of the proper divisors of m is n, and vice versa. The smallest such pair is (220, 284). It is still unknown if there are infinitely many amicable pairs, but Erdős showed that the set of amicable numbers has density zero. This means, roughly speaking, that they are quite rare.

The man without boundaries

By now the picture of Paul Erdős, the great problem solver and problem poser, must have taken shape. But he not only posed problems, he also sought out people to pose the problems to. He offered sums of money as encouragement for students to think about problems. Much of this money was his own, he gave freely to numerous non-mathematical charities and causes as well, keeping hardly any money for himself.

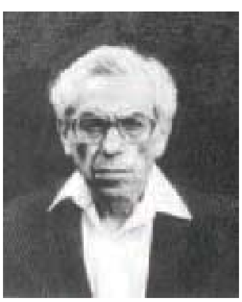

Figure 5: Two images of Paul Erdós.
Source: http://www-history.mcs.st-and.ac.uk/Mathematicians/Erdos.html

When he published his first paper in 1932 Erdős was merely 18. He continued to publish until 2003, almost 7 years after his death as some straggling papers continued to be published posthumously! He published 1521 papers in all, collaborating worldwide with a staggering number of mathematicians. How could he do this?

Erdős was prolific because his life was wholly devoted to mathematics. He did not have a job, a regular place of stay, or more possessions than he could carry with him in his two (half empty!) suitcases. He travelled from university to university, from mathematician to mathematician, working until his collaborator was exhausted, and then moving on. He did not cultivate human contact outside of his mathematical interactions, with the exception of his mother, whom he loved dearly. He didn't have to cook, clean or keep house; he had a cadre of people who looked after him, saw to it that he had food, shelter and, when necessary, a visa for his next destination.

Even the language of Paul Erdős was idiosyncratic. To him, children were 'epsilons', people 'died' when they stopped doing mathematics, and people 'left' when they actually died. He didn't lecture, he 'preached', and when he was ready to do mathematics, 'his brain was open'. To him, God was the 'Supreme Fascist'. But there was something that *was* divine for him: he used to speak of **The Book** in which all beautiful theorems and proofs was written down; the job of the mathematician was only to find them. When he found a really elegant argument, he would exclaim, "Ah, that's from The Book!"

Paul Erdős had a hard life. Born in 1913 in Budapest, Hungary, he was a child at the time of World War I, and the years after the war were worse. Jews were not allowed to attend university, and Erdős had to pass a national examination in 1930 before he was exempt from these 'fascist' rules. He attended the University Pazmany Peter in Budapest from 1930 to 1934 and then, fleeing the repressive regime in Hungary, went to Manchester in England for research. His mathematical wanderings began and he worked also in Cambridge, London, Bristol and other places.

By 1938 he could no longer safely return to Hungary because of Hitler's control of Austria, and Erdős spent a year at the Institute for Advanced Study in Princeton University in the USA. After a year, he left Princeton, and started wandering, university to university, mathematician to mathematician, and conference to conference. In 1945 he received word that most of his extended family had been killed in Auschwitz and that his father had died of a heart attack in 1942. He visited Hungary, and spent time in England and the USA. But by 1954, he had problems with the USA which refused to issue an entrance visa for him, alleging communist sympathies. Eventually his entry was eased, after petitions from mathematicians, but all this made Erdős sceptical of nations and boundaries.

Erdős collaborated with 509 authors, nearly twice as many as the next most well connected mathematician. He collaborated so much that the most accepted measure of connectedness is the **Erdős number (EN):** the simple distance connecting a person with Erdős by co-authorship. Erdős himself has EN zero; the lucky 509 co-authors have EN one, and those who have co-written an article with one of this group has EN two (there are 6984 of them), and so on. [1]

If ever there was a mathematician who knew no boundaries, national or subject-wise, it was Paul Erdős, the ultimate problem solver and problem poser.

Further reading

- A. Baker, A. Bollobas and A. Hajnal, *A Tribute to Paul Erdős*, Cambridge University Press, 2012.
- Deborah Helligman, *The Boy Who Loved Math*: *The Improbable Life of Paul Erdős*, Roaring Brook Press, 2013.
- Paul Hoffman, *The man who loved only numbers*, Hyperion, 1999.
- Bruce Schechter, *My Brain is Open: The mathematical journey of Paul Erdős*, Simon and Schuster, 2000.

1. The author of this article is proud to be among the 26,422 who have an Erdős number of three.

R Ramanujam is a researcher in mathematical logic and theoretical computer science at the Institute of Mathematical Sciences, Chennai. He has an active interest in science and mathematics popularization and education, through his association with the Tamil Nadu Science Forum. He may be contacted at jam@imsc.res.in.

A well-known quote, and a favourite among mathematicians, is:

A mathematician is a machine for turning coffee into theorems

*This meta-theorem has been widely ascribed to **Paul Erdős**, but most likely it originated from another Hungarian mathematician, **Alfréd Rényi**, who was a long-time friend and colleague of Erdős's.*

*A third Hungarian, **Paul Turán**, added the following:*

Weak coffee is suitable only for lemmas.

feature

I dare to find a proof

Area of a Cyclic Quadrilateral

Brahmagupta's Theorem

A surprising but true fact: sometimes a 'low-tech' proof of a theorem is less well-known than the 'hi-tech' one. In this article we see an example of this phenomenon.

Sadagopan Rajesh

When students ask me a relevant question, I am reminded of this conversation between a mother and child. Child: "Mummy, why are some of your hairs turning grey?" Mother (trying to make best use of the question): "It is because of you, my dear. Every bad action of yours turns one of my hairs grey!" Child (innocently): "Now I know why grandmother has only grey hairs on her head!"

I try to answer relevant questions by students in an appropriate way. When students asked me for the proof of Heron's formula which they had found in their textbook (but without proof), I gave them a proof using concepts they know, similar to the one given in *At Right Angles* (Vol. 1, No. 1, June 2012, page 36), and suggested they look up some internet resources. Later, they told me that they had come across Brahmagupta's formula (for area of a cyclic quadrilateral), had noted its similarity to Heron's formula, but had found the proof used ideas from trigonometry. They asked whether the theorem can be proved using geometry and algebra. I took up the challenge and found such a proof. Here it is.

The theorem is due to the Indian mathematician Brahmagupta (598–670 A.D.) who lived in the central Indian province of Ujjain, serving as the head of the astronomical observatory located there. It was Brahmagupta who wrote the important and influential work *Brahmasphutasiddhānta*. (This is the first mathematical text to explicitly describe the arithmetic of negative numbers and of zero.)

Brahmagupta's formula gives the area of a cyclic quadrilateral (one whose vertices lie on a circle) in terms of its four sides.

Here is the statement of the theorem.

Theorem (Brahmagupta). *If $ABCD$ is a cyclic quadrilateral whose side lengths are a, b, c, d, then its area σ is given by $\sigma = \sqrt{(s-a)(s-b)(s-c)(s-d)}$ where $s = \frac{1}{2}(a+b+c+d)$ is the semi-perimeter of the quadrilateral.*

Note the neat symmetry of the formula. We shall prove it using familiar concepts in plane geometry such as: (i) properties of a circle (ii) properties of similar triangles (iii) Heron's formula for the area of a triangle, according to which the area of a triangle ABC with sides a, b, c is equal to $\sqrt{s(s-a)(s-b)(s-c)}$ where $s = \frac{1}{2}(a+b+c)$ is half the perimeter of the triangle.

Before offering a proof let us pass the formula through a 'check list' of simple tests.

- *Is the formula dimensionally correct?* Yes; the quantity within the square root is the product of four lengths, so the quantity $\sigma = \sqrt{(s-a)(s-b)(s-c)(s-d)}$ has the unit of area.
- *Is the formula symmetric in the four quantities a, b, c, d?* Yes. (It would be strange if the formula 'preferred' one quantity to another. An example of a formula which is dimensionally correct but not symmetric in a, b, c, d would be the following: $\sqrt{(s-\frac{1}{3}a)(s-\frac{1}{2}b)(s-\frac{1}{4}c)(s-d)}$.)
- *Does the formula give correct results when one side shrinks to zero?* Suppose that $d = 0$. This means that vertices A, D of the quadrilateral have collapsed into each other, and the figure is a triangle (with vertices A, B, C) rather than a quadrilateral. The Brahmagupta formula now reduces to $\sqrt{s(s-a)(s-b)(s-c)}$ which is simply Heron's formula for area of a triangle — a known result.

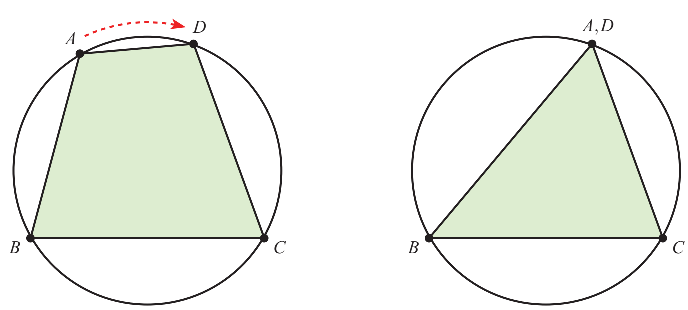

FIGURE 1. Here, vertices *A*, *D* have coalesced into each other (hence $d = 0$)

- *Does the formula yield the correct result for a rectangle, which is a special case of a cyclic quadrilateral?* It does: if the rectangle has dimensions $a \times b$, then $s = a + b$, and the formula yields $\sigma = \sqrt{b \cdot a \cdot b \cdot a} = ab$, which is correct.

We see that the formula has passed all the tests; this increases our confidence in it (but of course, these steps are not a substitute for a proof). It is in general a useful exercise to subject a formula to tests like these.

A final comment: the formula gives the area of a *cyclic* quadrilateral in terms of its sides. Implicitly such a formula makes the claim that if the sides of a quadrilateral are fixed, and we are told that the quadrilateral is cyclic, then its area gets fixed. This is so, and it can be proved. For a general quadrilateral there cannot be a formula for area only in terms of its four sides, for the simple reason that the four sides alone cannot fix the quadrilateral. (To see why, think of the quadrilateral as made of four jointed rods having the given lengths. Such a shape is obviously not rigid, so the area is not fixed.)

Proof of the formula

Let $ABCD$ be a cyclic quadrilateral. Since we know that the Brahmagupta formula works for rectangles, there is nothing lost by assuming that $ABCD$ is not a rectangle. In this case at least one pair of opposite sides of the quadrilateral are not parallel to each other. We shall suppose that AD is not parallel to BC, and that lines AD and BC meet when extended at point P as shown in Figure 2. (Under the assumption that AD is not parallel to BC, this will be the case if $AB < CD$. If $AB > CD$, then AD and BC will meet on the 'other' side of the quadrilateral. The third possibility, that $AB = CD$, cannot happen since we have assumed that AD and BC are not parallel to each other.)

Elementary circle geometry shows that $\triangle PAB \sim \triangle PCD$; for we have $\measuredangle PAB = \measuredangle PCD$ and $\measuredangle PBA = \measuredangle PDC$; and the angle at P is shared by the two triangles. Let a, b, c, d be the lengths of AB, BC, CD, DA; let u, v be the lengths of PA, PB; and let e, f be the lengths of the diagonals AC, BD respectively (see Figure 2). Our strategy will now be the following:

Step 1: Find u, v in terms of a, b, c, d, using the similarity $\triangle PAB \sim \triangle PCD$.

Step 2: Find the area of $\triangle PAB$ using Heron's formula.

Step 3: Find the area of $\triangle PCD$, once again using the similarity $\triangle PAB \sim \triangle PCD$.

Step 4: Find the area of the quadrilateral, by subtraction.

Sounds simple, doesn't it? Here's how we execute the steps.

Steps 1 & 2: Let the coefficient of similarity in the similarity $\triangle PAB \sim \triangle PCD$ be k. Since the sides of $\triangle PAB$ are u, v, a, while the corresponding sides of $\triangle PCD$ are $v + b, u + d, c$, we have:

$$v + b = ku, \qquad u + d = kv, \qquad c = ka. \tag{1}$$

Hence we have:

$$k = \frac{c}{a}, \qquad u - v = \frac{b - d}{k + 1} = \frac{a(b - d)}{c + a}, \qquad u + v = \frac{b + d}{k - 1} = \frac{a(b + d)}{c - a}. \tag{2}$$

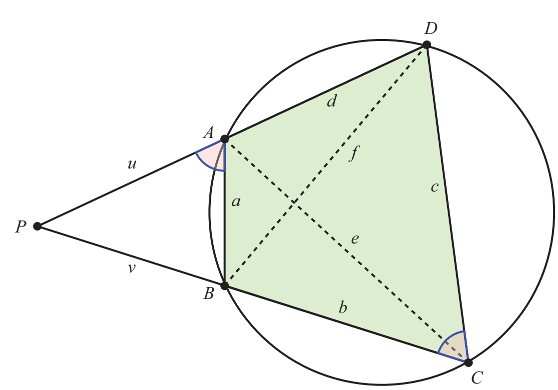

FIGURE 2.

Therefore the semi-perimeter s' of $\triangle PAB$ is given by:

$$2s' = a + u + v = a + \frac{a(b + d)}{c - a} = \frac{a(b + c + d - a)}{c - a}. \tag{3}$$

Hence we have the following relationship between s' and the semi-perimeter s of quadrilateral $ABCD$:

$$s' = \frac{a(s-a)}{c-a}. \tag{4}$$

To compute the area of $\triangle PAB$ we need expressions for the following:

$$2s' - 2a = u + v - a = \frac{a(b+d)}{c-a} - a = \frac{a(d+a+b-c)}{c-a} = \frac{2a(s-c)}{c-a},$$

$$2s' - 2u = a + v - u = a - \frac{a(b-d)}{c+a} = \frac{a(c+d+a-b)}{c+a} = \frac{2a(s-b)}{c+a},$$

$$2s' - 2v = a + u - v = a + \frac{a(b-d)}{c+a} = \frac{a(a+b+c-d)}{c+a} = \frac{2a(s-d)}{c+a}.$$

Hence the area Δ' of $\triangle PAB$ is given by

$$\Delta' = \sqrt{s'(s'-a)(s'-u)(s'-v)} = \sqrt{\frac{a(s-a)}{c-a}\,\frac{a(s-c)}{c-a}\,\frac{a(s-b)}{c+a}\,\frac{a(s-d)}{c+a}}. \tag{5}$$

This simplifies to:

$$\Delta' = \frac{a^2}{c^2-a^2}\sqrt{(s-a)(s-b)(s-c)(s-d)}. \tag{6}$$

We have now found the area of $\triangle PAB$.

Step 3: The scale factor in the similarity $\triangle PAB \sim \triangle PDC$ is $k = c/a$. So the area Δ'' of $\triangle PCD$ is $k^2 = c^2/a^2$ times the above expression; that is,

$$\Delta'' = \frac{c^2}{c^2-a^2}\sqrt{(s-a)(s-b)(s-c)(s-d)}. \tag{7}$$

Step 4: The area σ of quadrilateral $ABCD$ is equal to $\Delta'' - \Delta'$. This simplifies to:

$$\sigma = \sqrt{(s-a)(s-b)(s-c)(s-d)}, \tag{8}$$

and we have proved Brahmagupta's formula.

Exercise. Derive the following formulas for u and v:

$$u = \frac{a(ad+bc)}{c^2-a^2}, \qquad v = \frac{a(ab+cd)}{c^2-a^2}.$$

References

[1] http://www-history.mcs.st-andrews.ac.uk/Biographies/Brahmagupta.html
[2] http://en.wikipedia.org/wiki/Brahmagupta
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Set theory revisited

As easy as PIE

Principle of Inclusion and Exclusion – Part 2

In Part–I of this article we solved some problems using the PIE or the 'Principle of Inclusion and Exclusion'. We saw how the law $|A \cup B| = |A| + |B| - |A \cap B|$ *generalizes, and we used the PIE to find a formula for Euler's totient function* $\varphi(N)$ *which counts the number of integers in the set* $\{1, 2, \ldots, N\}$ *which are coprime to* N*. Now in Part–II we use the PIE to find a generalization of the formula connecting the* gcd *and* lcm *of two numbers. We also discuss a problem about a secretary who loves mixing up job offers sent to applicants, and another problem concerning placement of rooks on a chessboard.*

feature

B Sury

I. The Möbius function

You would have noticed in the first part of this article (PIE-I) that the same kind of sum has been coming up repeatedly, in which terms are alternately positive and negative. A convenient way of writing such sums is through the use of a function called the *Möbius function*, written $\mu(n)$ and read aloud as 'mew of n'. It is defined as follows: $\mu(1) = 1$, and:

- If n is the product of unequal prime numbers, then $\mu(n) = 1$ if the number of primes is even, and $\mu(n) = -1$ if the number of primes is odd. So $\mu(p) = -1$ for any prime p; $\mu(pq) = 1$ for any two unequal primes p, q; and so on. Here is a more compact way of writing this: if n is the product of r distinct primes, then $\mu(n) = (-1)^r$. Examples: $\mu(5) = -1$, $\mu(10) = 1$, $\mu(30) = -1$.
- If n is divisible by the square of any prime number, then $\mu(n) = 0$. Examples: $\mu(4) = 0$, $\mu(12) = 0$.

Using this function, an expression such as

$$N - \left(\frac{N}{p} + \frac{N}{q} + \frac{N}{r} + \cdots\right) + \left(\frac{N}{pq} + \frac{N}{qr} + \frac{N}{pr} + \cdots\right) - \cdots$$

can be written compactly as

$$\sum_{d|N} \mu(d) \frac{N}{d}.$$

Hence, we have:

$$\varphi(N) = \sum_{d|N} \mu(d) \frac{N}{d}. \qquad (1)$$

Incidentally, the name Möbius is popularly known in another context — the so-called *Möbius strip*, which will be a topic for another day.

The Möbius function has numerous nice properties which make it a very useful function in number theory and combinatorics.

II. Relation between GCD and LCM of several numbers

To demonstrate how unexpectedly useful the PIE formula can be, we describe a nice application of the formula. Here is the context. We all know the pleasing formula that relates gcd ('greatest common divisor', also known as 'highest common factor') and lcm ('lowest common multiple') of any two positive integers a and b:

$$\gcd(a,b) \times \operatorname{lcm}(a,b) = ab. \qquad (2)$$

This formula relates the gcd and lcm of two integers a, b. Here is the corresponding formula for the case of three integers. If a, b, c be any three positive integers, then:

$$\operatorname{lcm}(a,b,c) = \frac{abc \times \gcd(a,b,c)}{\gcd(a,b) \times \gcd(b,c) \times \gcd(a,c)}. \qquad (3)$$

For the general case we need the following result which is actually the PIE in another incarnation (though it may not look like it):

Theorem (PIE′). *If $n_1, n_2, \cdots, n_r$ is a finite sequence of positive integers, then*

$$\begin{aligned}\max(n_1, \cdots, n_r) &= \sum_i n_i - \sum_{i<j} \min(n_i, n_j) \\ &\quad + \sum_{i<j<k} \min(n_i, n_j, n_k) \\ &\quad - \cdots + (-1)^{r-1} \min(n_1, \cdots, n_r). \qquad (4)\end{aligned}$$

Here, 'max' and 'min' stand for maximum and minimum respectively. The symbol $\sum_{i<j}$ means: 'the sum over all pairs of indices i, j where $i < j$. Similarly for the symbol $\sum_{i<j<k}$ and others like it. The formula may look mysterious, so it will help if we examine it more closely.

- Take the case of two positive integers a, b. Then the claim is that

  $$\max(a,b) = a + b - \min(a,b).$$

  This is clearly true.

- Take the case of three positive integers a, b, c. Then the claim is that

  $$\begin{aligned}\max(a,b,c) = a + b + c &- \min(a,b) \\ &- \min(a,c) - \min(b,c) \\ &+ \min(a,b,c).\end{aligned}$$

  To see why this is true, suppose (there is no loss of generality in assuming this) that $a \le b \le c$. The above claim then reduces to the following:

  $$c = a + b + c - (a + a + b) + a,$$

  which is clearly true.

- Take the case of four positive integers a, b, c, d where (without any loss of generality, as earlier) $a \le b \le c \le d$. Then the claim reduces to the following claim:

  $$\begin{aligned}d &= a + b + c + d \\ &\quad - (a + a + a + b + b + c) \\ &\quad + (a + a + a + b) - a,\end{aligned}$$

  which is clearly true. The general case may be similarly reasoned out and is left as an exercise.

To convince ourselves that the above can indeed be useful in unexpected ways, let us look at a set

$a_1, a_2, \cdots, a_r$ of positive integers. Then we will show the following:

$$\operatorname{lcm}(a_1, \cdots, a_r) = \frac{\left(\prod_i a_i\right)\left(\prod_{i<j<k} \gcd(a_i, a_j, a_k)\right)\cdots}{\left(\prod_{i<j} \gcd(a_i, a_j)\right)\left(\prod_{i<j<k<l} \gcd(a_i, a_j, a_k, a_l)\right)\cdots}. \tag{5}$$

This will be deduced from statement (4) about maxima and minima. To see the connection, consider the prime numbers dividing the a_i's. Then, clearly: *the exponent of a prime p dividing the* gcd *of a collection of numbers is equal to the minimum of the exponents of p dividing the numbers, and the exponent of a prime p dividing the* lcm *of a collection of numbers is equal to the maximum of the exponents of p dividing the numbers*.

Thus, if $p^{n_1}, \cdots, p^{n_r}$ are the powers of a fixed prime p dividing the numbers $a_1, \cdots, a_r$, then the gcd of the a_i's is exactly divisible by $p^{\min(n_1,\cdots,n_r)}$, and the lcm of the a_i's is exactly divisible by $p^{\max(n_1,\cdots,n_r)}$. Let us use the short form $\operatorname{Ord}_p(N)$ for the largest integer e such that p^e divides N. Then if we raise p to each of the terms of the equality

$$\begin{aligned}\max(n_1, \cdots, n_r) &= \sum_i n_i - \sum_{i<j} \min(n_i, n_j) \\ &+ \sum_{i<j<k} \min(n_i, n_j, n_k) \\ &- \cdots + (-1)^{r-1} \min(n_1, \cdots, n_r),\end{aligned}$$

(to see why, you need to use repeatedly the fact that $p^{a+b} = p^a \times p^b$ and $p^{a-b} = p^a \div p^b$), we obtain

$$\operatorname{Ord}_p(\operatorname{lcm}(a_1, \cdots, a_r)) = \operatorname{Ord}_p\left(\frac{\left(\prod_i a_i\right)\left(\prod_{i<j<k} \gcd(a_i, a_j, a_k)\right)\cdots}{\left(\prod_{i<j} \gcd(a_i, a_j)\right)\left(\prod_{i<j<k<l} \gcd(a_i, a_j, a_k, a_l)\right)\cdots}\right).$$

We have obtained expression (5) for the lcm of the a_i's.

III. The secret(ary) adversary

Here is another well-known problem concerning a particularly careless (or perhaps mischievous) secretary. The scenario is that a rich person writes a letter each to Alka, Beena, Chanda and Deepa offering different financial scholarships to each, but the secretary puts each letter in a wrongly addressed envelope. The financier is naturally cross and asks the secretary to correct his mistake. However, the secretary *again* puts each letter in a wrong envelope! How many ways can he make such a mistake? A bit of counting (which we leave as an exercise for you) shows that the number is 9.

What is the best way to figure out this number if there are n people and n envelopes (and each letter must go to the wrong person)? Once again, the PIE comes to the rescue. The total number of ways of distributing n letters among n persons (one letter to each person) is of course $n!$. Let N_1 be the number of ways of distributing the letters so that at least one person (it could be any of the n persons) gets his or her correct letter; let N_2 be the number of ways of distributing the letters so that at least two persons get their correct letters; let N_3 be the number of ways of distributing the letters so that at least three persons get their correct letters; and similarly for N_4, N_5, ... (Note that by this notation we could say that $N_0 = n!$.) Then the PIE tells us that the number of ways of distributing the letters so that no one gets their letter is

$$N_0 - N_1 + N_2 - N_3 + N_4 - + \cdots + (-1)^n N_n.$$

Computing N_1, N_2, ...is easy. Suppose that at least r people receive their correct letters. Let us look at a *fixed* set of r people. For the remaining $n - r$ persons no restriction has been placed, so the number of ways of distributing the letters is $(n-r)!$. This is so for each fixed set of r persons, and there are $\binom{n}{r}$ such sets; hence $N_r = \binom{n}{r} \times (n-r)!$. It follows that the number of possibilities in which when *no one* receives their correct letter is

$$n! - \binom{n}{1}(n-1)! + \binom{n}{2}(n-2)! - \binom{n}{3}(n-3)! + \cdots + (-1)^n \binom{n}{n} 0! = n! \sum_{r=0}^{n} \frac{(-1)^r}{r!}$$

This is called the *derangement number* and it is denoted by D_n; so $D_n = n! \sum_{r=0}^{n} (-1)^r / r!$.

Here are the values of the first few such numbers:

n	1	2	3	4	5	6	$\cdots$
D_n	0	1	2	9	44	265	$\cdots$

IV. Chess-'bored' Rooks?

The next example we mention is to do with a chess board. We know that there are 8! ways of placing 8 rooks on a chess board such that no two attack each other. This is because on the top row, one can place a rook on any one of the 8 places; the second rook can be placed on the second row on any one of the 7 columns other than the column containing the first one. Then, the third rook can be placed on the third row on any of the 6 columns not containing either of the two rooks, etc.

Now, what if we fix a subset T of the n^2 squares in a $n \times n$ chess board where the rooks do not like to sit (let us say these seats are 'boring')? That is, we place n mutually non-attacking rooks on the chess board such that none of the rooks are on the T-squares. How many ways can this be done? (Of course, this will depend on T.)

Let us look at an example where the chessboard has size 4×4. Denote the 16 squares by ordered pairs (i, j) where $1 \le i, j \le 4$. Suppose $T = \{(1,1), (2,2), (3,3), (3,4), (4,4)\}$. Counting carefully gives us 6 possibilities of placing 4 non-attacking rooks with no rook on any of the 5 T-squares. Indeed, the possible arrangements are these:

$$(1,2), (2,4), (3,1), (4,3)$$

$$(1,3), (2,4), (3,1), (4,2)$$

$$(1,4), (2,3), (3,1), (4,2)$$

$$(1,4), (2,1), (3,2), (4,3)$$

$$(1,3), (2,4), (3,2), (4,1)$$

$$(1,4), (2,3), (3,2), (4,1)$$

In general, let us look at an $n \times n$ chessboard and a fixed subset T of squares. Let T_r denote the number of ways of placing r non-attacking rooks on T. Then, by the PIE, the number N of ways of placing n mutually non-attacking rooks such that none of them lies on a T-square is given as

$$N = n! - (n-1)!\,T_1 + (n-2)!\,T_2 - \cdots + (-1)^n T_n.$$

The proof of this is left to the reader as an exercise.

V. Deep waters

Finally, we draw attention to some connections of the Möbius function with prime numbers at a basic but deep level. One of the great discoveries of the great Carl Friedrich Gauss is a prediction known as the *prime number theorem*. At the ripe old age of 15 (in 1792), Gauss conjectured that the number $\pi(x)$ of prime numbers not exceeding a given number x is 'asymptotic' to $x/\log(x)$. By 'asymptotic', one means here that the ratio $\pi(x) \div x/\ln x$ gets arbitrarily close to 1 as x gets arbitrarily large.

More precisely, he predicted that $\pi(x)$ is asymptotic to the following integral:

$$\int_1^x \frac{dt}{\ln t}.$$

This most amazing statement became a theorem only a century later when it was proved simultaneously and independently by Hadamard and by de la Valle Poussin. The remarkable fact is that this theorem is equivalent to the simply-stated assertion that

$$\frac{1}{x}\sum_{n \le x} \mu(n) \to 0 \quad \text{as } x \to \infty.$$

Of course, this is only neat as a statement. Proving it is just as difficult as proving the prime number theorem!

At this point of time, is there any single problem in mathematics which could be held as a show-piece in that it embodies the most difficult of open problems in mathematics? If such a thing is at all admissible, the winner would certainly be the so-called *Riemann hypothesis* stated by Gauss's student, the great Bernhard Riemann (1800–1840). We do not state it here as it is not easy to do so in simple terms. However, the equivalent statement in terms of the Möbius function is the following:

Conjecture. For any constant $t > 1/2$, there exists a constant $C > 0$ such that

$$\sum_{n \le x} \mu(n) \le Cx^t \quad \text{for all } x > 0.$$

But I would not advise readers to try proving this!

Exercises

(1) Let n be any positive integer exceeding 1. Show that the sum $\mu(d)$ over all the divisors d of n equals 0.

Example: Take $n = 6$. Its divisors are $1, 2, 3, 6$, and their μ-values are $1, -1, -1, 1$, whose sum is 0.

(2) Let n be any positive integer exceeding 1. Show that the sum $|\mu(d)|$ over all the divisors d of n equals 2^k where k is the number of distinct prime divisors of n.

Example: Take $n = 6$. Its divisors are $1, 2, 3, 6$, and their $|\mu|$-values are $1, 1, 1, 1$, whose sum is 4. The number of distinct prime divisors of 6 is 2, and $2^2 = 4$.

(3) In proving that

$$\begin{aligned}\max(a, b, c) = a + b + c &- \min(a, b)\\ &- \min(a, c) - \min(b, c)\\ &+ \min(a, b, c),\end{aligned}$$

we said: "there is no loss of generality in assuming that $a \le b \le c$". Why is there 'no loss of generality' in assuming that $a \le b \le c$?

(4) Try proving the general relation (4). (It is not as difficult as it looks!)

Further reading

i. V Balakrishnan, *Combinatorics: Including Concepts Of Graph Theory* (Schaum Series)

ii. I Niven, H S Zuckerman & H L Montgomery, *An Introduction to the Theory of Numbers* (John Wiley, Fifth Edition)

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feature

PHTs ...Primitive and beautiful

Harmonic Triples

Part–2

Read how the same simple relationship connects the side and diagonals of a regular heptagon and prove this using the triple angle identities from trigonometry. Then prove the same result using the little known Ptolemy's theorem. And finally, learn how to generate these lesser known triads — Primitive Harmonic Triples.

Shailesh Shirali

In Part I of this article we introduced the notion of a *primitive harmonic triple* ('PHT') as a triple (a, b, c) of coprime positive integers satisfying the equation

$$\frac{1}{a} + \frac{1}{b} = \frac{1}{c}.$$

Examples: $(3, 6, 2)$ and $(6, 30, 5)$. We had listed various geometric and physical contexts in which this equation surfaces. We had also mentioned that the equation arises in connection with the diagonals of a regular 7-sided polygon. We start by studying this problem.

Diagonals of a regular heptagon

Given a regular heptagon, one can draw $\binom{7}{2} = 21$ different segments connecting pairs of its vertices. But these 21 segments come in just three different lengths: its diagonals come in two different lengths, and then there is the side of the heptagon.

Let a and b be the lengths of the longer diagonal and the shorter diagonal (respectively), and let c be the side of the heptagon, so $a > b > c$ (see Figure 1); then the claim is that $1/a + 1/b = 1/c$. We provide two proofs for this claim.

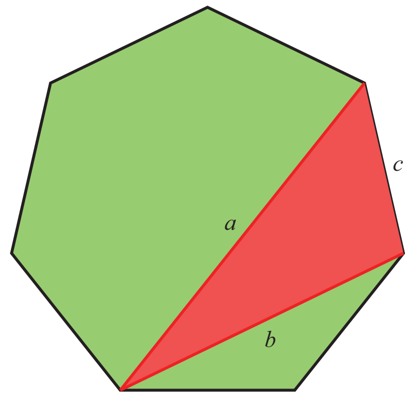

FIGURE 1. Regular heptagon and inscribed triangle; claim: $1/a + 1/b = 1/c$

A trigonometric proof. The angle subtended by each side of a regular heptagon at the centre of the circumscribing circle is $360^\circ/7$. It follows that in the shaded triangle shown in Figure 1, with sides a, b, c, the angles are (respectively) $720^\circ/7$, $360^\circ/7$ and $180^\circ/7$. For convenience let us denote $180^\circ/7$ by θ; then the angles of the triangle are $4\theta, 2\theta, \theta$, and the lengths of the sides opposite these angles are, by the sine rule, proportional to $\sin 4\theta, \sin 2\theta, \sin\theta$ respectively. So the claim that $1/a + 1/b = 1/c$ is equivalent to the claim that if $\theta = 180^\circ/7$, then:

$$\frac{1}{\sin 4\theta} + \frac{1}{\sin 2\theta} = \frac{1}{\sin\theta}, \tag{1}$$

and this is what we now establish. Using the well-known double- and triple-angle identities we rewrite the identity in various equivalent forms:

$$\begin{aligned}
\frac{1}{\sin 4\theta} + \frac{1}{\sin 2\theta} = \frac{1}{\sin\theta} \quad &\Leftrightarrow \quad 1 + \frac{\sin 4\theta}{\sin 2\theta} = \frac{\sin 4\theta}{\sin\theta} \\
&\Leftrightarrow \quad 1 + 2\cos 2\theta = 4\cos\theta\left(2\cos^2\theta - 1\right) \\
&\Leftrightarrow \quad 1 + 2\left(2\cos^2\theta - 1\right) = 4\cos\theta\left(2\cos^2\theta - 1\right) \\
&\Leftrightarrow \quad 8\cos^3\theta - 4\cos^2\theta - 4\cos\theta + 1 = 0.
\end{aligned}$$

Hence the relation $1/a + 1/b = 1/c$ is equivalent to the following: if $\theta = 180^\circ/7$, then

$$8\cos^3\theta - 4\cos^2\theta - 4\cos\theta + 1 = 0. \tag{2}$$

So if we prove (2) we also prove (1). To prove (2) we note that since $7\theta = 180^\circ$, we have the relation $3\theta = 180^\circ - 4\theta$, and therefore $\sin 3\theta = \sin 4\theta$. This yields, via the double- and triple-angle identities:

$$\begin{aligned}
&3\sin\theta - 4\sin^3\theta = 2\sin 2\theta\cos 2\theta = 4\sin\theta\cos\theta\left(2\cos^2\theta - 1\right), \\
&\therefore 3 - 4\sin^2\theta = 4\cos\theta\left(2\cos^2\theta - 1\right) \qquad [\text{since } \sin\theta \neq 0], \\
&\therefore 3 - 4\left(1 - \cos^2\theta\right) = 4\cos\theta\left(2\cos^2\theta - 1\right), \\
&\therefore 8\cos^3\theta - 4\cos^2\theta - 4\cos\theta + 1 = 0.
\end{aligned}$$

Thus (2) is established, and hence (1).

A proof using Ptolemy's theorem. There is an elegant proof of the above equality using 'pure geometry', but it requires the use of a theorem which is not so well known at the high school level (though it ought to be better known, as it is such a nice and useful result). The theorem is due to Ptolemy. Here is its statement: *If PQRS is a cyclic quadrilateral, then its sides obey the following equality:* $PQ \cdot RS + PS \cdot QR = PR \cdot QS$. That is, the sum of the products of pairs of opposite sides equals the product of the diagonals. Ptolemy's theorem can be applied in many kinds of settings and yields many nice results. (Of course, we first need to identify a suitable cyclic quadrilateral.) (*Editor's note:* Ptolemy's theorem will be taken up in the 'Geometry corner' of a subsequent issue of *At Right Angles*.)

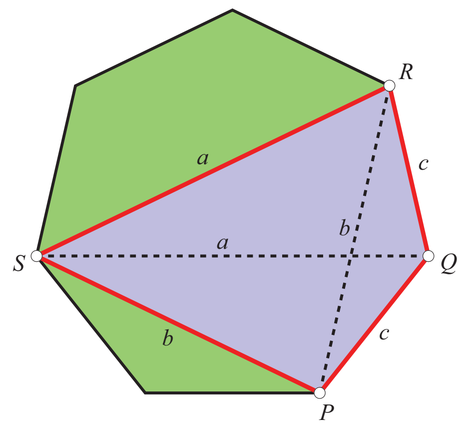

FIGURE 2. Regular heptagon and a cyclic quadrilateral inscribed in it

Here, we simply apply Ptolemy's theorem to cyclic quadrilateral $PQRS$ whose vertices P, Q, R, S are chosen as shown in Figure 2. (The quadrilateral is cyclic since any regular polygon is cyclic, and P, Q, R, S are vertices of a regular heptagon.) Note that PQ and QR are sides of the heptagon (both have length c), RS is a 'long' diagonal (length a), SP is a 'short' diagonal (length b), and its diagonals QS and PR have lengths a and b respectively. Hence by Ptolemy's theorem:

$$bc + ac = ab.$$

Dividing through by abc, we get $1/a + 1/b = 1/c$, as claimed. (Simple, no? But it did require spotting a suitable quadrilateral ....)

Generation of PHTs

Now we take up the question of how to generate primitive harmonic triples in a systematic and mathematically 'nice' way. (So we avoid 'brute force enumeration'.) To avoid listing the same solution more than once (i.e., listing both (a, b, c) and (b, a, c); clearly if one of these is harmonic, so is the other), we shall assume right through that $a \le b$.

At the start we draw attention to a feature about PHTs which makes them different from PPTs. In the case of a Pythagorean triple it is easy to show that if two numbers of the triple are multiples of some number k, then so must be the third number of the triple; e.g., consider the triple $(6, 8, 10)$. Strangely, this property does not hold for harmonic triples! A nice example is the harmonic triple $(10, 15, 6)$; here, each pair of numbers shares a factor exceeding 1, but this factor fails to divide the third number in the triple. The same is true for the PHT $(21, 28, 12)$. (Nevertheless we call such triples 'primitive', because there is no factor common to all the three numbers.)

Systematic generation of harmonic triples. As with PPTs, there are many ways in which we can track down the full family of PHTs. We use an approach based on factorization.

We first clear fractions and get the relation $c(a + b) = ab$. We write this as:

$$ab - ac - bc = 0. \qquad (3)$$

If we try to factorize $ab - ac - bc$ we find there is a 'term missing': the expression is 'almost' equal to $(a-c)(b-c)$ but not quite. So we put it in the missing term (which is clearly c^2) and write $ab - ac - bc + c^2 = c^2$. Factorizing this we get:

$$(a-c)(b-c) = c^2. \tag{4}$$

From this we see that $a - c$ and $b - c$ are a pair of *complementary factors* of c^2. (Two factors of a number are called 'complementary factors' if their product equals that number; e.g., 2 and 5 are complementary factors of 10.) Right away we get a method of generating solutions to the harmonic equation — the **'method of complementary factors'**. We express it algorithmically as follows.

(i) Select any positive integer c.

(ii) Write c^2 as a product $u \times v$ of two positive integers, with $u \le v$.

(iii) Let $a = c + u$ and $b = c + v$.

(iv) Then (a, b, c) is a harmonic triple in which $a \le b$. To check that it is harmonic:

$$\begin{aligned}\frac{1}{a} + \frac{1}{b} &= \frac{1}{c+u} + \frac{1}{c+v} = \frac{1}{c+u} + \frac{1}{c + c^2/u}\\ &= \frac{1}{c+u} + \frac{u}{c(c+u)} = \frac{c}{c(c+u)} + \frac{u}{c(c+u)} = \frac{c+u}{c(c+u)} = \frac{1}{c}.\end{aligned}$$

The triple may not be primitive, so we must work out how to ensure this. But it is clear that by selecting all possible values of c, and by factorizing c^2 is all possible ways, we will get all possible harmonic triples. Here are two worked examples.

- Let $c = 6$; then $c^2 = 36$. Choose the factorization $c^2 = 2 \times 18$. This yields $a = 6 + 2 = 8$ and $b = 6 + 18 = 24$, and we get the harmonic triple $(8, 24, 6)$. Note that it is not primitive.
- We again let $c = 6$, but change the factorization to $c^2 = 4 \times 9$. Now we get $a = 6 + 4 = 10$ and $b = 6 + 9 = 15$, and we get the harmonic triple $(10, 15, 6)$, which is primitive.
- Let $c = 10$, and choose the factorization $c^2 = 2 \times 50$. This yields $a = 10 + 2 = 12$ and $b = 10 + 50 = 60$. We get the triple $(12, 60, 10)$. Note that it is not primitive.
- Let $c = 10$, and choose the factorization $c^2 = 4 \times 25$. This yields $a = 10 + 4 = 14$ and $b = 10 + 25 = 35$. We get the triple $(14, 35, 10)$, which is primitive.

It appears that *for (a, b, c) to be primitive, we must choose the factorization $c^2 = u \times v$ such that u and v are coprime*. This is so and we take up the proof in Part III of this series. (We pose this as a problem for you, below.) Table 1 gives a list of a few primitive harmonic triples generated this way.

$(2, 2, 1)$,	$(3, 6, 2)$,	$(4, 12, 3)$,	$(5, 20, 4)$,
$(6, 30, 5)$,	$(7, 42, 6)$,	$(8, 56, 7)$,	$(9, 72, 8)$,
$(10, 15, 6)$,	$(10, 90, 9)$,	$(14, 35, 10)$,	$(18, 63, 14)$,
$(21, 28, 12)$,	$(22, 99, 18)$,	$(24, 40, 15)$,	$(30, 70, 21)$,
$(33, 88, 24)$,	$(36, 45, 20)$,	$(44, 77, 28)$,	$(55, 66, 30)$,
$(60, 84, 35)$,	$(65, 104, 40)$,	$(78, 91, 42)$,	$(105, 120, 56)$.

TABLE 1. Some PHTs

Questions to ponder

(1) We stated above that for (a, b, c) to be primitive, we must choose the factorization $c^2 = uv$ in such a way that u and v are coprime. Why should this be so?

(2) In Table 1, note the triples $(2, 2, 1)$, $(3, 6, 2)$, $(5, 20, 4)$, $(6, 30, 5)$, ... Each of these has the same form. Find a formula that generates these PHTs, and show that each such triple is primitive.

(3) Add to the list of triples to Table 1 and study the table carefully. Try to find some interesting features that the triples share. (In Part III — the concluding part — of this series we will explore some properties of PHTs.)

(4) Some PHTs can be 'realized' as triangles; for example, there exist triangles with sides $2, 2, 1$ and $10, 15, 6$ respectively. On the other hand there does not exist a triangle with sides $6, 30, 5$; nor does there exist a triangle with sides $5, 20, 4$. (Reason: Each of these violates the triangle inequality.) What extra condition is needed in the factorization method which will yield a PHT that can be realized as a triangle?

(5) On examining the primitive harmonic triples in Table 1, we notice that the following steps sometimes yield a triple which is harmonic:

   (i) Choose any two integers a and c, with $a > c$.

   (ii) Let $g = \gcd(a, c)$, and let $b = ac/g^2$.

   (iii) Then the triple (a, b, c) is sometimes harmonic.

   For example, take $a = 14$, $c = 10$; then $g = \gcd(14, 10) = 2$, so $b = 14 \times 10/4 = 35$. We may verify that the triple $(14, 35, 10)$ is harmonic (indeed, it is a PHT).

   On the other hand, take $a = 14$, $c = 12$; then $g = 2$, so $b = 14 \times 12/4 = 42$. But the triple $(14, 42, 12)$ is not harmonic.

   When do these steps yield a harmonic triple? Obtain a complete answer.

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In October 2012, the Nobel Prize in Economics was awarded to Prof Alvin E. Roth and Prof Lloyd S. Shapley "for the theory of stable allocations and the practice of market design". The official citation reads in part as follows (from http://www.nobelprize.org/nobel_prizes/economics/laureates/2012/press.html):

Stable Allocations – from Theory to Practice

This year's Prize concerns a central economic problem: how to match different agents as well as possible. For example, students have to be matched with schools, and donors of human organs with patients in need of a transplant. How can such matching be accomplished as efficiently as possible? What methods are beneficial to what groups? The prize rewards two scholars who have answered these questions on a journey from abstract theory on stable allocations to practical design of market institutions.

Lloyd Shapley used so-called cooperative game theory to study and compare different matching methods. A key issue is to ensure that a matching is stable in the sense that two agents cannot be found who would prefer each other over their current counterparts. Shapley and his colleagues derived specific methods [the Gale-Shapley algorithm] that ensure a stable matching.

Alvin Roth recognized that Shapley's theoretical results could clarify the functioning of important markets in practice. He and his colleagues demonstrated that stability is the key to understanding the success of [some] market institutions. Roth was later able to substantiate this conclusion in systematic laboratory experiments. He helped redesign existing institutions for matching new doctors with hospitals, students with schools, and organ donors with patients. These reforms are all based on the Gale-Shapley algorithm, along with modifications that take into account specific circumstances and ethical restrictions.

[The] combination of Shapley's basic theory and Roth's experiments and design has generated a flourishing field of research. This year's prize is awarded for an outstanding example of economic engineering.

It may not be obvious where the mathematics in this work lies. The theory to which the Nobel citation refers comes from a paper Shapley wrote in 1962 with Prof David Gale, College Admissions and the Stability of Marriage. It deals with the problem of a college "having to decide how many and which applicants to admit to most nearly achieve a desired quota." The authors define what it means for an assignment of applicants to colleges to be 'unstable' and what it means for a stable assignment to be 'optimal'. Then they consider a special case in which there are as many applicants as colleges, and all quotas are unity. This quickly brings to mind the marriage scenario, in which brides and grooms must be matched. They prove that, no matter how the members of a community comprising n men and n women rank potential spouses, a stable matching does exist. Their proof gives an iterative procedure for finding such a stable matching.

At the end of their paper the two authors write that they have "abandoned reality altogether and entered the world of mathematical make-believe." But they add, "It is our opinion that some ideas introduced here might usefully be applied to certain phases of the admissions problem." Over the decades it has become clear that this conclusion is a huge understatement, for the Shapley-Gale algorithm has been applied to many contexts since then, just as the Nobel citation states. Thus the work may be regarded as yet another illustration of "the unreasonable effectiveness of mathematics".

The following article by Devang S Ram Mohan is a whimsical and simplified look at the work of Shapley, Gale and Roth, loosely based on an expository talk given by Prof Manjunath Krishnapur of the Department of Mathematics, Indian Institute of Science, Bangalore. Some of you may not be happy at the use of marriage as a context to discuss a mathematical concept! But we ask you to bear with us and read on ….

feature

Mathematical make-believe?

Mat(c)h made in Heaven

Devang S Ram Mohan

The Nobel Prize presentation ceremony recently concluded in Oslo, Norway, and like most others, I only managed to read the first two paragraphs of any article on the achievements of these men and women. Being a student of mathematics, and due to the absence of a Nobel prize in that field (rumoured to be due to a disagreement between Alfred Nobel and mathematician Mittag Leffler), I find myself drowning in the technical jargon present in all such write ups. I was thus circumspect when I saw a notice announcing a talk – requiring no prior knowledge of the subject – on the Nobel Prize winning work of Alvin Roth (Economist) and Lloyd Shapley (Mathematician/ Economist).

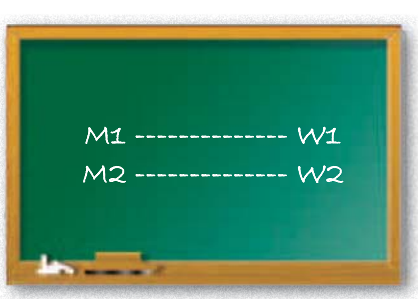
Walking into the packed hall, I went to the back of the room and seated myself so as to be able to make a quiet exit in case I disagreed with the notice on what “no prior knowledge required” meant. In walked our speaker for the day (henceforth referred to as Professor), looking pleased at the large turnout. Setting his notes down on the table he addressed the crowd of eager faces.

"As I mentioned in my mail inviting you all here, I am not an expert on this subject. In fact, this talk is more to celebrate the unprecedented event that I actually understood the work of some economists and I would like to share my excitement with you. Feel free to ask me as many questions as you wish and in turn, allow me the freedom to not know the answer at times!"

I chuckled quietly to myself, pleased at the informal beginning to the proceedings.

"So," he began again, "today we're going to discuss the work of Roth and Shapley. They were just recently awarded the Nobel Prize in Economics for their work on the matching problem. The basic problem, initially worked on by Shapley (along with a mathematician by the name of David Gale) is as follows. Suppose you have n men and n women in a room, each man has a list which rates each of the n women according to who he likes more and similarly, each woman has a corresponding list of men. Suppose a marriage consists of pairing up a man and a woman, i.e., to each man a unique woman is associated and vice versa. For those of you with a mathematics background, a bijection (one to one, onto correspondence) is set up between the men and the women. You are allowed to divorce your spouse if you prefer the husband (correspondingly wife) of another person to your current partner and that person also prefers you to their wife (correspondingly husband)."

He probably sensed our brains furiously trying to wrap itself around the idea, because he soon picked up a chalk and wrote on the board:



"M1 can divorce W1 only if W2 is higher than W1 on his preference list AND W2 prefers him to M2. Now, the question is, is there a marriage arrangement such that all n men and women are matched, and no one wants (or in this case, is permitted) a divorce!"

"Such an arrangement IS possible, and not just that, there is an algorithm by which you can get this 'stable' arrangement, but we'll come to that in a moment. Let me first give you a slightly different example, and one where a stable arrangement is NOT possible. This is the *roommate problem*."

"Suppose you have four people A, B, C, and D who have to share two rooms (two in each room). Again, they all have their own preference lists and the conditions by which you can change rooms is analogous to the divorce scenario in the previous example. Now take these as your preference lists and work out that a stable arrangement is not possible and tell me what the difference between the two examples is."

I whipped out my notebook and began to scribble furiously on the last page, determined not to lose track of things.

Person	Preference
A	B > C > D
B	C > A > D
C	A > B > D
D	C > A > B

I wrote out the various possibilities:

Possible pairs in Room 1	Therefore pairs in Room 2	Preferences Room 1	Preferences Room 2
A & B	C & D	A is happy, B prefers C	C prefers B, D is happy
A &C	B & D		
A &D	B & C		

I thought to myself, "In 1), B and C will be better suited, in 2), A and B will want to room together and in 3), A and C will want to share. So there is no stable arrangement! But what is the difference between this and the marriage problem?!"

"Anyone figured out the difference yet?" asked the Professor.

To my dismay, someone's hand shot up. "Here A can choose from B, C, D whereas in the earlier problem, the men can only choose from the woman and vice versa. It's a modern day marriage problem sir!" he quipped.

Happy with the participation of the audience, the Professor replied smiling, "That's right! In the Marriage Problem, the men ONLY rank the women, and the women can ONLY choose from the men, which is not the situation in the roommate conundrum! Okay, so now that we've established that it isn't a trivial problem that we're attempting to understand, let's think about this algorithm that our economist friends have come up with."

The chalk reappeared in his hand and he began to write again. I fidgeted around, trying to find the optimum angle to look at the board from, kicking myself for my seating choice. I managed to find a position just as he finished his visit to the board.

"Suppose there are 3 men and 3 women, for simplicity's sake," he said, now walking up and down the length of the board, all the while looking at his audience. "Suppose that each man proposes to his favourite lady, and each lady considers all the proposals she receives (possibly none), scrutinizes them and keeps the one which is highest on HER list and rejects the rest. Note that she does not say 'Yes' to the one she keeps, she just tells him 'you're in contention, but hold your horses, I may change my mind yet'. Now all the men who are depressed at the outright rejection get another chance, and they propose to their second favourite woman, and the same procedure repeats itself."

He paused as if for dramatic effect before exclaiming, "This simple technique is the algorithm!"

There was a murmur around the audience as everyone spoke to those sitting beside them, looking slightly comical. Excited and serious is not an expression that the human face has learnt to master!

A faint voice from the back of the room slowly piped up... mine. "Sir, I can see that it seems to give a stable arrangement *(Refer to Box I)*, but how are we guaranteed that this procedure will ever end, and even if it does, it need not be unique, right?"

"Good question! To answer the first part, take this example and try and work it out for yourself and see that there is nothing that you have done that is specific to this example.

Man	Preference
M1	W1>W3>W2
M2	W1>W3>W2
M3	W3>W2>W1

Woman	Preference
W1	M2>M3>M1
W2	M3>M1>M2
W3	M2>M1>M3

(Answer on Page 30)

Here is my reasoning on why the suggested algorithm yields a stable arrangement:

Suppose M (for man) is not married to W (for woman) but yet prefers her to his own wife. We show that W cannot prefer M to her husband. Since M likes W more than his wife, at some point during the algorithm, M would have proposed to W. Since M and W are not together, that means that W rejected M's proposal in favour of someone she liked more! Thus, W must like her current husband more than she liked M and hence there is no instability in our arrangement!

Box I: Reasoning for stability of the arrangement

A more detailed yet easily understood explanation is available in the American Mathematical Monthly where D Gale and L S Shapley published their work."*(Refer to Box II.)*

As mathematicians, however, we will never be satisfied with a proof just because it seems to work in a particular example. On getting back to my room after the talk, I looked up the original paper by Roth and Shapley in the American Mathematical Monthly. The argument is simple. First of all, note that eventually (in fact in $n^2 - 2n + 2$ stages), every girl must have received a proposal.

Suppose some girl hasn't received a proposal. Then, since the number of boys and girls are the same, there must be at least one girl who at that point has at least two proposals. Thus, she must reject all but one and the rejected (and dejected) boys must now propose again. Since no boy can propose to the same girl more than once, every girl MUST receive a proposal sooner or later! And once the last girl receives a proposal, the period for "courtship" is over and the procedure must end and each girl must accept the boy she has on her string!

Box II: Why the procedure will end!

"As for your second question, this solution may not be unique! Suppose that in some parallel universe the women are the ones proposing and the men accepting/ rejecting. Then, using the same algorithm we'll get another marriage arrangement that might well be different. In fact, if you notice, the 'proposing party' gets a favourable result as opposed to the other! This is because the 'proposing party' is in fact going out there in order to look for the best possible deal for them while the 'accepting party' is waiting to decide from the offers they receive! I'm sure there is a life lesson here somewhere but I'm not here to lecture on philosophy!"

The room burst into laughter, mostly at the joke, but partly to express their happiness at having understood the lecture thus far.

"Many years later, Dr Roth (along with a number of fellow academics) modified the Gale Shapley algorithm in different instances and applied the work to a number of areas such as matching hospitals and medical students. Just about ten years or so back, he was asked to sort out the chaotic New York City High School application system. As is the case with any great piece of work, a number of other people have taken up this idea and tried to make it even more streamlined. One such case involves the example of hospital and medical students. A possible aim is to find a system in which if students lie about their preferences, it may not yield a solution in their favour *(Refer to Box III).* Another could be accommodating for married students wanting to be in the same hospital (or town) as their respective spouses. It is a case of taking the above algorithm and making it more attuned to the eccentricities of the real world."

I found myself quite excited, not least because I finally had somewhat of a tangible answer to people asking me what it was I could do after learning so much maths. My customary "the world is at my feet" sort of answer was getting stale to my ears!

"So that was what I wanted to discuss regarding 'the marriage problem'. Now, if nobody minds, there is another, unrelated but all the same interesting topic that I would like to discuss. How are we doing on time?"

I glanced at my watch and found that while we were trying to play match maker, nearly an hour had passed! Expectedly, everyone vociferously nodded their assent and we continued.

"Okay, we now discuss a slightly different problem. Suppose you have a particular town, and in that town a finite number of schools. Let this sheet of paper represent the town," he says holding a colourful sheet; "let the dots from which the colours are radiating be the schools, and let each point on the sheet be a child. Yes, I know that's unrealistic but just bear with me. Each school has a fixed capacity. The question now becomes, how does one allocate students to schools? The natural idea is to use distance:

students who are closer are preferred to those further away. You may object and say that a student may be equidistant from two schools (or vice versa) but this would form what is called 'a set of measure zero'."

I tried to recall my course in measure theory, one semester ago suddenly felt like an eternity away. "Measure zero essentially means of negligible size, like the integers as a subset of real numbers, or, for the poetically inclined, the stars in the sky," I remember my teacher saying and quickly brought my attention back to the lecture.

"So what's happening in this picture is, each school branches out radially until it fills up its quota, and if two school 'kingdoms' touch, then neither of them moves any further in that direction but continue to expand in other directions. Analogous to the divorce concept in the previous problem, a child X may change schools if there is a school A that:

a. Is closer to him than his present establishment, B

b. Has a student Y who is further away from him

Then, much to the displeasure of Y and his parents, he will be asked to leave A, and X will be enrolled in his place. The question then becomes, is there a stable arrangement for this question? The answer to this too is Yes, and, anticipating your next question, this arrangement will be unique! The reason is that, unlike in the marriage scenario, here the criteria for the 'preference lists' is the same for both students and schools! So if men and women find a parameter and an unambiguous rating system on which everyone agrees, then we can find a unique stable arrangement that is at the same time the best and worst possible for both parties!"

Someone seated in the front row asks, "But sir, in this particular diagram, if you notice, there are disjointed bits for some colours. Is it possible for every colour to be 'connected' in some sense?"

"Definitely! This is the next logical question to wonder about and that is exactly what researchers wondered. I won't go into the details of this; perhaps we can have another seminar sometime on this question where we can discuss this question at length. On this colourful note I will end today's lecture. I hope you enjoyed yourself. For those of you who thought that this was a waste of your time, hopefully the samosas and tea outside will make it feel a little more worthwhile!"

Spontaneous applause broke out in the room and everyone was on his or her respective feet, some eager to get their hands on the promised samosa, but more in appreciation of a well-delivered and more importantly, reasonably well-understood lecture – a far from common event in the world of academia!

As I stood waiting for my share of the refreshments, I heard a remark, "If this Ph D thing doesn't work out, maybe I can use this algorithm to open my matrimonial site!" While that, I'm quite certain, wasn't the aim of Messrs. Roth and Shapley, I too realised that Maths, or any subject for that matter, is far more interesting when you look at the problem at hand in a broader perspective, rather than being caught up with whether the expression in line 23 should have a minus sign or not.

(Answer: M2-W1, M1-W3 and M3-W2 will live happily ever after!)

References

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One particular such "real world eccentricity" is people wanting to subvert the system to obtain results in their favour! That is to say, suppose one man knew the preference list of all the others, would changing his own preference list skew the final arrangement in his favour? The answer to this question is far from obvious, but it is yes! In fact, Roth proved that there is no stable arrangement for which telling the truth is the best strategy for all parties concerned! Let us take the following case as an example to see how one may influence the final arrangement in their favour.

Men	True Preference	Women	True Preference
M1	W1>W2>W3>W4	W1	M4>M3>M2>M1
M2	W1>W3>W2>W4	W2	M4>M1>M3>M2
M3	W1>W2>W4>W3	W3	M1>M2>M4>M3
M4	W3>W4>W2>W1	W4	M2>M1>M4>M3

An easy verification shows that the Gale Shapley Algorithm will now yield the following stable result (under the Women propose scenario):

W2-M4, W3-M1, W4-M2, W1-M3

Now, suppose M4 is not the righteous man we believe him to be and he decides to try to subvert the system. Armed with the knowledge of the preference lists, M4 cunningly changes his list to: W3>W4>W1>W2.

Men	New Preference	Women	True (=New) Preference
M1	W1>W2>W3>W4	W1	M4>M3>M2>M1
M2	W1>W3>W2>W4	W2	M4>M1>M3>M2
M3	W1>W2>W4>W3	W3	M1>M2>M4>M3
M4	W3>W4>W1>W2	W4	M2>M1>M4>M3

The Gale Shapley Algorithm now yields the arrangement given by:

W2–M1, W3–M2, W4–M4, W1–M3

Note now, that as compared to the previous arrangement, M4 is now married to W4 as opposed to W2. Since W4 is higher on his 'true' preference list, he has achieved a more favourable result by giving a different rating list!

The reason that our algorithm is still viable in the real world, despite the large number of miscreants trying to find loopholes in the system, is that the volume of information required to foresee the possibilities, and to find a way around it is enormous! Consider the High School application system mentioned earlier. For an applicant to subvert the system, they need to know the preference list of the applicants and (potentially) that of the schools as well. Roth and Rothblum proved that provided the information available to applicants is sufficiently limited, he or she cannot gain by submitting a list which reverses the true ordering of two schools (as M4 did earlier).

In non-cooperative game theory, such a situation (one in which each player is assumed to know the equilibrium strategies of the other players, and no player has anything to gain by changing only his or her own strategy unilaterally) is known as Nash Equilibrium.

Box III

Quiz Question:

Find the odd man out from the following?

ART ◻ MUSIC ◻ POETRY ◻ MATHEMATICS

The answer is … ***None of the above!***

If you agree with that claim, here are some quotations from mathematicians who think likewise.

- It is impossible to be a mathematician without being a poet in soul.
  Sophia Kovalevskaya

- The mathematician's patterns, like the painter's or the poet's must be beautiful; the ideas, like the colors or the words must fit together in a harmonious way. Beauty is the first test: there is no permanent place in this world for ugly mathematics. … I am interested in mathematics only as a creative art.
  G.H.Hardy

- Mathematics, rightly viewed, possesses not only truth, but supreme beauty – a beauty cold and austere, like that of sculpture, without appeal to any part of our weaker nature, without the gorgeous trappings of paintings or music, yet sublimely pure and capable of a stern perfection such as only the greatest art can show.
  Bertrand Russell

- A mathematician who is not also something of a poet will never be a complete mathematician.
  Karl Weierstrass

- By the help of God and with His precious assistance, I say that Algebra is a scientific art. The objects with which it deals are absolute numbers and measurable quantities which, though themselves unknown, are related to 'things' which are known, whereby the determination of the unknown quantities is possible. Such a thing is either a quantity or a unique relation, which is only determined by careful examination. What one searches for in the algebraic art are the relations which lead from the known to the unknown, to discover which is the object of Algebra as stated above. The perfection of this art consists in knowledge of the scientific method by which one determines numerical and geometric unknowns.
  Omar Khayyám, "Treatise on Demonstration of Problems of Algebra" (1070)

- Submitted by Srirangavalli Kona

Viewpoints

Angles and Demons

Presenting two rarely discussed facets of angles which straddle two different areas. The first part goes straight into the difficulties of measurement, and the second part discusses alternative ways of measuring angles. Read this to bring more to your class than just the historical need to measure angles and its applications in real life.

in the classroom

The Editors

The difficulties children encounter (during a formal study of angles) might imply that angle and turn measure need not be introduced to young children. However, there are valid reasons to include these as goals for early childhood mathematics education. First, children can and do compare angle and turn measures informally. Second, use of angle size, at least implicitly, is necessary to work with shapes; for example, children who distinguish a square from a non-square rhombus are recognizing angle size relationships at least at an intuitive level. Third, angle measure plays a pivotal role in geometry throughout school and laying the groundwork early is a sound curricular goal. Fourth, the research indicates that although only a small percentage of students learn angles well through elementary school, young children can learn these concepts successfully. Source: [1]

Reading further in [1], one sees a learning trajectory for angle measurement which starts with an intuitive angle builder (2–3 years), an implicit angle user (4–5 years), an angle matcher (6 years), an angle size comparer (7 years), and an angle measurer (8+ years).

This article focuses on angle measurement, which according to the above trajectory should be taught in the third standard but which continues to give students difficulties two or even three years later

To most grown-ups, angles present no difficulty. An angle has a vertex and two arms spread apart to a certain degree called the 'measure' of the angle; a handy instrument called a protractor can be used to measure that degree. This definition is present in many textbooks. It seems such a simple concept that one cannot imagine anyone having difficulty with it. Workbooks dedicate a page or two to an introduction on angles and hurry on to problems on drawing and measurement and naming parts of an angle. But ask the students to measure the angle of an inverted cone and most of them struggle with orienting their protractors correctly. Or show two equal angles with different arm lengths and ask them to say which is greater; a good many of them will pick the one with the bigger arms. Or study the situation depicted in Figure 1, where the student thinks that the baseline has to be completely covered by the protractor.

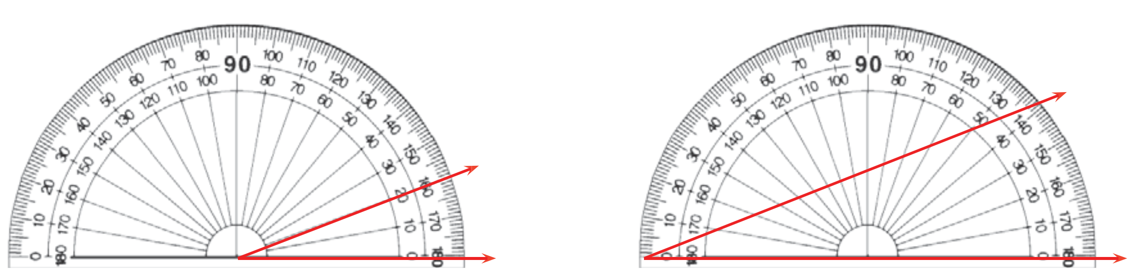
Why do such misconceptions arise? Is it because, from the beginning, we inundate children's minds with words like vertex, line segment, ray and so on and neglect practical tasks associated with measurement? Pick up a protractor and examine it. Is it really easy to use with its mass of lines and markings (clockwise and counter-clockwise) and numbers? In fact, it is so complicated that it is a miracle that children learn to use it at all!

In this article we present a sequence of ideas that introduces young learners to angles. It is guided by the belief that anything which relates to the tangible world of children is going to have better learning outcomes than otherwise.

Playing with angles. Angles have been defined from two perspectives — as a 'shape' formed by two rays extending from one point, or as a 'rotation' or 'turn'. Students sometimes think of these as different concepts. Activities dealing with angles should encompass both the notions, so that students appreciate the intrinsic meaning of the term.

Using a circle of paper folded into quarters (and therefore with rounded edges), the teacher can demonstrate how to make a right angle. By aligning it against different angles, students can grasp the right way to compare two angles (Figure 2). The device also serves as a rudimentary protractor. The same shape can be folded or unfolded to form smaller and bigger angles. After this, introducing the terms 'acute' and 'obtuse' is a matter of association.

An interesting use of this folded shape is to illustrate invariance of angle with arm length, a concept even middle-school students sometimes



A 20˚ angle . . .   . . . or a 40˚ angle?

Figure 1. 20˚ or 40˚? Some students would say it is the latter.

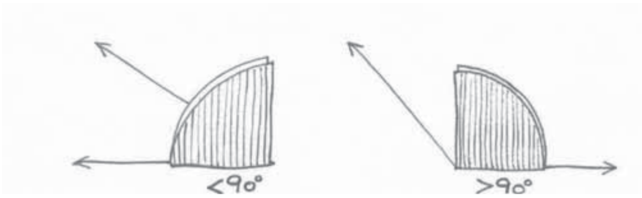


Figure 2.

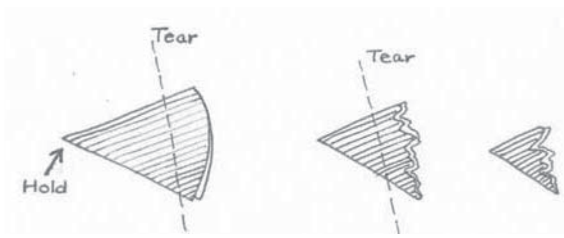


Figure 3.

grapple with. Hold it at the vertex and tear the paper (Figure 3). QED!

Another way is to use a string and two straws (Figure 4). Not only can the invariance of angle with arm length be demonstrated by moving the straws along the arms, but one can also demonstrate the idea that an angle can be formed 'in the imagination', with the vertex not visible. This is a difficulty that students in grades 9 and 10 experience while studying the topic of 'Heights and Distances' in trigonometry.

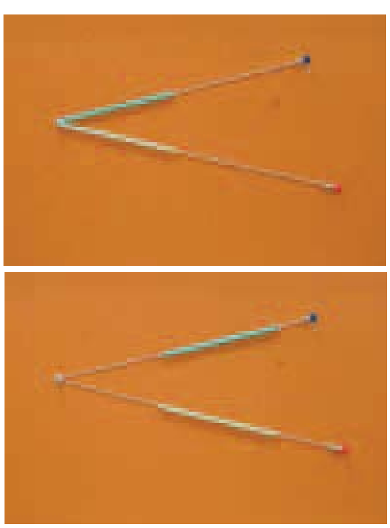


Figure 4.

Do you have a class with any kinesthetic learners? Then introduce the notion of angle with the help of a game: 'Angle Yoga'. Starting from the zero position and keeping one arm fixed, call out 'Right', 'Acute' or 'Obtuse' and move the other arm accordingly. As they do this, children realize several things, e.g., children with different arm lengths can show the same angle; there can be a range of correct angles for acute and obtuse angles; the fixed arm need not be horizontal or vertical; angles can be oriented differently. Most importantly, they learn the art of estimation using their arms to form an angle.

An interesting way of measuring rotation is to use the classroom door (Figure 5). The teacher marks angles on the floor from 0° to 90°, at intervals of 15° or 30°. This becomes a self-learning tool; children interact with it and learn. They may not immediately understand what the degree symbol is or why some markings are missing. Some may wonder what happens if the door opens even further: how does one go about measuring the angle then?

Figure 5. Source: http://business.outlookindia.com/printarticle.aspx?267253

(Which makes me wonder: Why are protractors not made with a slim metal strip that swivels from the centre and opens up from 0° to 180°?)

At this point, terminology such as *ray*, *vertex* and *line segment* can be introduced. Since students are familiar with the measurement of length using the iteration of a unit, the degree as a unit of measurement of angles should be acceptable to them. Geogebra is an excellent tool for students to understand the concept of rotation.

Different ways of measuring angles.

Now that students have learnt to measure angles using the protractor, they can investigate other ways of measuring angles and the advantages and disadvantages of these methods.

Maybe ancient geometers measured angles by fitting a line segment between the arms at a standard distance? Let's see what this leads to.

Suppose that, in order to measure $\measuredangle AOB$, points C and D are marked, one on each arm, 1 unit length from the vertex, and segment CD is drawn. Then the length of CD is taken to be a measure of $\measuredangle AOB$ (Figure 6). We call this the **chord method** to measure angles.

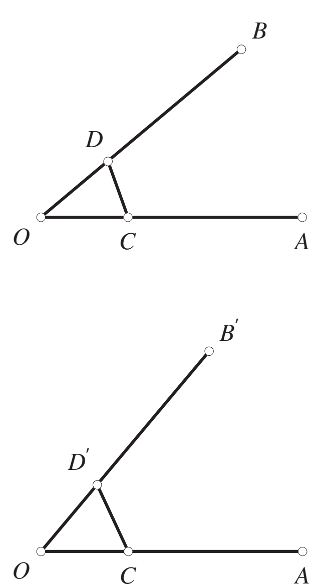


Figure 6. Here OC = OD = OD′.
If CD′ > CD then $\measuredangle$AOB′ > $\measuredangle$AOB, and conversely

It can be checked that this approach does preserve the *order relation*. In other words, if $\measuredangle AOB < \measuredangle AOB'$ then $CD < CD'$; and conversely. To see why, we apply the 'inequality form of the SAS congruence theorem' which states the following (we only consider the form applicable to isosceles triangles, as that is all we need): Let $\measuredangle ABC$ and $\measuredangle PQR$ *be isosceles, with* $AB = AC = PQ = PR$. *Then: if* $\measuredangle A < \measuredangle P$, *then* $BC < QR$; *and if* $BC < QR$, *then* $\measuredangle A < \measuredangle P$. This can be proved using pure geometry, but we leave the proof to you. (Some readers may prefer the following trigonometric proof. In an isosceles ΔABC in which $b = c$, we have $a = 2b \sin A/2$. Since b is fixed and $\sin x$ is an increasing function of x over the interval from 0° to 90°, it follows that a increases when $\measuredangle A$ increases from 0^0 to 180^0; and conversely. The same conclusion is reached if we use the cosine rule which yields: $a^2 = 2b^2(1\text{-}\cos A)$, but now we use the fact that $\cos x$ is a decreasing function of x over the interval from 0° to 180°.)

So the chord method of measuring angles preserves the order relation. But it fails to pass a second test which is as important: *additivity*. To see what this means, consider a pair of adjacent angles, $\measuredangle AOB$ and $\measuredangle BOC$, which share an arm OB (Figure 7). Since $\measuredangle AOC$ is the union of $\measuredangle AOB$ and $\measuredangle BOC$, and there is no 'overlap' between the latter two angles, it is reasonable to demand that the measure of $\measuredangle AOC$ should be equal to the sum of the measures of $\measuredangle AOB$ and $\measuredangle BOC$. But does this requirement hold good for the chord measure?

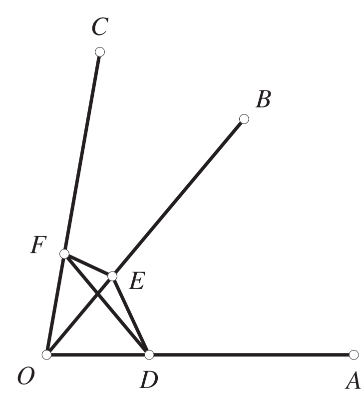


Figure 7.

Let D, E, F be points on the rays OA, OB, OC such that $OD = OE = OF = 1$ unit. By definition, the chord measures of $\measuredangle AOB$, $\measuredangle BOC$ and $\measuredangle AOC$ are the lengths DE, EF and DF, respectively. Is it true that $DE + EF = DF$? Clearly not. In fact we will always have $DE + EF > DF$, for two sides of a triangle are together always greater than the third side (here applied to ΔDEF). So the sum of the measures of $\measuredangle AOB$ and $\measuredangle BOC$ is *greater* than the measure of $\measuredangle AOC$. We see from this line of reasoning that the chord measure of an angle fails the test of additivity.

(Note: The above argument assumes that points *D, E, F* in Figure 7 do not lie in a straight line. But how can we be sure that they do not lie in a line? If we do not provide a justification for this, then what we have said is incomplete. Readers are asked to find a proof on their own.)

We do not know whether the ancients considered chord length as a candidate for angle measure. The measure they did adopt is the one we use today, and it possesses both the desired attributes —the order relation and the additivity property. It is based on **arc length.** Here, given an $\measuredangle AOB$, we mark points *C* and *D*, one on each arm, at 1 unit length from the vertex, and draw the circle with centre *O* and passing through *C* and *D*. Then *the length of arc CD is taken to be a measure of* $\measuredangle AOB$ (Figure 8).

Let us see how additivity is handled by this definition. In Figure 9 we see $\measuredangle AOB$ and $\measuredangle BOC$ which share an arm *OB*. As in Figure 7 the two angles have no overlap. Their arc measures are the lengths of arcs *DE* and *EF* which are both part of the circle with radius 1 unit, centred at *O*. The arc measure of $\measuredangle AOC$ is the length of the single arc *DF*. Is the length of arc *DF* equal to the sum of the lengths of arcs *DE* and *EF?* Clearly yes, as the arcs are all part of a single circle, and arc *DF* is simply the union of the two smaller non-overlapping arcs.

Arc measure of an angle is less natural than chord measure, but one begins to appreciate its elegance and advantages as one studies it more deeply.

In conclusion we may say that constructing, interpreting and recognizing shortcomings or gaps in definitions are all teaching and learning opportunities where teacher and learner can work together for greater understanding. It is when we see the inadequacies of attempted definitions that we begin to recognize the beauty and economy of existing definitions.

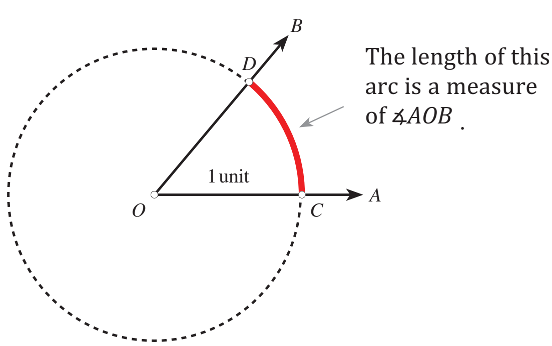


Figure 8.

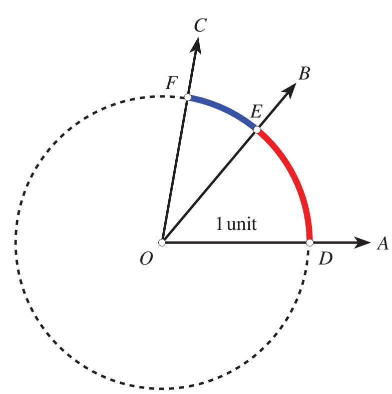


Figure 9.

Acknowledgements.

This article is the outcome of several absorbing discussions on different platforms. Anupama, a math resource person at the University Resource Centre of Azim Premji University presented a paper on 'Angles' at a seminar, following which there was an animated discussion in the online math learning group, focusing on misconceptions that students may have regarding angles and how a teacher can address them. The math learning group gratefully acknowledges the contributions of Dr. Ravi Subramaniam (HBCSE), Dr. Shailesh Shirali (CoMaC), Dr. Hridaykant Dewan (VBS), Ramchandar Krishnamurthy (APF), Rajveer Sangha and Jyothi Thyagarajan to this discussion.

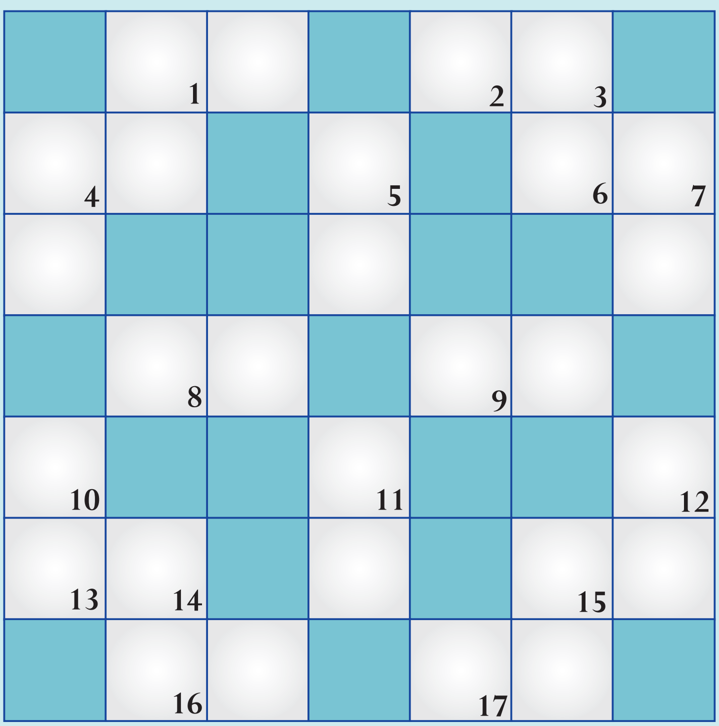
References

[1] *From Learning and Teaching Early Math —The Learning Trajectories Approach Studies in Mathematical Thinking, by Clements and Sarama*

number crossword-3

When the students of DRIK (Dwaraknath Reddy Institutes for Knowlededge) Patashala, Chittor came across the number crosswords in At Right Angles, (Issue I-1 and Issue I-2), they got down to solving them with great enthusiasm. But they ran into unexpected difficulties – not with the mathematics but with the language used. Though they studied in an English medium school, understanding the language in which the clues were framed seemed a challenge! However, when with a little help they managed to solve the crossword, there was no doubting the fact that number crosswords had found many enthusiastic converts. Their teachers too realised that here was an interesting way to improve language as well as mathematical skills.

When this incident was recounted to the creator of the crosswords, Mr D.D. Karopady, he came up with the interesting suggestion of having the next crossword created by these very students, which we now present to you for solution!



CLUES ACROSS :

1. 'L' IN ROMAN NUMERALS
2. 1221 TIMES ------ IS EQUAL TO 111111
4. 7D PLUS 5
6. 15D+7D+12D-25
8. LAST TWO DIGITS OF THE REVERSED MEASURE OF A LINEAR ANGLE
9. THE PRODUCT OF 15D AND THE FIRST DIGIT OF 7D
13. THE LCM OF 12 AND 30
15. 12D MINUS 30
16. 3D TIMES 3
17. 15D PLUS 5^2 MINUS 5

CLUES DOWN :

1. 4A MINUS 10
3. CUBE ROOT OF 2197 PLUS 6
4. PRIME NUMBER MADE WITH 1 AND 6
5. LAST TWO DIGITS OF THE MEASURE OF A GROSS
7. 13A PLUS 2
10. 4 SQUARED
11. SUM OF DIGITS OF THE RAMANUJAN NUMBER
12. 4 CUBED MINUS CUBE ROOT OF 4096
14. LOWEST AND HIGHEST NUMBERS AMONG 2,5,0,3,4,1
15. FIFTH PRIME NUMBER

DRIK Patashala in Chittoor, Andhra Pradesh was established in 2006 with the vision to ensure that children from urban slums and villages accessed their rights to education. The school now has 84 children chosen from the most neglected circumstances. Experiential learning, exposure trips, lots of music, dance, art and games are all part of an evolving and empowering curriculum which takes education beyond schooling for these children.

The path to mastery

Teacher: Leader, Supporter, Enabler

… is never smooth

SNEHA TITUS

It is often the kind teacher who is supportive and helpful, and ensures compliance using charm and persuasiveness, about whom effusive essays are written and who is remembered for the way he or she guided the students towards self-confidence and often, examination success. But how often does this teacher steer the student to mastery of the subject?

I was reminded of the stereotype of the kind teacher when I read an article by Alan Wigley[i] in which he describes two models for teaching mathematics. The first, *the path smoothing model*, is one practised by teachers who use *'the essential methodology of smoothing the path for the learner'*. This is how Alan Wigley describes this model:

1. *The teacher states the kind of problem on which the class will be working.*
2. *The teacher classifies the subject matter into a limited number of categories and presents them one at a time.*
3. *Pupils are led through a method for tackling the problems. The key principle is to establish secure pathways for the pupils. Thus it is important to present ways of solving problems in a series of short steps; often only one approach is considered seriously. Teachers question pupils, but usually in order to lead them in a particular direction.*
4. *Pupils work on exercises to practise the methods given aimed at involving learners more actively. These are usually classified by the teacher and graded for difficulty. Pupils repeat the processes until they can do so with the minimum of error.*
5. *Revision: Longer term failure is dealt with by returning to the same or similar subject matter throughout the course.*

Wigley goes on to explain that most teachers do provide insights into the concept they are teaching but under pressure to 'cover' the syllabus, they move on to the serious business of doing the exercises given. On reading this article, I was reminded of my experiences of teaching the chapter 'Maxima and Minima' in grades 11-12. I usually introduced the chapter with an interesting problem, such as the swimmer in distress who had to be reached in the least possible time by the life-guard. I would outline the stages of the solution and explain the theory at each stage. This would be followed by a series of problems solved in class, of increasing complexity. Having taught the chapter for many years, I was aware that there were some students who found this kind of problem rather difficult, though they were good at other sections of the course and had no difficulty with the topic of differentiation. Where they stumbled was in *understanding* what they were doing in the problem. Typically, the wording of the problem caused the difficulty: students could not distinguish between what they were given and what they had to prove. Once the problem was unfolded, they sped along the path to the solution.

I now see that the strategy I developed was a path smoothing model. Having recognized the boulder in the path (not the 'calculus' or 'small stone'!), I devised a series of steps which worked infallibly for all maxima and minima problems. I will use a familiar problem to illustrate the steps: 'Given a rectangular sheet of paper 9 inches by 12 inches, form a box by cutting congruent squares from the corners, folding up the sides and taping them to form an open box. To make a box with maximum capacity, how large should the squares be?' Here was my seven step path:

1. Identify the variable to be maximized or minimized (in this case, the volume *V*).
2. Write a formula for the variable (*V=length x breadth x height=lbh*).
3. Write the variable in terms of one variable only ($V= x(9-2x)(12-2x)$; here x is the side of one of the squares cut from each corner).
4. Differentiate the variable with respect to this variable. (In this case find $\frac{dv}{dx}$)
5. Set the derivative equal to zero and find the value of the independent variable at the turning point. ($\frac{dv}{dx} = 12x^2-84x+108$ which yields $x = 1.69, 5.30$)
6. Check by differentiating again and substituting these values of x in the second derivative, which value gives a maximum volume and which a minimum volume. (At $x = 1.69$, second derivative is $24*1.69-84< 0$, hence maximum.)
7. Go back to the question and give the specific information required. (In this case it was the size of the square cutouts which would be of side 1.69 cm and area approx. 2.85 square cm.)

Undoubtedly I was smoothing the path to good performance by providing the students with such a structured approach. As students used the seven steps for all the problems in the book, it appeared that most of them had mastered the content and that I had helped them to do so. There were some who never attained a degree of comfort with the topic – and whenever they approached me for a tutorial I would guide them through more problems with the same algorithm in my eagerness to prove that it was infallible. Very few of the students who had problems with this approach ever got comfortable with the topic, and they tended to shy away from this section in examinations.

Alan Wigley goes on to describe two different approaches to teaching and learning mathematics and how they seem to lie in watertight compartments. Here they are:

Exploration	Instruction
Invented methods	Given methods
Creative	Imitative
Reasoned	Rote
Informal	Formal
Progressive	Traditional
Open	Closed
Process	Content
Talking (pupil)	Talking (teacher)
Listening (teacher)	Listening (pupil)

The important point Wigley makes is about how lessons fall fully into one category or the other. For example, my lesson would clearly fall into the second category. Obviously, I needed to heed his advice to create classes that ensured conceptual understanding and enabled students to develop their own procedures. How could I do this in the time available? I paid heed to Wigley's advice to follow the 'challenging model' the features of which are given below:

The teacher presents a challenging context or problem and gives pupils time to work on it and make conjectures about methods or results. Often the teacher will have an aspect of the syllabus in mind, but this may not be declared to pupils at this stage.

An important word here is challenge. *The problem must be pitched at the right level, not too difficult, but more importantly, not too easy.*

A second important word is time. *It is crucial to give sufficient time for pupils to get into the problem – to recognise that it poses a challenge and that there may be a variety of approaches to it – so that discussion begins.*

Here again the role of the teacher is crucial – initially, in drawing out pupils' ideas. The syllabus may require the learning of more formal processes. The stimulus for this may be a harder mathematical problem and may require exposition by the teacher. However, the pupil will have the context of previous work to which more advanced techniques can be related.

A variety of techniques is used to help pupils to review their work, and to identify more clearly what they have learned and how it connects together. Longer term failure is dealt with by ensuring that any return to the same subject matter encourages a different point of view and does not just go over the same ground in the same way. The model places a strong emphasis on the learner gaining new insights, and the time required for reflection is considered to be fully justified.

The actual sub-unit began with group work on maxima and minima. The class was divided into groups of 4 and each group was given a problem. I deliberately used problems in which the dependent variable was a function of more than one variable.

In the initial 90 minute class, each group first worked on understanding the problem. After discussing what data the problem gave and what they were being asked to find out, the students decided on a method to represent the problem. At the end of the class, each group gave a short presentation on the problem: how they represented it, and how they used the model to collect data. For example, the group working on the box problem said that they would actually construct different boxes by cutting squares of different sizes from sheets of the given dimension. The group working on the swimming pool problem planned to create a simulation, and since exact rates could not be used, they would make a table of data using the given rates. For each group I reiterated the importance of explaining the need for optimization. Note that this class was spent in studying the problem and in listening to the other groups, and not on the solution to the problem.

Much before beginning the unit on maxima and minima, I had given a lot of emphasis to the concept of dependent and independent variable. I ensured that this idea was introduced while studying functions and revisited while creating tables and plotting graphs. In the next class, we used graphing software to understand the characteristics of turning points when the dependent variable was plotted against the independent variable. This software helped students to understand the reason why rate of change equals 0 at the turning point. They were also able to observe the sign change of the first derivative. Observations and conclusions were noted down in a worksheet which accompanied the exercise. In a class discussion following this exercise, the conclusions were discussed and noted down formally. This was followed by simple problems on maxima and minima from the textbook where the dependent variable was expressed in terms of one variable only.

Once the models were ready, the students were able to see the visual connect between the data given and the constraints specified. For example, in the box problem, students were able to measure and understand that the height of the box formed was the side of the square. This and other observations helped them write the volume in terms of the side *x* of the square cut out. I found this was crucial for them to understand that the given constraints allowed them to express the dependent variable in terms of one variable only. With the level of algebra that most students had drilled into them from high school, this was not a problem if one simply did a series of intricate steps that gave the desired result. But making the models helped students 'see' the implications of the constraints. Also, during the group discussion, peers observed and questioned and added their remarks. If a particular group was not able to proceed, suggestions were invited from other groups. Rarely did I have to intervene. Based on the common points from their problem, class mates were able to give constructive suggestions which helped the group in distress. Each group was able to arrive at the point where an expression for the dependent variable in terms of one single independent variable allowed a graph to be drawn. In the next class, we used these graphs and their learning so far on maxima and minima to complete the problem using differentiation and a formal algebraic procedure. For homework, each group had to do the remaining groups' problems.

Working in groups on a concrete or semi-concrete model helped students understand the problem and its solution. Certainly, some students still had doubts. But instead of countering their doubts with the same algorithm each time, I was able to use a variety of stimuli to understand as well as clarify their doubts. Eventually, I did share the 7 step plan with them. But this was after they had done a sufficient number of problems and they could connect each step to why they did this step. For the students who were comfortable with the topic, I encouraged to experiment with more difficult versions of the problem. For example, from an article on the box problem[ii] :-

a. If we use a square sheet of paper, does a common relationship exist between the side of this square paper and the side of the square cutout?
b. If the piece of paper we start with is an equilateral triangle, how do we cut the corners so that we can then fold up the sides and get a box with an equilateral triangle for base? What is the relationship between the side of the original equilateral triangle and the height of the lateral sides of the box in order for the box to have maximum volume?

Conclusion

While resources such as the Mathematics Teacher give teachers plenty of food for thought, it is the experience of modifying material to suit our particular need that makes the journey challenging and interesting. I was happy because I was able to incorporate elements of challenge, cooperative work and creativity, and at the same time preserve the rigor of the mathematics. I was also able to deliver a differentiated program of learning based on the student's mastery of the topic as well as comfort level with areas such as model making, simulating, use of graphing software, communicating and presenting. Finally, students were able to both reflect and critique on the experience. And where kindness dictated my 7 step approach, I was able to teach my students a better understanding of problem solving with this exercise. My thanks to Alan Wigley for having challenged me!

i Mathematics Teacher MT141 December 1992

ii 'Thinking Out of the Box' Mathematics Teacher Volume 95, No. 8, November 2002

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How to …

Solve a Geometry Problem

Part–2

Continuing our informal, short self-help guide on solving geometry problems. In the second part of this series, Ajit Athle describes some strategies which help in solving geometry problems and demonstrates how these strategies are used in solving an intriguing problem.

Ajit Athle

George Pólya once remarked, "Geometry is the science of correct reasoning on incorrect figures." But drawing an accurate figure is often an important first step in solving a problem in geometry, because it may reveal an unsuspected relationship — perhaps an equality of a pair of angles, or a pair of sides, or the perpendicularity of a pair of sides; the possibilities are many. In the same vein the use of colour can help — marking different parts of the figure in different ways. Any approach is permissible if it helps you to spot relationships which are otherwise nearly invisible. In this edition of 'Geometry Corner' we solve a challenging and intriguing problem.

An incircle & median problem

In $\triangle ABC$, the median AM to side BC is trisected by the incircle, i.e., $AP = PQ = QM$. Find the ratios $AB : BC : CA$. (See Figure 1.) Try to solve the problem before reading on.

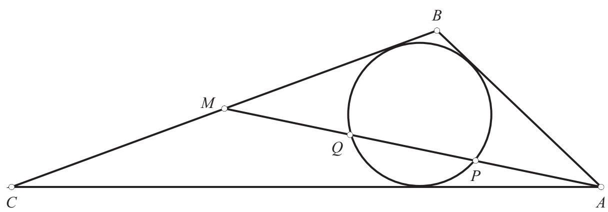


FIGURE 1. Problem concerning an incircle and a median

Solution to the problem

Let E, F, G be the points of contact of the incircle with the sides AB, BC, CA of the triangle (Figure 2).

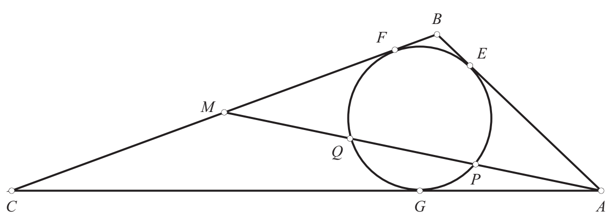


FIGURE 2.

A good beginning would be to take $BC = 2a$, as that would make $BM = MC = a$. Let us also take $BF = d$. By the equal tangents theorem, $BE = d$. By the Power-point theorem,

$$MF^2 = MQ \cdot MP, \qquad AE^2 = AP \cdot AQ,$$

and since $AP = MQ$ and $MP = AQ$, we have $MF = AE$. Let $AP = m$; then $MQ = QP = m$, hence $MF^2 = 2m^2$, and $MF = m\sqrt{2}$. But $MF = a - d$ as well, therefore:

$$(a-d)^2 = 2m^2. \tag{1}$$

Further, as AM is a median we may employ the theorem of Apollonius to $\triangle ABC$ to give:

$$\begin{aligned} AB^2 + AC^2 &= 2\left(BM^2 + AM^2\right) \\ &= 2\left(a^2 + 9m^2\right). \end{aligned} \tag{2}$$

Since $AE = MF = a - d$, it follows that $AB = a$. Also,

$$\begin{aligned} AC &= AG + CG = AE + CF \\ &= a - d + 2a - d = 3a - 2d. \end{aligned}$$

Substituting for AB and AC in (2) and then combining (1) and (2), we obtain the following quadratic equation:

$$5d^2 - 6ad + a^2 = 0.$$

This is easily factorized and solved to yield:

$$d = a, \quad \text{or} \quad d = \frac{a}{5}.$$

Of these, the former has to be rejected as it is inconsistent with the given conditions (it would make $MF = 0$). Hence, $d = a/5$ and this yields $AC = 3a - 2a/5 = 13a/5$. Therefore:

$$AB = a, \quad BC = 2a, \quad CA = \frac{13a}{5},$$

giving us the required proportions, $AB : BC : CA = 5 : 10 : 13$.

Note how knowledge of the equal tangents theorem, the intersecting chords theorem (or "power of a point" theorem) and Apollonius' theorem helped us to arrive at the answer.

Appendix: Some standard theorems of plane geometry

Equal tangents theorem: *Given a circle $\mathcal{C}$ and a point P outside the circle, let PA and PB be the two tangents that can be drawn from P to $\mathcal{C}$. Then $PA = PB$.*

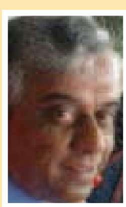
The theorem has a natural extension to three dimensions, with 'sphere' taking the place of 'circle'.

Intersecting chords theorem: *Given a circle $\mathcal{C}$, let two chords AB and CD meet at a point P. Then $PA \cdot PB = PC \cdot PD$.*

Remark. The result is true even if P lies outside the circle, or if one of the chords is tangent to the circle. The theorem has a natural converse. The value of $PA \cdot PB$ is called the *power of P with respect to the circle $\mathcal{C}$*.

Theorem of Apollonius: *Given a triangle ABC, let D be the midpoint of BC. Then $AB^2 + AC^2 = 2\left(AD^2 + BD^2\right)$.*

The result follows easily from the Pythagorean theorem; see if you can prove it. There are also easy and natural proofs using vectors; using coordinate geometry; and using trigonometry. The theorem has a generalization called Stewart's theorem.



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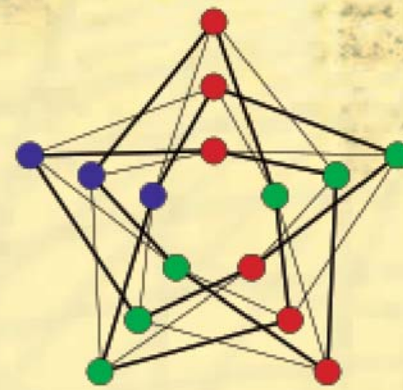
MAJOR ADVANCE TOWARDS PROVING THE

'TWIN PRIMES' CONJECTURE



Who can fail to be charmed by the primes? They are elusive and mysterious in their ways, and hide their secrets very well. They have been known since ancient times, and many statements can be made about them that can be understood even by primary school children. Yet, proving these same statements can be a task that may baffle the mightiest mathematician.

One such is the ***Goldbach Conjecture*** – the claim that *every even number after 2 can be written as the sum of two primes*; e.g., 50=43+7. It remains unproved, though mathematicians have come very close. Another is the ***Twin Primes Conjecture.*** 'Twin primes' are pairs of primes that differ by 2; for example, the pairs (3,5), (5,7), (11,13), (41,43) and (101,103). Such pairs thin out as the numbers get larger. Thus there are 205 such pairs below 10,000; 137 pairs in the next block of 10,000 numbers; 125 pairs in the next such block; then 124, followed by 114; and so on. Studying these data we may wonder whether the source 'dries up' eventually; that is, whether no more prime twins $(n,n+2)$ exist for n beyond some point. But this seems never to happen. Larger and larger 'twins' keep getting discovered! Currently the largest known such pair is:

$$(3756801695685\times2^{666669}-1, \quad 3756801695685\times2^{666669}+1).$$

Progress in proving the conjecture has been slow but not nil. One curious positive result is the following: If S is the set of all primes p which are part of a twin prime pair,

$$S=\{3,5,7,11,13,17,19,29,31,41,43,59,61,71,73,101,103,107,109,137,139,\ldots\},$$

then the sum Σ 1/p of the reciprocals of the numbers in S is finite. We even know the sum! – it is roughly 1.902. This theorem implies that the twin primes, even if they are infinite in number, are 'thin' in their distribution (else the sum would not be finite).

The twin prime conjecture may be restated as follows: If the successive primes are written as p_1, p_2, p_3, etc then the difference $p_{i+1} - p_i$ between successive primes is less than 3 infinitely many times. If we replace the '3' in this by a larger number, say 5, we get a statement which is weaker than the original one. Denote the statement "The difference $p_{i+1} - p_i$ is less than k infinitely many times" by $St(k)$, so $St(3)$ is the claim that there are infinitely many twin primes. Note that $St(3)$ implies $St(5)$ but not the other way round; so if we prove $St(5)$ we will have proved something significant but not quite the Twin Primes conjecture. The larger the value of k, the weaker the statement; yet, the difficulty of proving $St(k)$ for any k has been so great till now that it must be considered a significant achievement even if k is quite large.

Precisely such an achievement has been announced by Prof Yitang Zhang of the University of New Hampshire (USA). He claims to have proved $St(k)$ for k = 70 million! This may seem very far from the Twin Prime conjecture but it is still a very significant achievement. At the time this note is going to press, experts seem to be of the view that the proof is correct; indeed, his results have been described as being "of the first rank." In a following issue of *At Right Angles* we shall say more about Zhang's result.

Maths Club

A fascination for counting

TANUJ SHAH

We have been running a Maths Club in our school for a number of years, for class 7 students. We meet for one hour every week. We find that at this age, students have a need to explore the subject at a greater depth and a great desire to venture out and make connections with real life applications. They are also able to appreciate the aesthetic aspects of the subject at this age. The club is open to everyone irrespective of their mathematical ability. The aim of the Maths Club is to open up the ways in which students perceive Mathematics; to help them see the beauty and power of the subject. One topic that never fails to fascinate the children is that of counting. It is accessible for children of all abilities, and reveals patterns very quickly.

It is important to provide interesting narratives and contexts. We start off by talking about how blind people are able to read. Somebody suggests that the letters could be raised on a paper. Somebody disagrees, saying it is not a very efficient way of reading and would take a long time if the blind had to feel each letter. At this point somebody interjects and says that they have seen Braille books, which consists only of dots. The children are told that each character in Braille is represented within a 2×3 rectangle by raising dots in a particular pattern. There is only a single way in which each character can be represented. For example the letter 'm' would be represented as in Figure 1.

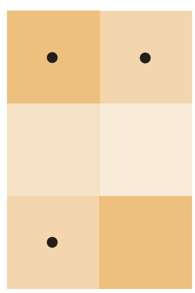


Figure 1

Now the children are given the task of finding out how many different characters could be represented in this way on a 2×3 rectangular grid, i.e., how many different patterns of dots and blanks are there. After some time we gather together and discuss how everyone went about the task and the difficulties that they encountered. It is observed that there are a large number of possibilities and it may not be easy to verify if we have covered them all. Somebody suggests that it may be worth doing things systematically: let's look at the possibilities of 0 dots, 1 dot, 2 dots etc. separately and then add them all up. There is only one way of representing the rectangle with zero dots. For 1 dot we can see that it can be put in any of the 6 spaces in the rectangle, so therefore there are 6 possibilities. For 2 dots one needs to go about it systematically by keeping one dot fixed in each of the 6 spaces and then adding the 2nd dot and being careful that there is no double counting. Here we find 15 possibilities (through systematic counting we can see that it is 5+4+3+2+1). As we write down the possibilities children start noticing an interesting pattern with reflective symmetry. Somebody suggests that the case for 4 dots is the mirror image of 2 dots, only that the dots and empty spaces are reversed. We display the results in a table:

Number of dots	0	1	2	3	4	5	6
Number of ways	1	6	15	20	15	6	1

The final sequence looks like this:
1, 6, 15, 20, 15, 6, 1. The total number of possibilities turns out to be 64, though somebody suggests that we should ignore the box with zero dots, as this would be confusing for blind people. So we settle on 63 possibilities. A question is posed, whether Braille can work for other kind of rectangles apart from 2×3. We agree to explore this question in the following week.

When we meet again, they are shown a blank flag with four stripes. Then the question, how many different ways can we colour the flag if we are only allowed to use the colours black and red, is posed (it is made clear that none of the stripes can be left blank). We gather together after five minutes and discuss how every one is going about the task. Quite a few remember the previous week's suggestion of doing it systematically, starting with one colour i.e. how many flags with 0 red stripes, 1 red stripe, 2 red stripes, etc. They are then given some more time to work it out. Those who finish quickly are given the task of exploring a flag with 5 stripes. As we get together we observe that for the 4 striped flag we get a reflective pattern: 1, 4, 6, 4, 1, and the total number of possibilities is 16. Somebody notices that this looks very similar to last week's question. We see that in both the situations there were 2 possibilities —a dot or a blank in the first one, and red and black in the second. Discovering this has a powerful effect on the children since situations with two possibilities can now be modelled in this manner. The question raised at the end of the first session can now be answered. We could use different sized rectangles for writing Braille; however we would not be able to cover all the characters with a rectangle smaller than 2×3. Those who attempted the 5-stripe problem tell us that there are 32 possibilities.

However it would not be enough for the alphabet and all the punctuations to be shown. We decide to put all our findings down and see if we can spot any more patterns.

Four-stripe problem	1	4	6	4	1		
Five-stripe problem	1	5	10	10	5	1	
Six-stripe problem	1	6	15	20	15	6	1

Most are quick at recognising the Pascal's triangle, which they have been exposed to earlier. There is still more interest in this topic and so there is a promise of more to follow.

In the third session, we explore the number of paths that can be taken from one end of a rectangular grid to another, without backtracking. We start in one corner cell (S) and move either down or right until we reach the corner cell that is diagonally across (E). At each step one has a choice of going down or to the right (see Figure 2). How many possible paths are there?

Start		
		End

Figure 2

An interesting pattern emerges if one starts writing down the number of ways to each cell in the grid as shown in Figure 3.

Start	1	1	1
1	2	3	4
1	3	6	10
1	4	10	20

Figure 3

The children are quick to spot Pascal's triangle yet again, and fascinated at the way it turns up in such unexpected ways. Later when children attempt expanding binomial expressions with different indices, they will have great satisfaction in spotting these patterns again. If time permits, then one could explore the problem of tossing different number of coins and looking at the outcome of heads and tails.

This series of three lessons would have given them pleasure in spotting patterns and developed their abilities to spot and deal with similar situations in counting.

While working on the Braille problem there are lots of interesting asides one can talk about depending on the interest level. One can talk about the biographical account of Louise Braille, mention that earlier the rectangular grid used to be larger and this made it difficult to read, until Louise Braille introduced the standard 2×3 grid which is now used. There is also a separate language for representing Mathematics and reading a music score for the blind.

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techspace

Connections between Geometry and Number Theory

In this article we explore connections between specific numbers and geometry, revealing new connections that you, dear reader, might have overlooked or not have seen before. We do the explorations by using modern technology, illuminating the strength of technology when working with mathematical investigations.

Thomas Lingefjärd

Some oddities of the number 7

A well-known fact is that a week has 7 days. "In six days God made the heaven and the earth, the sea, and all that is in them, but He rested the seventh-day. Therefore the Lord blessed the Sabbath day and made it holy." So the creation story points out 7 as a special number.

The Egyptians had seven original and higher gods; the Phœnicians had seven kabiris; the Persians had seven sacred horses of Mithra; the Parsees seven angels opposed by seven demons, and seven celestial abodes paralleled by seven lower regions. The seven gods were often represented as one seven-headed deity. The whole heaven was subjected to the seven planets; hence, in nearly all the religious systems we find seven heavens.

An important cognitive ability within humans is memory span. Memory span often refers to the longest possible list of items (e.g., colours, digits, letters, words) which a person can repeat immediately after a presentation, in the correct order. Millar (1956) has shown that the memory span of humans often is approximately 7 ± 2 items.

According to the theory of biorhythms, a person's life is affected by rhythmic biological cycles which affect one's ability in various domains, such as mental, physical, and emotional activity. These cycles begin at birth and oscillate in a steady sine wave fashion throughout life; by modelling them mathematically, a person's level of ability in each of these domains can be predicted approximately from day to day.
The emotional biorhythm model is a 28-day cycle. Here too the number 7 plays a role.

Mathematically interesting connections

The number 7 is prime, and Archimedes discovered its approximate kinship to the circle. He realized that a circle's circumference can be bounded from below and from above by inscribing and circumscribing regular polygons and computing the perimeters of the inner and outer polygons. By so doing, he proved that

$$3\frac{10}{71} < \pi < 3\frac{1}{7}$$

The first prime which is not 1 more than a power of 2 is 7: thus, $2=2^0+1$, $3=2^1+1$, $5=2^2+1$, but $7=2^3-1$.

A regular polygon with 7 sides is the first regular polygon which cannot be constructed by traditional Euclidean methods using straightedge and compass alone. (After 7 the next two such numbers are 9 and 11.)

The repeating portion of the decimal fraction corresponding to 1/7 is 142857 (that is, 1/7 equals 0.142857 142857 …). We have furthermore that:

142857 x 1	=	142857
142857 x 2	=	285714
142857 x 3	=	428571
142857 x 4	=	571428
142857 x 5	=	714285
142857 x 6	=	857142

The same figures come back in different order! We also see that we can express 1/7 as a geometric sum defined as

$$a\sum_{n=0}^{\infty} k^n = \frac{a}{1-k}$$

where $a = 0.14$ and $k = 0.02$. (The sum evaluates to 0.14/0.98 which simplifies to 1/7.)

Remember the ancient Egyptian and Archimedes approximation for π through 22/7 = 21/7 + 1/7 = 3.142857142857…

Given an integer k, a positive integer x is said to be *k-transportable* if, when its left most digit is moved to the units place (i.e., 'left to right'), the resulting integer is kx.

The integer 142857 is 3-transportable since

$$428571 = 3\times142857.$$

Kahan (1976) proved that for $k > 1$ there are no such integers unless $k = 3$, and the 3-transportable integers all belong to one of the following two sequences:

142857, 142857142857, 142857142857142857, …

285714, 285714285714, 285714285714285714, …

The following strange connection between algebra, geometry and the fraction 1/7 was shown to the author by the Swedish mathematician Andrejs Dunkels in 1988. Dunkels challenged us to show that if we combine six overlapping pairs of digits in 142857, and thereby get the following Cartesian points in the plane (1, 4), (4, 2), (2, 8), (8, 5), (5, 7) and (7, 1), these six points lie on an ellipse.

This astonishing fact was first pointed out in 1986 by Edward Kitchen, who encouraged readers of *Mathematics Magazine* (problem section) to prove the fact noted above. See Figure 1 constructed with GeoGebra. The problem is easily solved by Dynamical Geometry software (e.g., GeoGebra or Geometer's Sketchpad), but in the October 1987 issue of the magazine the problem was solved by hand by John C. Nichols, Thiel College, Pennsylvania.

It is well known that five arbitrary points satisfy a conic equation given by

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0.$$

We have six coefficients to determine; but they are determined up to multiplication by a non-zero constant (that is, if the six numbers are scaled up by a common constant, we get the same conic), which means that five points determine the conic (provided that no four of them lie on a line; if three of the points lie on a line, the conic is a union of two lines).

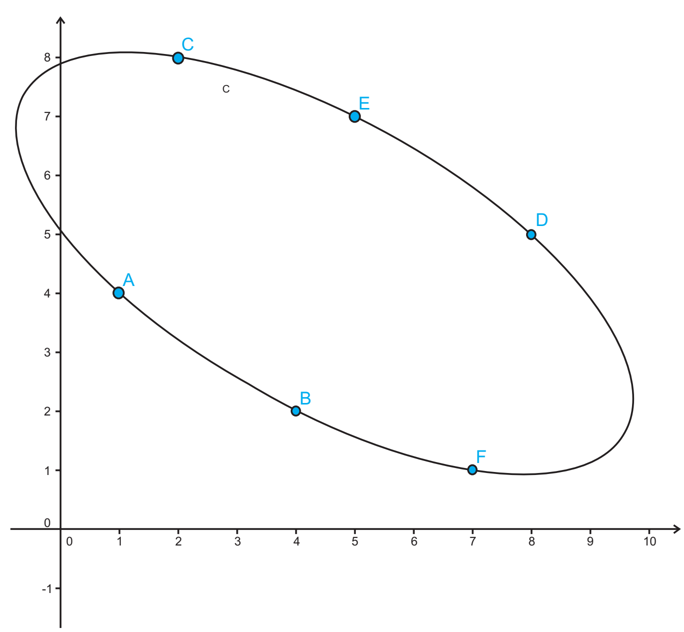

Figure 1: The 1/7 ellipse, where A = (1, 4), B= (4, 2), C = (2, 8), D = (8, 5), E = (5, 7), F = (7, 1).
Its equation is: $19x^2 + 36xy + 41y^2 - 333x - 531y + 1638 = 0$.

The fact that in the '1/7 ellipse' the sixth point too lies on the conic rests on a symmetric relation that holds between the six points; specifically, on the fact that 142 + 857 = 999, which yields the following relation (the significance of this will be seen presently):

$$(1,4) + (8,5) = (4,2) + (5,7) = (2,8) + (7,1) = (9,9).$$

What other reciprocals have the same qualities? What will for instance happen if we combine the points (14, 28), (28, 57), (57, 14) with the points (42, 85), (85, 71), (71, 42)? (These are obtained by taking 2-digit combinations from the decimal expansion of 1/7.) It happens that these six points too lie on an ellipse; see Figure 2.

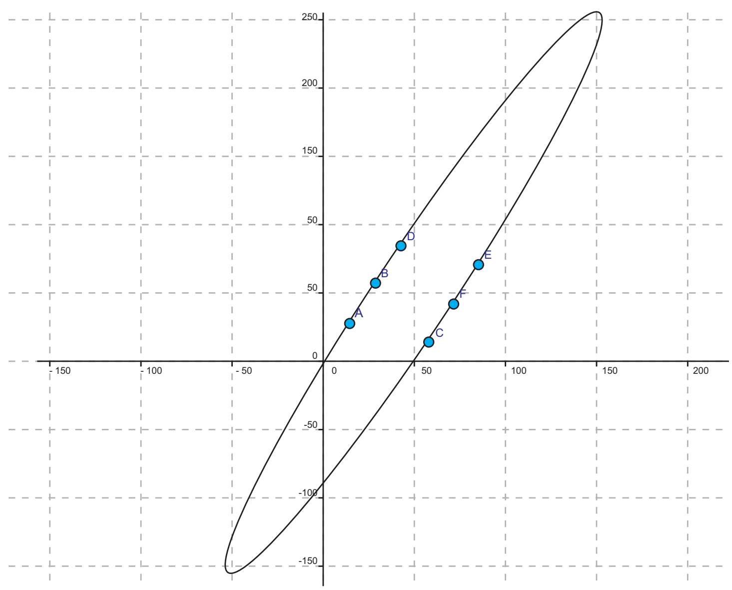

Figure 2: Variant of the 1/7 ellipse,with A = (14, 28), B= (28, 57), C = (57, 14), etc.
Its equation is: $165104\,x^2 - 160804\,x\,y + 41651\,y^2 - 8385498\,x + 3836349\,y + 7999600 = 0$.

Generalizing the question

The Shippensburg University problem solving group (1987) investigated all 'period six reciprocals' (i.e., those whose digital forms have a six-digit repetend, like 1/7) and found that reciprocals of 13 and 77 yield hyperbolas, the reciprocals of 39, 63, 91, 143, 273, 429, 693 and 819 yield ellipses, while the reciprocals of 21, 117, 189, 231, 259, 297, 351, 407, 481 and 777 do not yield a conic at all.

Mathpuzzle (December 2006) cited Chris Lomont: "Out of curiosity, I found a lot more of these ellipses. One with more points is the 1/7373 ellipse, 1/7373 = 0.00013653... which gives seven points (0,0), (0,1), (1,3), (3,0), (3,5), (5,6), (6, 3) on an ellipse. To get 8 points on a single ellipse I found that the fraction 4111/3030303 works. I've yet to find more on a single ellipse. I'm unaware of any proof that can be done, although integer points on curves are much studied."(Web reference).

The first 6 pairs of numbers in several decimal fractions lie on an ellipse (e.g. 23/91 or 75/91) or on a hyperbola (e.g. 2/13 or 36/91).

Further, one might investigate the effect of considering for the coordinates not just single digits but blocks of digits of various lengths (2, 3, ...). I found that the blocks of length 2 of several reciprocals including 1/7, 1/13, 1/77, 1/91 and 1/819 yield conics but the blocks of length 2 of 1/7373 (period 8 reciprocal with 7 points on a conic) do not yield a conic. Moreover, blocks of length 3 of the reciprocals 1/7, 1/13, 1/77, 1/91 yield the straight line $y = -x + 999$, whereas blocks of length 3 of 1/819 yield the straight line $y = -x + 222$.

For example blocks of length 2 of 1/13 yield a hyperbola with the equation see (Figure 3; the caption shows how the coordinates of the points are computed):

$$-4013x^2 + 36478xy - 53117y^2 - 1408374x + 3452922y + 7074800 = 0.$$

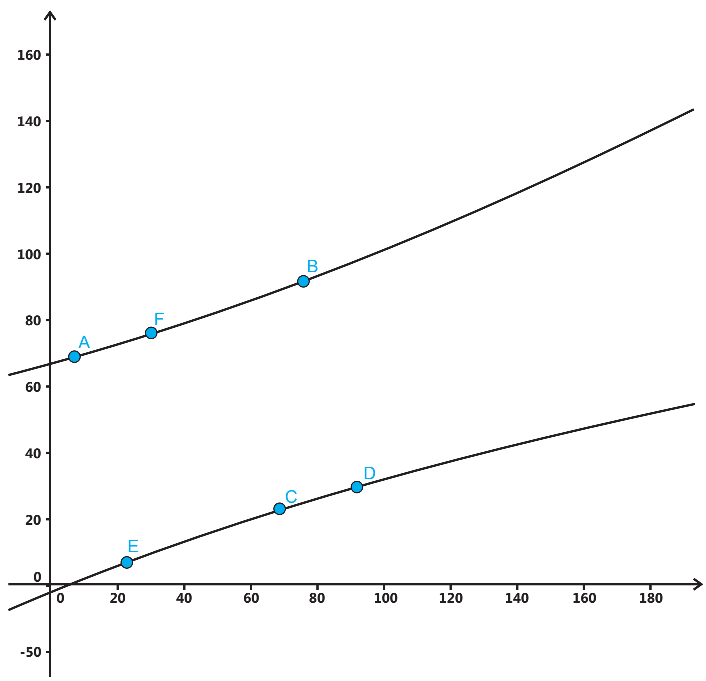

Figure 3: Points produced from blocks of length 2 from 1/13 = 0.076923…, yield a hyperbola. Here, A = (07,69), B = (76,92), C=(69,23), D = (92, 30), E = (23, 07), F = (30, 76)

The centres of the conics of 1/7 and 1/13 are all located at (9/2, 9/2) whereas the centres of the conics connected with blocks of length 2 are located at (99/2, 99/2).

Analysis: One way to look at digital-conics is that if you have four numbers $a, b, c, d,$ then the following six points necessarily lie on a central conic with centre $(d/2, d/2)$:

$$(a, b),\ (b, c),\ (c, d-a),\ (d-a, d-b),\ (d-b, d-c),\ (d-c, a).$$

Which particular conic manifests (hyperbola or ellipse) depends on the values of $a, b, c, d.$ However it seems difficult to make a precise correlation. It would be an interesting project to explore this correlation further.

In the case of 1/7 we have $a = 1$, $b = 4$, $c = 2$, $d = 9$ which as noted earlier draws on the fact that 142 + 857 = 999, and in a similar way the following six points lie on a conic:

$$(14,28), (42,85), (28,57), (85,71), (57,14), (71,42).$$

This can be seen as

$$(a,b),\ (d-c,d-a),\ (b,c),\ (d-a,d-b),\ (c,a),\ (d-b,d-c),$$

with $a = 14$, $b = 28$, $c = 57$, $d = 99$; in this case the coordinates are permuted in two different cycles of length 3 while we in the previous case had one single cycle of length 6. It is worth noting that in both cases the centre of the ellipse lies at $(d/2, d/2)$.

If we go back to the example of the 1/7 ellipse, the ellipse can be described as

$$19(2x-9)^2 + 36(2x-9)(2y-9) + 41(2y-9)^2 = 1224.$$

Also other 6-tuples of numbers can be used. They do not need to be different; for example, the number 112332 with ($a = b = 1$, $c = 2$, $d = 4$) gives 6 points that lie on the ellipse $3(x-y)^2 + (x+y-4)^2 = 4$.

If pairs of *triplets* of a period six reciprocal lie on the same line, the slope of the line must be $s = -1$. This is so because the first and fourth point have the same coordinates but in reverse order: $x_1 = y_4$ and $x_4 = y_1$ which gives $s = -1$.

What happens if we multiply 7 with 13, giving 91? We have: 1/91 =0.010989 ...which yields the points (0, 1),(1, 0), (0, 9), (9,8), (8, 9) and (9, 0). See Figure 4.

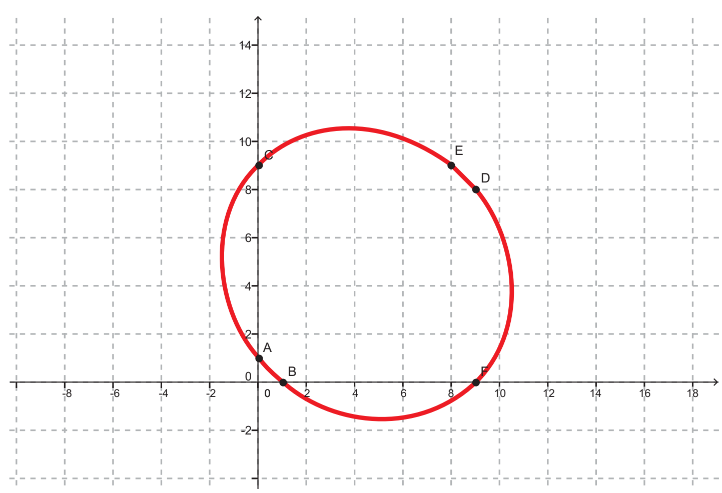

Figure 4: The ellipse built on the fraction 1/91

So far we have worked mainly on numbers generating fractions. What if we ask the question the other way round? For instance, is there a fraction $1/n$ with a cycle of length eight that yields an ellipse? It is not that hard to find that 1 / 73 = 0.01369863013 gives eight points (0, 1), (1, 3), (3, 6), (6, 9), (9, 8), (8, 6), (6, 3), (3, 0). See Figure 5.

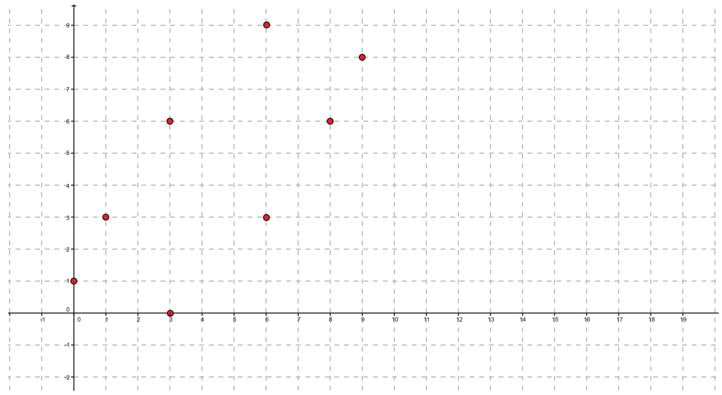

Figure 5: points derived from the fraction 1/73

Once the points are sketched, we see that they lie on an oval with centre (9/2, 9/2). But they do not all lie on an ellipse, as we find if we ask GeoGebra to fit a conic to the points. But if we omit (9, 8) and (0, 1) we do get an ellipse that not only goes through the six remaining points, but also through (0, 0), (1, –1), (8, 10) and (9, 9). We call this a *ten point ellipse.* The equation is
$3(2x-9)^2 + 2(2y-9)^2 - 4(2x-9)(2y-9) = 81$. The midpoint is at (9/2, 9/2). See Figure 6.

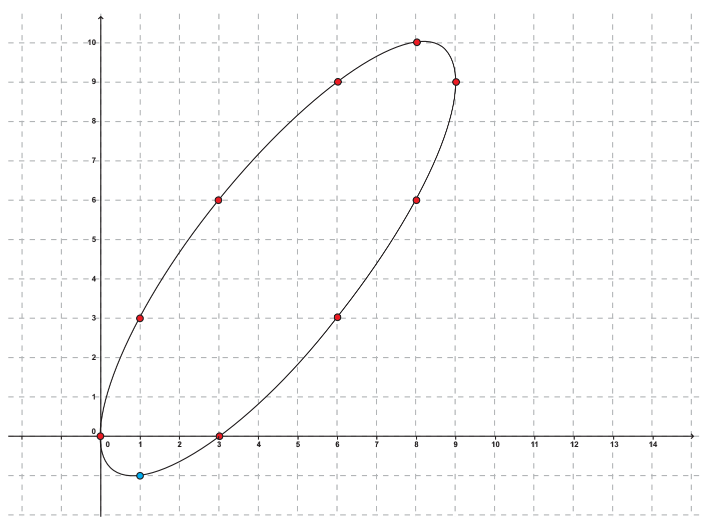

Figure 6: An ellipse partly derived from the fraction 1/73

In Figure 7, you find an ellipse with 18 outspread integer points on the periphery. Isn't that beautiful?

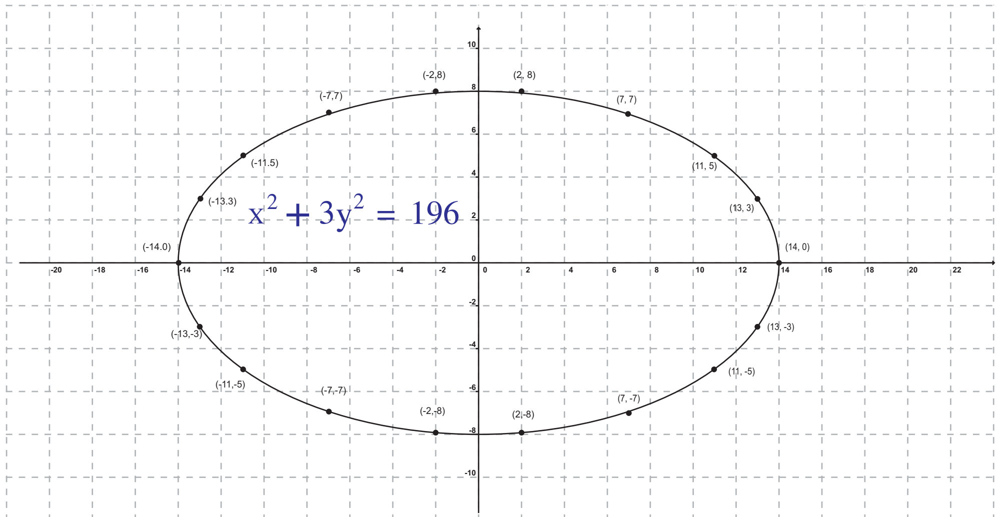

Figure 7: An ellipse with 18 integer points

Obviously we can go in many different directions. I leave further investigations to the reader.

Acknowledgements

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Worksheet

I. Investigate reciprocals of positive integers. Which reciprocals have 6 digit repetends? Make a record of these.

II. Focus on one particular reciprocal, say the reciprocal of 7. Here are some sample exercises:

1. Fill in the values for *a,b,c* so as to give a set of points that lie on a straight line:(142, 857), (285, 714), (428, *a*), (*a, b*), (857, *c*). What is its equation?
2. Do the points (1, 4),(4, 2), (2, 8) , (8, 5) and (5, 7) lie on a straight line?

   If they do, what is the equation of the line?

   If not, use Geogebra to investigate if they lie on a conic (remember that 5 points lie on a conic if no set of four points are in a straight line).

   If they do lie on a conic, check whether (7, 1) lies on the conic. Explain your finding.
3. Do the reflections of these points in the line $y = x$ lie on a conic? What is its centre, if so?
4. Substitute values for a, b, c and d and plot the 6 points (a, b), (b, c), $(c, d-a)$, $(d-a, d-b)$, $(d-b, d-c)$, $(d-c, a)$.
   - What happens if $a = b = c$?
   - If a, b, c are distinct, why do these points give a conic centred at $(d/2, d/2)$?
   - Will the same conclusion hold if a, b, c, d are two digit numbers?

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Notes added by the Editors

- The facts that (i) 7 is the smallest prime not of the form '1 more than a power of 2' and (ii) 'a regular polygon with 7 sides is the first regular polygon which cannot be constructed by traditional Euclidean methods' are, surprisingly, connected closely. We know this from the work of Carl F Gauss on the *Fermat primes.* Specifically, the result proved by Gauss is this:

  For odd integers $n \geq 3$, a regular n-sided polygon can be constructed using ruler-and-compass if and only if n is the product of distinct Fermat primes. (A 'Fermat prime' is a prime of the form $2^k + 1$; for example, 3, 5 and 17.)

  We shall elaborate on this connection in a subsequent article.

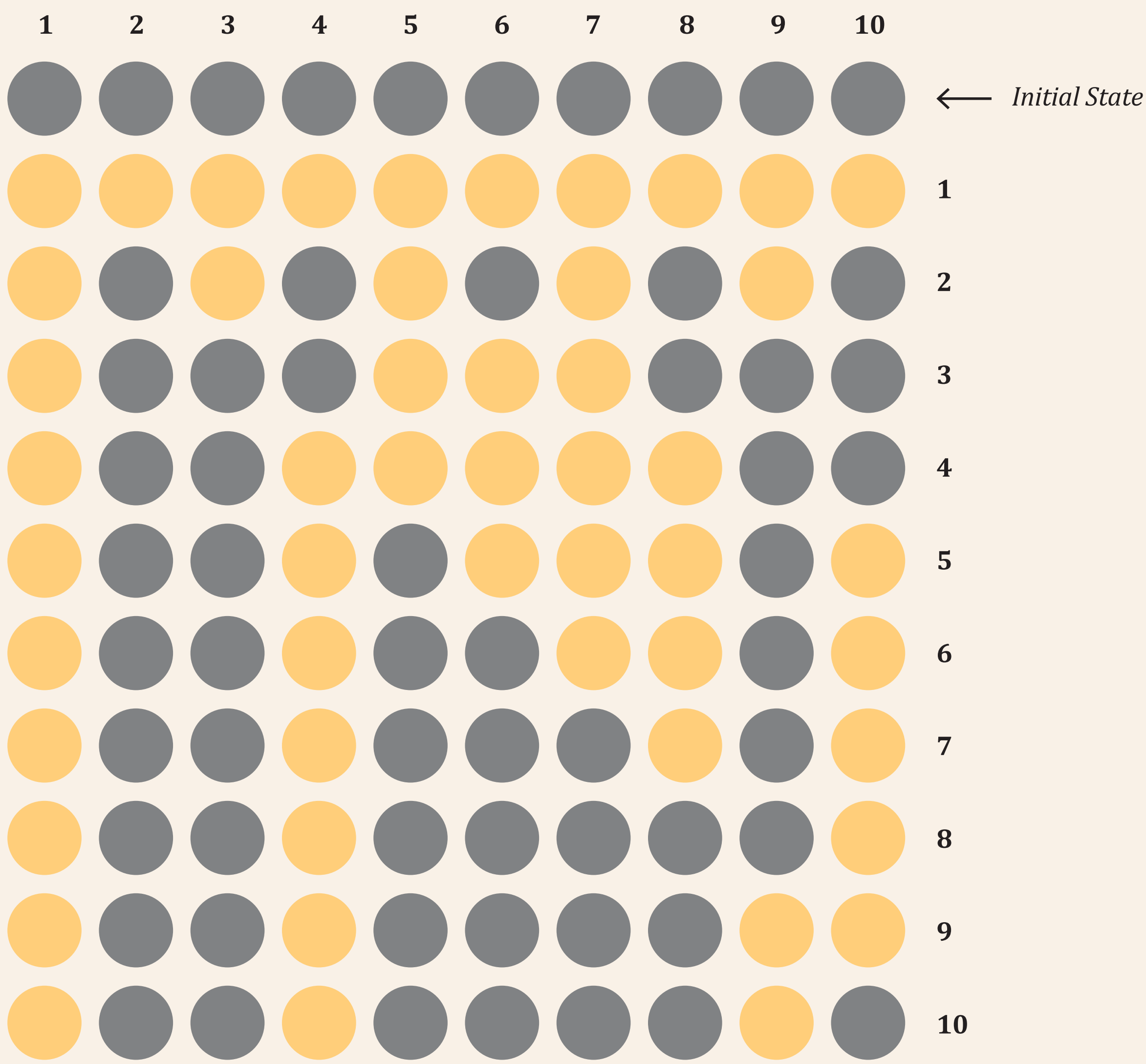
- All figures in the above article have been produced using GeoGebra. The 'conic' tool of GeoGebra is easy to use, and we urge the reader to explore further. The syntax is this: if *A, B, C, D, E* are given points, then the conic *k* defined by them is produced by the following command: *k* = Conic (*A, B, C, D, E*). The graphical interface can be used as well.

- The claim that the points $P = (a,b)$, $Q = (b, c)$, $R = (c, d - a)$, $S = (d - a, d - b)$, $T = (d - b, d - c)$, $U = (d - c, a)$ lie on a central conic with centre $M = (d/2, d/2)$ follows from the symmetric nature of this set of points. Let Γ be the conic passing through *P, Q, R, S, T*. (This conic is well-defined subject to some mild restrictions on *a, b, c, d.* For example, we should not have $a = b = c$.) Observe that *PS* and *QT* have the same midpoint, *M*. Let *M* be the origin of a new coordinate system, and let the equation of Γ in this system be $f(x, y) = 0$ where *f* is quadratic. Using the fact that in this system *P* and *S* have the origin as mid-point, as also *Q* and *T*, we argue that the coefficients of *x* and *y* in $f(x, y)$must be zero. Hence $f(x, y)$ must have only terms in the second degree (i.e., x^2, y^2, xy). But this implies that if any point lies on the conic, so does the point whose coordinates are the negatives of the first point. Since *R* lies on the conic, this implies that *U* does too. This justifies the claim.

Dumble - Door to the Rescue!

Preamble: Riddles are fun when they are told as stories. The act of exploring the mathematics behind them can be even more exhilarating. In this article, we tackle one such popular riddle and deconstruct the mathematics behind it.

Did you notice the pattern at the bottom of pages 37, 39, 41, 43, …, 57? Here is a composite view:



The visual is a simple version of a popular puzzle known simply as "100 doors in a row" or "door toggling puzzle". Since it does not appear to have a fancy origin story behind it, we came up with a fable of our own. We call it "The Dumble-Door":

"Voldemort has captured Potter and suspended him over a cauldron of boiling potion. Dumbledore watches helplessly as his protégé is being slowly lowered into the magical potion which will turn him into a raccoon. However, Voldemort will release Potter if Dumbledore is able to solve the following riddle.

There are 100 doors in a row in Voldemort's castle. All of them are closed. There is one guard who has the keys to all the doors. But he works in a strange manner. At first he opens all the doors one by one, all 100 of them. He then walks back to the starting position and this time he visits every second door and closes them back. The third time, he visits every third door (3rd, 6th, 9th, etc.) and toggles it (that is, if he finds it closed, he opens it, and vice versa). The fourth time he toggles every fourth door, and so on. Dumbledore has to tell Voldemort how many doors will remain open in the end.

The guard is old and frail and Potter is getting closer to the cauldron with each turn of the lever. If Dumbledore were to wait for the guard to finish his job, he would have no chance at rescuing his protégé. Is there a way to tell the answer quicker?"

To solve it, let's go back for a while to the grid of dots above. It is clear that out of the 10 dots, the ones which end up orange are 1, 4, 9, or the first three perfect squares. This pattern of perfect squares is easily generalizable to N. Thus for the Dumble-Door riddle above with 100 doors, exactly 10 will remain open in the end ($10^2=100$ is the largest perfect square less than or equal to 100). But what is the mathematics behind this?

To find that out, let's explore the problem a bit more through an activity. Complete the following table by filling in the sequence of states and turn numbers for each door. Three examples have been given.

Note for the teacher: When doing this as a classroom activity, extend the table to 30 so that students have enough data to figure out the pattern.

Door no.	State changes (starting from all 'Closed')	Turn at which state changes
1	Open	1
2		
3		
4		
5		
6	Open, Close, Open, Close	1, 2, 3, 6
7		
8		
9	Open, Close, Open	1, 3, 9
10		

If you have correctly filled in the table, you must have observed the following:

1. A *door is toggled at every turn which is a divisor of the door number*. Thus: Door 6 is toggled at the 1st, 2nd, 3rd and 6th turn. Door 5 toggles only at the 1st and 5th turn.
2. For a door to end up open, it must be toggled an odd number of times. (Remember that all doors are opened in the first turn.)
3. Only door numbers with an odd number of divisors will be left open.

The problem now reduces to this:

Which numbers have an odd number of divisors including 1 and themselves?

The Fundamental Theorem of Arithmetic and Law of Combinations now come handy. Consider a number like 12. We know it has 6 divisors (1, 2, 3, 4, 6, 12). How can we calculate the number of divisors without enumerating all of them?

Note that 12 can be written in terms of its prime factors as $2^2 \times 3$. Any divisor of 12 can choose from 2's in three ways (none, one or two) and 3's from two ways (none or one). Thus total number of ways a divisor can be chosen is $3 \times 2 = 6$.

In general, for a number $X = p^a q^b r^c \ldots$ where $p,q,r,\ldots$ are primes and $a,b,c,\ldots$ are positive integers, the total number of divisors is

$$D = (a+1)(b+1)(c+1)\ldots$$

So D is odd only when each of the factors $(a+1)$, $(b+1)$, $(c+1)$, … is odd. Thus $a,b,c,\ldots$ must all be even. But if they are all even, X is a perfect square!

It follows that any perfect square has an odd number of divisors, and we conclude that only the door numbers which are perfect squares will remain open.

Hence, resQED!

Dumble-Door to the Rescue is based on a story suggested by Prithwijit De (HBCSE). AtRiA acknowledges the contribution of Rajveer Sangha who suggested the idea of the running gag and presented the article in its present visual form.

problem corner

Fun Problems

C⊗MαC

k-transportable numbers

In the article *Connections between Geometry and Number Theory* (elsewhere in this issue of *At Right Angles*) the author refers to the notion of 'k-transportable numbers' — numbers with the property that if the left-most digit is shifted to the right-most end, then the number thus obtained is k times the original number. He quotes a result due to S. Kahan that the only integral value of k exceeding 1 for which such a number exists is $k = 3$. We prove this result here (our proof is different from Kahan's), and show a surprising way for generating such numbers. A more apt name for such numbers than the one given would be *cyclic numbers*, and we study this more general notion in the next section.

Let $A = \overline{a_1 a_2 a_3 \dots a_n}$ be a k-transportable number, and let $B = \overline{a_2 a_3 \dots a_n a_1}$, where k is a positive integer ($k \neq 1$; of course, $k < 10$). Then we have:

$$B = kA.$$

Construct the following two infinite recurring decimals, whose 'repetends' (i.e., the portions that repeat indefinitely) are the numbers A and B respectively. That is:

$$x = 0.A\,A\,A\ \dots,$$

$$y = 0.B\,B\,B\ \dots.$$

Both are 'pure' recurring decimals. Since $B = kA$, it follows that $y = kx$. Now consider the effect of multiplying x by 10. Noting the 'shift in the decimal point' we see that this yields

$$10x = a_1.B\,B\,B\ \dots,$$

a number with integer part a_1, and recurring portion the same as that of y. Hence we have:

$$10x = a_1 + y.$$

Since $y = kx$ this yields $10x = a_1 + kx$, and solving this for x we get:

$$x = \frac{a_1}{10 - k}.$$

Since x is a pure repeating decimal fraction, this relation puts restrictions on the value of k. Indeed we have $k \neq 2, 4, 5, 6, 8, 9$. We also have $k \neq 1$. So the possibilities for k are just 3, 7. Of these, $k = 7$ yields $1/(10 - k) = 0.333\dots$, which makes A a single digit number; this does not work out. Hence $k = 3$ (this was Kahan's result), and $x = a_1/7$. Since the repetend of 1/7 is 142857, we see that $A = a_1 \times 142857$, with a_1 chosen appropriately. Remembering that a_1 is also the left-most digit of A, we find that $a_1 = 1$ or 2; only these two choices work. Hence $A = 142857$ or 285714.

Obviously, repeating these blocks of digits will give more numbers with the same property. This justifies the claim made in the article that the only k-transportable integers are the following:

$$142857, \quad 142857142857, \quad 142857142857142857, \quad \dots,$$

$$285714, \quad 285714285714, \quad 285714285714285714, \quad \dots,$$

all of which are k-transportable with $k = 3$.

Cyclic numbers

The same idea can be used to solve the following: *Find a positive integer with the property that if its units digit is shifted to its left-most end, the new integer is twice the original one*. Denote the number by $A = \overline{a_1 a_2 a_3 \dots a_{n-1} a_n}$ (so it has n digits), and let $B = \overline{a_n a_1 a_2 a_3 \dots a_{n-1}}$; then $B = 2A$. Let x, y be pure recurring decimals defined as follows:

$$\begin{aligned} x &= 0.A\,A\,A\,A\ \dots \\ &= 0.\overline{a_1 a_2 a_3 \dots a_{n-1} a_n}\ \overline{a_1 a_2 a_3 \dots a_{n-1} a_n}\ \dots, \\ y &= 0.B\,B\,B\,B\ \dots \\ &= 0.\overline{a_n a_1 a_2 a_3 \dots a_{n-1}}\ \overline{a_n a_1 a_2 a_3 \dots a_{n-1}}\ \dots. \end{aligned}$$

Then $y = 2x$. If we multiply y by 10 we get a pure decimal recurring decimal whose repetend is the same as that of x:

$$\begin{aligned} 10y &= a_n.\overline{a_1 a_2 a_3 \dots a_{n-1} a_n}\ \overline{a_1 a_2 a_3 \dots a_{n-1} a_n}\ \dots \\ &= a_n + x. \end{aligned}$$

Since $y = 2x$ this yields: $20x = a_n + x$, and so:

$$x = \frac{a_n}{19}.$$

It therefore remains only to find the repetend of the fraction $1/19$, which we get by simple long division:

$$\frac{1}{19} = 0.\overline{052631578947368421}.$$

If we choose $a_n = 1$ we get $A = 052631578947368421$, which has 0 as its first digit; so we discard this solution. If we choose $a_n = 2$ we get $A = 105263157894736842$, and we have a possible answer:

$$A = 105263157894736842.$$

Please check that $105263157894736842 \times 2 = 210526315789473684$.

This means that 105263157894736842 *is the smallest possible solution to the problem*.

Other choices for a_n yield more solutions, all using the same repetend. Thus:

$$\begin{aligned} a_n &= 2 \text{ yields } A = 105263157894736842, \\ a_n &= 3 \text{ yields } A = 157894736842105263, \\ a_n &= 4 \text{ yields } A = 210526315789473684, \\ a_n &= 5 \text{ yields } A = 263157894736842105, \\ a_n &= 6 \text{ yields } A = 315789473684210526, \\ a_n &= 7 \text{ yields } A = 368421052631578947, \\ a_n &= 8 \text{ yields } A = 421052631578947368, \\ a_n &= 9 \text{ yields } A = 473684210526315789. \end{aligned}$$

That's a lot of solutions!

Given a positive integer N, let $f(N)$ denote the integer obtained by shifting its units digit to its left-most end. (Example: $f(1234) = 4123$.) A number N with the property that the ratio $f(N) : N$ is an integer, or a rational number with small numerator and denominator, is called a ***cyclic number***. The best known example of such a number is 142857 (for which the ratio is $5 : 1$). Such numbers are always associated with the repetends of pure recurring decimals (and that is what helps in finding them); but there is more: the numbers also have some very striking properties. Here is one, which crucially underlies the phenomenon explored in the article *Connections between Geometry and Number Theory*.

Let p be any prime number greater than 5, and let the recurring decimal corresponding to $1/p$ be computed; it will always be a pure recurring decimal. Let N be the repetend of this decimal. The number of digits in N could be odd or even. If the number of digits in N is even, say $2k$, then let A and B be the k-digit numbers obtained by 'slicing' N into two halves. *Then the sum $A + B$ is a number made up only of nines. That is, $A + B = 10^k - 1$.* Here are three examples of this remarkable phenomenon which goes by the name of *Midy's theorem*.

- If $p = 7$ then $1/p = 0.\overline{142857}$, so $N = 142857$ which has an even number of digits (with $2k = 6$). Slicing the repetend into two, we get $A = 142$ and $B = 857$. Observe that $A + B = 999 = 10^3 - 1$.
- If $p = 13$ then $1/p = 0.\overline{076923}$, so $N = 076923$ which has an even number of digits (with $2k = 6$). Slicing the repetend into two, we get $A = 076$ and $B = 923$. Observe that $A + B = 999 = 10^3 - 1$.
- If $p = 17$ then $1/p = 0.\overline{0588235294117647}$, so $N = 0588235294117647$ which has an even number of digits (with $2k = 16$). Slicing the repetend into two, we get $A = 05882352$ and $B = 94117647$. Observe that $A + B = 99999999 = 10^8 - 1$.

In a future issue of *At Right Angles* we shall explore this beautiful theorem and some of its extensions.

Problems for Solution

Problem II-2-F.1 Find a positive integer with the property that if its units digit is shifted to its left-most end, the new integer is 3 times the original one.

Problem II-2-F.2 Find a positive integer with the property that if its units digit is shifted to its left-most end, the new integer is 9 times the original one.

Problem II-2-F.3 Find a positive integer with the property that if its units digit is shifted to its left-most end, the new integer is $1\frac{1}{2}$ times the original one.

Solutions of Problems from Issue-II-1

Problem II-1-F.1 *Solve the cryptarithm* $\overline{EAT} + \overline{THAT} = \overline{APPLE}$.

It is immediate that $A = 1$ and $T = 9$. This yields $E = 8$ and $L = 3$. Also, $P = 0$ as the sum of a 3-digit number and a 4-digit number cannot exceed 11000. This yields $H = 2$, and now all the digits have been found: $819 + 9219 = 10038$.

Problem II-1-F.2 *Solve the cryptarithm* $\overline{EARTH} + \overline{MOON} = \overline{SYSTEM}$.

The answer for this cannot be unique because the variables H and N (the two units digits) can be swapped with no ill effects. Other than this indeterminateness, however, the solution is unique:

$$97258 + 4336 = 101594,$$
$$97256 + 4338 = 101594.$$

We leave the derivation to the reader.

Problem II-1-F.3 *Given that* $\overline{IV} \times \overline{VI} = \overline{SIX}$, *and* $\overline{SIX}$ *is not a multiple of* 10, *find the value of* $\overline{IV} + \overline{VI} + \overline{SIX}$.

Since $X \neq I, V$ it follows that $I \neq 1$, $V \neq 1$. Since $\overline{IV} \times \overline{VI} > 101(I \cdot V)$ and $\overline{SIX}$ is a three-digit number, it follows that $I \cdot V < 10$. Since $I > 1$, $V > 1$, $I \neq V$ we get $\{I, V\} = \{2, 3\}$ or $\{2, 4\}$. The latter does not yield a solution since $24 \times 42 > 1000$, but the former does fit: $32 \times 23 = 736$. So the code is: $I = 3$, $V = 2$, $S = 7$, $X = 6$, giving $\overline{IV} + \overline{VI} + \overline{SIX} = 32 + 23 + 736 = 791$. Note that the information that '$\overline{SIX}$ is not a multiple of 10' has turned out to be superfluous.

Problem II-1-F.4 *Explain why the numbers* 1, 121, 12321, 1234321, 123454321, *...are all perfect squares.*

It is immediate that $1 = 1^2$, $121 = 11^2$, $12321 = 111^2$, and so on. To see why the digits build up in that pattern simply examine the underlying long multiplication. For example, here is 111×111:

$$\begin{array}{rccccc}
 & & & 1 & 1 & 1 \\
\times & & & 1 & 1 & 1 \\
\hline
 & & & 1 & 1 & 1 \\
 & & 1 & 1 & 1 & \\
 & 1 & 1 & 1 & & \\
\hline
 & 1 & 2 & 3 & 2 & 1 \\
\hline
\end{array}$$

Of course, the pattern will break after the number of digits exceeds 9.

Problem II-1-F.5 *Explain why the numbers* 1089, 110889, 11108889, 1111088889, *...are all perfect squares.*

We observe that $1089 = 33^2$, $110889 = 333^2$, $11108889 = 3333^2$, and so on. Let us see why this pattern persists. Let

$$A_n = \underbrace{333 \ldots 3}_{n \text{ threes}}.$$

Then:

$$A_n^2 = (333 \ldots 3)^2 = (111 \ldots 1) \times (999 \ldots 9)$$

$$= (111 \ldots 1) \times (10^n - 1)$$

$$= \underbrace{111 \ldots 1}_{n \text{ ones}}\underbrace{000 \ldots 0}_{n \text{ zeroes}} - \underbrace{111 \ldots 1}_{n \text{ ones}}$$

The subtraction clearly yields the number

$$\underbrace{111 \ldots 1}_{(n-1) \text{ ones}}\ 0\ \underbrace{888 \ldots 8}_{(n-1) \text{ eights}}\ 9,$$

which has the stated form.

A 'Least Sum' Divisibility Problem

In this short note we solve the following problem from the Regional Mathematics Olympiad (RMO) of 2006.

Given that a and b are positive integers such that $a + 13b$ is divisible by 11 and $a + 11b$ is divisible by 13, find the least possible value of $a + b$.

Attempting to solve the problem by 'brute force' does not seem satisfactory; we need a more insightful approach. We shall look for a way to generate pairs (a, b) of positive integers having the required divisibility properties, and thereby find the pair with least sum.

Since $11 \mid a + 13b$ it follows that $11 \mid a + 2b$. (Recall that '|' is the symbol for divisibility; e.g., we have $4 \mid 12$ but $5 \nmid 11$.) Similarly, since $13 \mid a + 11b$ we have $13 \mid a - 2b$. Let

$$\begin{cases} a + 2b = 11x, \\ a - 2b = 13y, \end{cases}$$

where x, y are integers. Solving this pair of simultaneous equations for a and b we get:

$$a = \frac{11x + 13y}{2}, \quad b = \frac{11x - 13y}{4},$$

and hence:

$$a + b = \frac{33x + 13y}{4}.$$

Since a and b are integers we see that x and y are either both odd or both even, and their sum must be a multiple of 4. (For: $4 \mid 33x + 13y$, hence $4 \mid x + y$.) Also, since $a > 0$ and $b > 0$ we must have

$$x > 0, \quad -\frac{11x}{13} < y < \frac{11x}{13}.$$

In any case we must have $y < x$. (Note that y can be negative.) Subject to these conditions we list in Table 1 some of the possibilities for x and y, and hence for a and b. For each value of x we have listed all possible values of y that yield integer values for a and b.

x	y	a	b	$a + b$
3	1	23	5	28
4	0	22	11	33
5	3	47	4	51
5	−1	21	17	38
6	2	46	10	56
6	−2	20	23	43
7	5	71	3	74
7	1	45	16	61
7	−3	19	29	48

The table suggests that the least possible value of $a + b$ subject to the stated conditions is **28**. We justify that this is so by observing that since $-11x < 13y < 11x$, the value of $33x + 13y$ lies between $33x - 11x$ and $33x + 11x$, i.e., between $22x$ and $44x$, and hence that

$$\frac{11x}{2} < a + b < 11x.$$

So if $x > 6$ the value of $a + b$ cannot drop below 33, and if $x = 5$ the value of $a + b$ cannot drop below $27\frac{1}{2}$ (and hence it cannot drop below 28, as it is a integer).

Since we have already achieved a value of 28 with $x = 3$ and $y = 1$, this itself must be the least possible value.

A graphical view

It is possible to view this problem in graphical terms. We consider the problem posed in the following manner: *Given that a and b are positive integers such that $11 \mid a + 2b$ and $13 \mid a - 2b$, find the least possible value of $a + b$.*

In Figure 1 we have sketched the lines $a+2b = 11k$ for $k = 0, \pm1, \pm2, \pm3, \ldots$ (blue, dashed), and the lines $a - 2b = 13k$ for $k = 0, \pm1, \pm2, \pm3, \pm4, \ldots$ (red, dashed). Points with non-negative integer coordinates which are part of both families of lines have been shown as heavy black dots. These correspond to the pairs (a, b) of non-negative integers such that $11 \mid a + 2b$ and $13 \mid a - 2b$.

To find the solution with the least $a + b$ value we imagine the line $a + b = d$ drawn for increasing values of d (starting with $d = 0$), advancing across the plane; we want the least value of d for which the line passes through one of the heavy dots. It is clear that the point which this line will pass through is the one marked 'Desired point' in the graph.

From the graph we can also make out the next smallest value taken by $a + b$ (after 28). It is clearly $22 + 11 = 33$. And the one after that is $21 + 17 = 38$.

Remark. The graph reveals an important feature of the problem which the purely algebraic solution did not: the fact that the pairs (a, b) of non-negative integers which satisfy the given conditions fall on a family of lines with slope -6. Thus, we have the points $(23, 5)$, $(22, 11)$, $(21, 17)$, ...which lie on the line $6a + b = 143$; the points $(47, 4)$, $(46, 10)$, $(45, 16)$, $(44, 22)$, ...which lie on the line $6a + b = 286 = 2 \times 143$; and so on. (These lines have been shown in green.) Therefore we have the following interesting result which is far from obvious:

If a and b are non-negative integers such that $11 \mid a + 2b$ and $13 \mid a - 2b$, then we have $143 \mid 6a + b$.

It is a nice exercise to prove this property algebraically, without recourse to the graph.

In closing we make the following remark: Graphs — and pictures in general — often allow us to *see* things, to spot properties of various kinds. Once seen, they may be proved rigorously using algebra. But the initial seeing (a crucial first step) is far more difficult to come by if one sticks only to algebra. Herein lie the importance and power of diagrams and well drawn pictures.

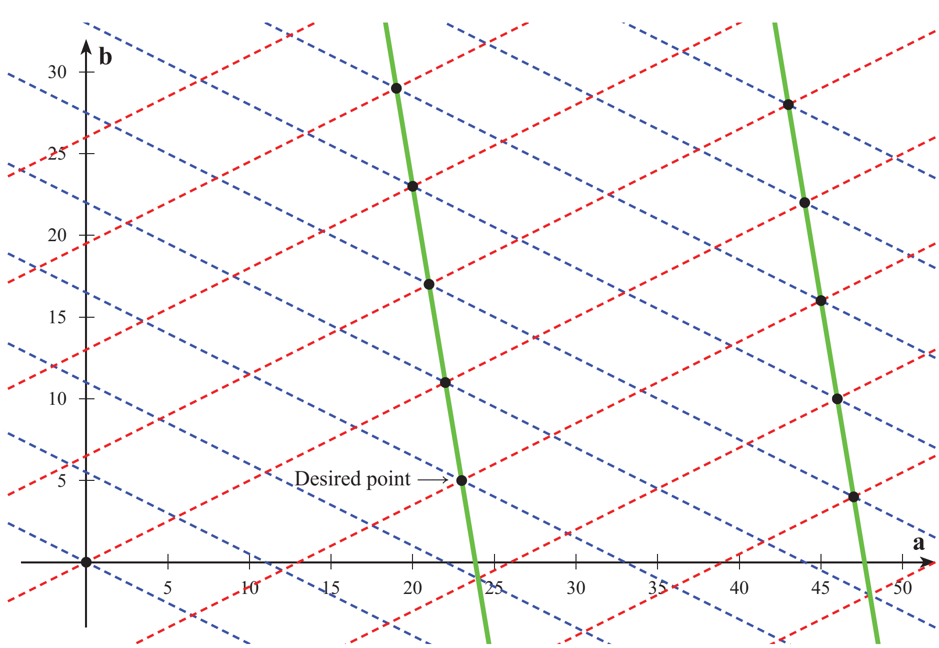


Figure 1.

problem corner

Problems for the Middle School

Problem Editor : R. ATHMARAMAN

Problems for Solution

Problem II-2-M.1
Find all natural numbers n such that the quantity

$$n^4 - 4n^3 + 22n^2 - 36n + 18$$

is a perfect square. (China Western Math Olympiad, 2002)

Problem II-2-M.2
A railway line is divided into 10 sections by the stations $A, B, C, D, E, F, G, H, I, J, K$. The distance from A to K is 56 km. A trip along any two successive sections never exceeds 12 km. A trip along any three successive sections is at least 17 km. What is the distance between B and G? (Swedish Math Contest, 1993)

Problem II-2-M.3
In right angled triangle ABC, with BC as hypotenuse, suppose $AB = x$ and $AC = y$ where x and y are positive integers. Squares $APQB$, $BRSC$ and $CTUA$ are drawn externally on the sides AB, BC and CA, respectively. When QR, ST and UP are joined, a convex hexagon $PQRSTU$ is formed. Let k be its area. Prove that $k \neq 2013$.

Problem II-2-M.4
The numbers $1, 2, 3, \ldots, n$ are arranged in a line in such a way that each number is either strictly bigger than all the numbers to its left, or strictly smaller than all the numbers to its left. In how many ways can this be done? (21-st Canadian Math Olympiad,1989)

Problem II-2-M.5
If a, b, c are real numbers such that $1/a + 1/b + 1/c = 1/(a+b+c)$, show that the following is true for any positive integer n:

$$\frac{1}{a^{2n+1}} + \frac{1}{b^{2n+1}} + \frac{1}{c^{2n+1}} = \frac{1}{a^{2n+1} + b^{2n+1} + c^{2n+1}}.$$

Solutions of Problems in Issue-II-1

Solution to problem II-1-M.1 *Two distinct two-digit numbers a and b are chosen ($a > b$). Their GCD and LCM are two-digit numbers, and a/b is not an integer. What could be the value of a/b?*

Let $c =$ GCD (a, b) and $d =$ LCM (a, b); let $a = a'c$ and $b = b'c$. Then: (i) $a' > b'$; (ii) a', b' are coprime; (iii) $d = a'b'c$; (iv) c, $a'c$, $b'c$, $a'b'c$ lie between 10 and 99; (v) a'/b' is not an integer. Since $c \geq 10$ and $a'b'c \leq 99$ we also have: (vi) $a'b' < 10$. So a', b' are digits.

Applying (i), (ii), (v), (vi) we find that just one pair is left: $(a', b') = (3, 2)$. It follows that $a/b = 3/2$.

We can say more. We have: $a = 3c, b = 2c, d = 6c$. Since $c, 2c, 3c, 6c$ are two-digit numbers, it follows that $10 \le c \le 16$. Hence the possibilities for (a, b) are the following: $(30, 20)$, $(33, 22)$, $(36, 24)$, $(39, 26)$, $(42, 28)$, $(45, 30)$ and $(48, 32)$.

Solution to problem II-1-M.2 *The sum of a list of 123 positive integers is 2013. Given that the LCM of those integers is 31, find all possible values of the product of those 123 integers.*

As the LCM of the numbers is 31, each number is a divisor of 31. As 31 is prime, its only divisors are 1 and 31. Hence each number in the list is 1 or 31. Let the number of 1s in the list be x, and the number of 31s be y. Then $x + y = 123$ and $x + 31y = 2013$. Solving these equations for x and y we get $x = 60$ and $y = 63$. So the list is:

$$\underbrace{1, 1, 1, \ldots, 1, 1}_{\text{60 of these}}, \underbrace{31, 31, 31, \ldots, 31, 31}_{\text{63 of these}}.$$

Solution to problem II-1-M.3 *Let a and b be two positive integers, with $a \le b$, and let their GCD and LCM be c and d, respectively. Given that $a + b = c + d$, show that: (i) a is a divisor of b; (ii) $a^3 + b^3 = c^3 + d^3$.*

Let $a = ca'$ and $b = cb'$; then a', b' are coprime, and $a' \le b'$. As the product of two numbers also equals the product of their GCD and LCM, we have $cd = a'cb'c$, i.e., $d = a'b'c$. Since $a + b = c + d$ it follows that $ca' + cb' = c + ca'b'$, i.e., $a' + b' = 1 + a'b'$. This leads to:

$$a'b' - a' - b' + 1 = 0, \qquad \therefore (a' - 1)(b' - 1) = 0,$$

hence at least one of a', b' equals 1. Since $a' \le b'$, it follows that $a' = 1$. Hence $a = c$, implying that a is a divisor of b, and $d = b$. Both (i) and (ii) now follow.

Solution to problem II-1-M.4 *Let a and b be two positive integers, with $a \le b$, and let their GCD and LCM be c and d, respectively. Given that $ab = c + d$, find all possible values of a and b.*

Since the product of two numbers also equals the product of their GCD and LCM we have $ab = cd$, hence $cd = c + d$. This may be written as $cd - c - d + 1 = 1$, giving $(c - 1)(d - 1) = 1$. Hence $c - 1 = 1 = d - 1$, i.e., $c = 2 = d$. As the GCD and LCM are both equal to 2, the numbers must be 2, 2. That is, $a = 2 = b$.

Solution to problem II-1-M.5 *Let a and b be two positive integers, with $a \le b$, and let their GCD be c. Given that $abc = 2012$, find all possible values of a and b.*

Let $a = ca'$ and $b = cb'$. Then a', b' are coprime. We are told that $abc = 2012$. Hence $a'b'c^3 = 2012$. Now the prime factorization of 2012 is $2012 = 2 \times 2 \times 503$. So we have $a'b'c^3 = 2 \times 2 \times 503$, with $\text{GCD}(a'b') = 1$. Since 2012 is not divisible by a cube larger than 1, it follows that $c = 1$, i.e., a, b are coprime. Since $a \le b$ (given), the possibilities for (a, b) are $(1, 2012)$ and $(4, 503)$.

Solution to problem II-1-M.6 *Let a and b be two positive integers, with $a \le b$, and let their GCD and LCM be c and d, respectively. Given that $d - c = 2013$, find all possible values of a and b.*

Let $a = ca'$ and $b = cb'$; then $d = a'b'c$, so the information given yields: $a'b'c - c = 2013$, i.e., $c(a'b' - 1) = 2013$. Hence c is a divisor of 2013. Now the prime factorization of 2013 is $3 \times 11 \times 61$. Hence the divisors of 2013 are the following: 1, 3, 11, 33, 61, 183, 671, 2013. (There are 8 divisors.) The possibilities are thus:

c	1	3	11	33	61	183	671	2013
$a'b' - 1$	2013	671	183	61	33	11	3	1
$a'b'$	2014	672	184	62	34	12	4	2

Each value of $a'b'$ in the last line leads to possible values of (a, b). If $a'b' = 2$ then $(a', b') = (1, 2)$, so $(a, b) = (2013, 4016)$. If $a'b' = 4$ then $(a', b') = (1, 4)$, so $(a, b) = (671, 2684)$. If $a'b' = 12$ then $(a', b') = (1, 12)$ or $(3, 4)$, so $(a, b) = (183, 2196)$ or $(549, 732)$. And so on — all the possibilities can be thus listed, one by one.

Problems for the Senior School

Problem editors: PRITHWIJIT DE & SHAILESH SHIRALI

problem corner

Problems for Solution

Problem II-2-S.1
A circle has two parallel chords of length x that are x units apart. If the part of the circle included between the chords has area $2 + \pi$, find the value of x.

Problem II-2-S.2
The prime numbers p and q are such that $p + q$ and $p + 7q$ are both perfect squares. Determine the value of p.

Problem II-2-S.3
Determine the value of the infinite series

$$\frac{1}{3^2+1} + \frac{1}{4^2+2} + \frac{1}{5^2+3} + \frac{1}{6^2+4} + \cdots .$$

Problem II-2-S.4
In trapezium $ABCD$, the sides AD and BC are parallel to each other; $AB = 6$, $BC = 7$, $CD = 8$, $AD = 17$. Sides AB and CD are extended to meet at E. Determine the magnitude of $\measuredangle AED$.

Problem II-2-S.5
You are told that the number 27000001 has exactly four prime factors. Find their sum. (Computer solution not acceptable!)

Solutions of Problems in Issue-II-1

Solution to problem I-2-S.1 *Drawn through the point A of a common chord AB of two circles is a straight line intersecting the first circle at the point C, and the second circle at the point D. The tangent to the first circle at the point C and the tangent to the second circle at the point D intersect at the point M. Prove that the points M, C, B, and D are concyclic.* (See Figure 1.)

Two cases are possible: (i) C and D are on the same side of the line joining the centres of the circle. (ii) C and D are on the opposite sides of the line joining the centres of the circle. In both cases we see that $\measuredangle MCD = \measuredangle CBA$ and $\measuredangle MDC = \measuredangle ABD$. Thus $\measuredangle CBD = \measuredangle CBA + \measuredangle ABD = \measuredangle MCD + \measuredangle MDC = 180^\circ - \measuredangle CMD$. Therefore points M, C, B, and D are concyclic.

Solution to problem I-2-S.2 *In triangle ABC, point E is the midpoint of the side AB, and point D is the foot of the altitude CD. Prove that $\measuredangle A = 2\measuredangle B$ if and only if $AC = 2ED$.* (See Figure 2.)

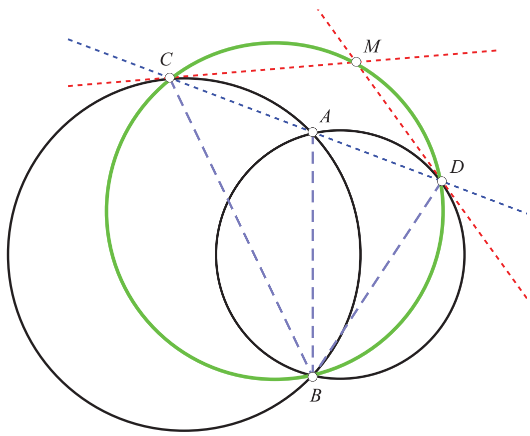


Figure 1. Showing that M, C, B, D are concyclic (Problem I-2-S.1)

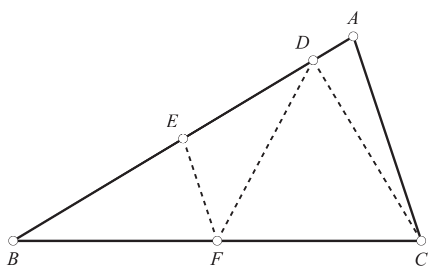


Figure 2. Showing that $\measuredangle A = 2\measuredangle B$ if and only if $AC = 2ED$ (Problem I-2-S.2)

Let F be the midpoint of BC. Therefore $EF \parallel AC$, and $AC = 2EF$. Also in right-angled $\triangle CDB$, F is the midpoint of the hypotenuse BC. Therefore $CF = DF = BF$.

In $\triangle BFD$, $DF = BF$. So $\measuredangle FDB = \measuredangle FBD = \measuredangle B$. Since $EF \parallel AC$, $\measuredangle FEB = \measuredangle A$. But $\measuredangle FEB = \measuredangle FDB + \measuredangle DFE$. That is, $\measuredangle A = \measuredangle B + \measuredangle DFE$. Now

$$\measuredangle A = 2\measuredangle B \iff \measuredangle DFE = \measuredangle B \iff EF = ED \iff AC = 2EF = 2ED.$$

Solution to problem I-2-S.3 *Solve the simultaneous equations: $ab + c + d = 3$, $bc + d + a = 5$, $cd + a + b = 2$, $da + b + c = 6$, where a, b, c, d are real numbers.*

Adding the four equations we obtain

$$(a + c)(b + d) + 2(a + c) + 2(b + d) = 16. \quad (1)$$

Adding the first two equations we obtain

$$(b + 1)(a + c) + 2d = 8. \quad (2)$$

Adding the last two equations gives

$$(d + 1)(a + c) + 2b = 8. \quad (3)$$

Subtracting (3) from (2) yields $(a + c - 2)(b - d) = 0$. Thus either $a + c = 2$ or $b = d$. But, if $b = d$ then $bc + d + a = cd + a + b$ which leads to $5 = 2$, an absurdity. Therefore $a + c = 2$. Now from (1) we get $b + d = 3$. So $c + d = 5 - (a + b)$. Therefore:

$$3 = ab + c + d = ab + 5 - (a + b), \quad (4)$$
$$2 = cd + a + b = (2 - a)(3 - b) + a + b$$
$$= 6 + ab - 2a - b. \quad (5)$$

These lead to:

$$a + b - ab = 2, \quad (6)$$
$$2a + b - ab = 4. \quad (7)$$

From (6) and (7), $a = 2$. Hence $c = 2 - a = 0$. Therefore $b = 2 - a - cd = 0$ and $d = 3 - b = 3$. It is easy to see that these values satisfy the given equations. Therefore $(a, b, c, d) = (2, 0, 0, 3)$.

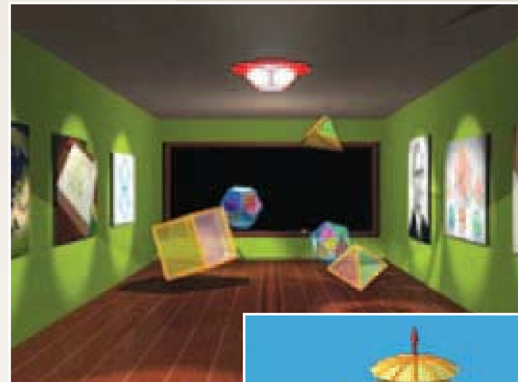
Solution to problem I-2-S.4 *Let x, y, a be positive numbers such that $x^2 + y^2 = a$. Determine the minimum possible value of $x^6 + y^6$ in terms of a.*

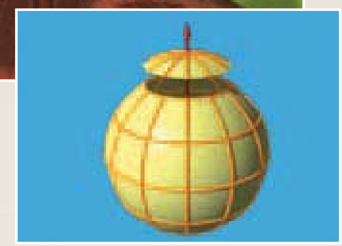
We have: $x^6 + y^6 = (x^2 + y^2)\{(x^2 + y^2)^2 - 3x^2y^2\} = a(a^2 - 3x^2y^2)$. Hence $x^6 + y^6$ attains its minimum value when x^2y^2 attains its maximum value. Since $x^2 + y^2 = a$, the maximum possible value of x^2y^2 is $(a/2)^2 = a^2/4$, with equality attained only when $x = y = \sqrt{a/2}$. Hence the minimum value of $x^6 + y^6$ is $a\left(a^2 - 3a^2/4\right) = a^3/4$.

Solution to problem I-2-S.5 *Let p, q and y be positive integers such that p is greater than q, and $y^2 - qy + p - 1 = 0$. Prove that $p^2 - q^2$ is not a prime number.*

Suppose $p^2 - q^2 = (p + q)(p - q)$ is a prime number. Then $p - q = 1$ and therefore $y^2 - qy + q = 0$. So $q = y^2/(y - 1) = y + 1 + 1/(y - 1)$.

As q and y are integers, so is $1/(y - 1)$. Since $y \geq 1$, it must be that $y = 2$. Hence $q = 4$ and $p = q + 1 = 5$, giving $p^2 - q^2 = 9$, which is not prime. A contradiction. Therefore $p^2 - q^2$ is not a prime number.





Review of 'Dimensions'

review

Dheeraj Kulkarni

The animation movie series "Dimensions" produced by Jos Leys, Étienne Ghys and Aurélien Alvarez is a free-to-download series available at the website http://www.dimensions-math.org. It is a highly refined exposition on the notion of *dimension* and the ideas centered round it. A beautiful combination of step-by-step build-up of mathematical ideas, highly creative use of animation and graphics, accompanied by melodious background music, makes it a unique exposition, perhaps one of the best expositions of mathematics available, and accessible to a wide audience. An appealing device that the authors use is the dramatized narration by actual mathematicians (present and past).

The series consists of nine chapters. Each chapter is just fourteen minutes long and builds on the preceding chapters. The series is densely packed with mathematical ideas and so provides a learning opportunity for students and teachers as well. Teachers can take away a wealth of experience on how to convey mathematical ideas.

We briefly describe the contents of each chapter. Chapter 1 starts with dimension two or 2D. In this chapter, Hipparchus explains that a pair of numbers is enough to describe the position of a point on the sphere. To describe a point on the earth, two parameters, namely the longitude and the latitude, are needed.

This is demonstrated with the help of animation of a tiny plane travelling on the surface of the earth. To draw an earth map using a globe on a table, we draw a ray from the North Pole (NP) to a point on the globe and locate its point of intersection with the plane of the table. This is known as the 'stereographic projection' of the point *p* onto the plane, and the process is known as 'stereographically projecting the sphere onto the plane'. See Figures 1 and 2.

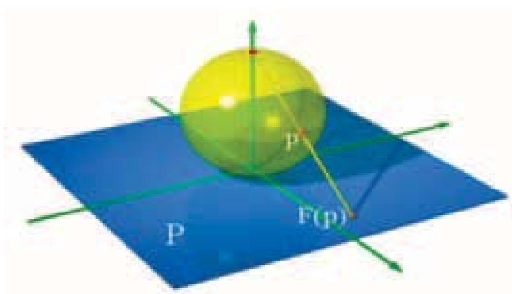


Figure 1. Stereographic projection

The film shows a wonderful animation of stereographic projections. One can see that the North Pole has no stereographic projection on the surface of the table, thus we say that it lies 'at infinity'. Note that stereographic projection does not preserve size. But angles between intersecting lines are preserved, hence directions are preserved. This gives an intuitive idea of the property of 'conformality' of this projection, i.e., 'preserving angles between curves'. The stereographic projection takes meridians to radii emanating from South Pole, and parallels to the concentric circles centered at the South Pole.

Figure 2. Stereographic projection of Earth

In Chapter 2, M.C. Escher tells us how 2D creatures on flat 2D surfaces might imagine 3D objects, giving us a hint on how to imagine 4D objects. The idea is to understand the three dimensions using cross sections on a surface, as though the 3D surface is passing through the flat 2D surface of the table (see Figure 3). To develop an understanding of platonic solids only through cross sections, we use the stereographic projections. By counting the number of the faces and edges and vertices, the creatures on the flat surface can develop an understanding of the 3D platonic solids. This is well demonstrated in the movie. This method of using cross sections helps prepare our imagination to understand the fourth dimension by using 3D cross sections of 4D objects.

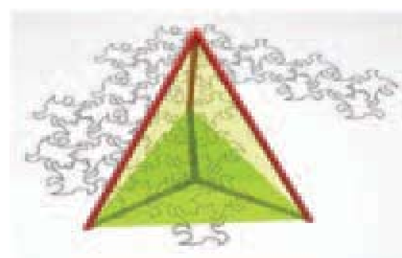


Figure 3. Cross section of a 3D Platonic solid

In Chapters 3 and 4, Swiss mathematician Ludwig Schläfli talks to us about imagining objects in 4D. He was one of the first to study geometry in higher dimensions. He shows us the idea of representing points in higher dimensional space by tuples of real numbers using a magic blackboard. Generalizing the idea of a line segment (a 1D simplex), an equilateral triangle (a 2D simplex) and a regular tetrahedron (a 3D simplex), he obtains a 4D 'simplex' on the board. To get a feel of a 4D object we must see 3D cross sections of the object like the flat creatures did in Chapter 2. The key idea is to describe the number of vertices, edges and 2D and 3D faces to describe the 4D simplex uniquely. In Chapter 4 we see stereographic projections of 4D simplexes to 3D space. We see faces blowing up because of rotation of the simplexes in 4D!

In Chapters 5 and 6, French mathematician Adrien Douady explains the notion of complex numbers and geometrical transformation of the plane in an interesting way.

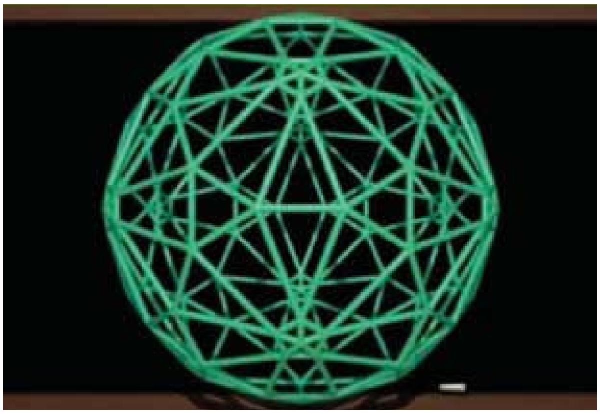


Figure 4. 3D cross section of a "600" simplex in 4D

The square root of −1 may be regarded as ¼ of a full turn since multiplication by −1 may be regarded as a half-turn. Once we understand this idea geometrically, we can visualize the addition and multiplication of complex numbers. This is shown with specific examples.

Combining these two notions – stereographic projection and complex numbers – it is seen that complex numbers are sufficient to describe all points on the sphere except the North Pole.

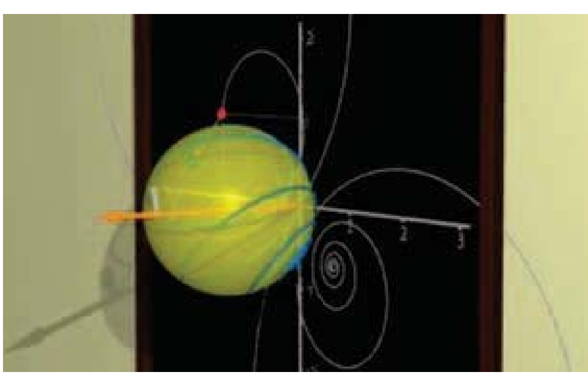


Figure 5. Sphere as a 'Complex Projective Line'

In Chapter 6, Douady describes transformations of the plane – dilations and rotations – using complex numbers. We see Douady's photograph getting dilated and rotated under the transformations. Similarity transformations are neatly encoded by complex numbers. We observe that even after the transformation we recognize Douady! Small shapes such as eyes and buttons preserve their shape. This is the geometric idea behind conformal maps.

The chapter then moves to studying dynamics of iterations, that is, the effects of applying the same transformation repeatedly; we see what happens to sets in the complex plane. This is nicely shown through animation. The famous Mandelbrot set is shown, and fractal structures discussed. One marvels at the beauty of the sets. Through iteration of simple functions one can produce rich and intricate structures.

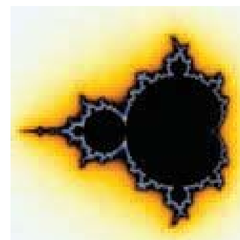


Figure 6. Mandelbrot Set   Figure 7. A Julia Set

In Chapters 7 and 8, mathematician Heinz Hopf describes the strange and non-intuitive idea of a 'fibration' using complex numbers. He shows us a beautiful arrangement of circles which together form a 3D sphere.

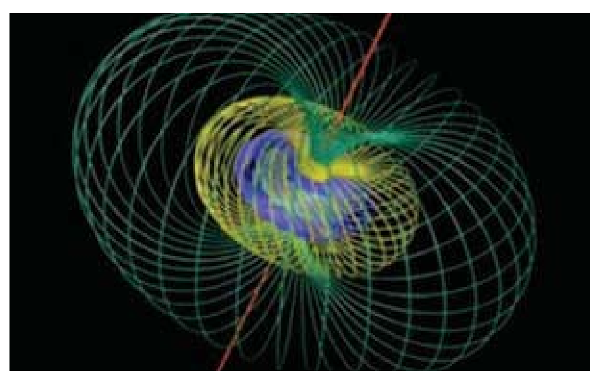


Figure 8. Hopf fibration

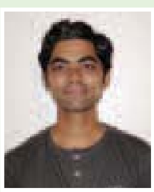
Hopf discovered one of the most important and well understood fibrations, known today as the 'Hopf fibration'. It is a smooth arrangement of circles, no two of which cut one another, forming a 3D sphere; each circle corresponds to a point in the 2D sphere.

In Chapter 8 we get a closer look at the Hopf fibration. The idea of formal proof is presented in Chapter 9 by mathematician Bernhard Riemann.

The chapter details can be found at http://www.dimensions-math.org/Dim_chap_E.htm. All images in this review are from the site http://www.dimensions-math.org/Dim_E.htm.

The author has the following suggestions for teachers. Since the series is densely packed with ideas, the series can be screened over the period of nine weeks, each week devoted to a chapter; the screening may be followed by a discussion of the main ideas. It may be helpful to pause and replay the video to understand the subtlety of ideas and their highly accurate presentations by the creators. Teachers will need to study the background material (found at the link given above) for the chapter before the screening.

For both students and teachers alike, the series will be an enjoyable journey to explore the notion of dimension. It will give a glimpse into how mathematical ideas are developed.



Dheeraj Kulkarni completed his Ph.D. in Mathematics from the Indian Institute Of Science, Bangalore, in 2012. He is currently visiting Georgia Tech, Atlanta, USA as a Post Doctoral Fellow. Apart from academics he enjoys sports such as football, cricket, swimming and long distance cycling. He also enjoys listening to music and going for long walks. He may be contacted at dheeraj.kulkarni@gmail.com.

What Is Your Corner?

A Game for Learning Place Value

Requirements:

Charts, Markers,
Sellotape/Blutack.
The number of participants varies at each level of the game.

Pre-requisite knowledge for players:

1. Addition: up to 4 digit numbers
2. The value of 10, 100, 1000 and the relationships between them (10×10=100, etc.)
3. Multiplication by 10, 100, 1000

In the mathematics class, place value is considered one of the 'hard spots' for the student as well as the teacher. I am glad you addressed this issue in the pullout of Issue II-1 of At Right Angles. Here is a simple game I have used to help children practise their understanding of place value up to four-digit numbers. The objectives of the game are not just to review place value but also to experience the learning of mathematics through physical movement and joint decision-making. Children move about within a room in which each corner has an associated place value, and learn how numbers are read aloud, how 'exchanges' are made between corners, and so on. The details are given below.

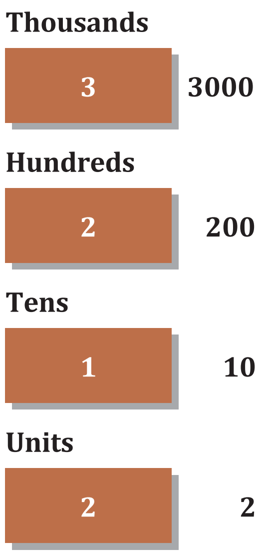
the Game

Level I

- Charts labeled 'Units', 'Tens', 'Hundreds' and 'Thousands' are pasted in the four corners of the room
- At this level, only 9 students should play to avoid having more than 10 students in a corner. This is preferable to informing students that there is a cap on the number of students in each corner.
- Students move about the room freely and when the facilitator claps, each chooses a corner to go to.
- After the students have chosen to stand in the four corners, the facilitator asks: 'What is the value of this corner now?' If there are 5 children in the units corner, their answer should be 5; if there are 5 children in the tens corner the answer should be 50; and so on. The children in each corner should focus on their corner. Of course the children will think about other corners, but the facilitator should ask them about their own corner. Answers given should be approved by the observers and the remaining groups and if the answer is unacceptable, the group which objects should explain their objection and arrive at the correct answer in discussion with the first group.
- At this level the facilitator should help students connect the value they arrive at, to simple multiplication with powers of ten; e.g., 2×10=20, 4×1000=4000, and so on

Level II

- At level II of the game too, only 9 or fewer students play.
- As before, the participants are in the centre of the room and when the facilitator claps they move to any of the four corners.
- After this we ask the students who are observing the game to record what they see in each corner and to use what they learned from level 1 to write on the black board, as shown in the example below:



Next, the observers add the values from the four corners and get 3212. The facilitator should have students play the game several times, changing the roles of the participants and observers. If there are no cases where there are zero children in a particular corner, the facilitator should suggest that they create some such cases so that they can study the meaning of zero in a multiple-digit number.

For example, 4000 + 300 + 0 + 2 = 4302. It is important that the student verbalises this.

Level III

Once the students are familiar with how to 'read' the numbers they have created by standing in corners, they should then go the 'opposite' way. The facilitator calls out a number like 2313, 7486, 2117, 307, 29, etc., and students decide how and where to stand to 'create' that number. The entire class participates at this level.

Level IV

- This level is similar to level II, except that a larger number of students should play so that more than ten students could end up in a corner.
- Apart from the placard bearing the name of the corner, each corner can be provided with unit squares in the units corner, strips of 10 squares in the tens corner, shapes measuring 10×10 in the hundreds corner, and stacks of ten 10×10 squares (held by rubber bands) in the 1000s corner.
- Extra manipulatives are placed in the centre of the room and one student can play the role of an 'exchange centre' or a 'bank'.
- Students can move about anywhere in the room and when the facilitator claps they have to go to any one corner among the four in the room.
- If there are more than 10 students in any corner, 10 of these students will have to take their 10 manipulatives to the 'exchange centre' and exchange it for 1 manipulative of the next higher place. This is taken by one student to the corner marking the next higher place, and the remaining nine students go back to the centre of the room.

- For example, if there are 12 students in the tens corner, then 10 of them take their 10 strips and exchange it for a 10×10 square which one of them takes to the hundreds corner. The remaining 9 students stay at the exchange centre.
- Movement to the next higher corner could cause the number there to increase to more than 10 and this in turn would need a visit to the exchange centre.
- If there are more than 10 students in the thousands corner then the facilitator must explain that the number is referred to as … thousand; e.g., fifteen thousand, if there are 15 students in the thousands corner.
- Once this is understood, then the game proceeds as in Level I and Level II.

RULES

- The game should be cooperative and not competitive; if students in one corner do not know the answer, the facilitator should encourage students from other corners to help them.
- The facilitator should have an element of recording so that some (if not all) configurations are recorded (perhaps with stick figures) in the notebooks. Later this can be represented in terms of squares, strips and grids. It is useful to have a video recording or a series of photos of the exchange process.
- The game can be noisy! While rapid movement is to be encouraged, students should discuss ground rules about pushing each other, shouting too much, and a common signal to 'Freeze!'

Maran is a resource person with the Azim Premji Foundation (Puducherry Field Institute). He has conducted workshops on "how to learn and how to teach" using modules and materials developed by his mentor Fr. Eugine SJ. He has travelled to Sweden and Denmark to support human rights campaigns, and has led a youth team on this project. Maran may be contacted at maran.c@azimpremjifoundation.org

The Closing Bracket . . .

This issue celebrates the life and work of Paul Erdős, one of whose great gifts was his love of working with people of all kinds. He had the uncanny knack of drawing out from different individuals just what they were capable of; choosing just the right level of problem to work on with each person. But this is also a gift that great teachers of mathematics have. In the previous issue of At Right Angles we had featured another great teacher of mathematics, George Pólya, and had presented his famous commandments for math teachers. Erdős would probably never thought of himself as a teacher, yet teaching is what he did through his itinerant activity.

Recently I was listening to a video-recording of a conversation with the mathematician Cornelius Lanczos (1893–1974) who was widely regarded as an outstanding teacher and expositor. (He was originally from Hungary but moved to USA and later to Ireland. He worked on relativity and served for a while as an assistant to Einstein.) Some quotes from that conversation have stayed in my mind. In response to a question posed to him whether outstandingly good lecturers are born and not made, he described how in his early years he was actually a very poor teacher because he was indulging in 'top-down lecturing' without putting himself in his students' shoes. Later when he came to realize how central teaching was to his existence, he evolved a style of teaching in which he prepared his lectures fully and wrote them out long hand before the lecture. When the teaching hour came he could speak freely without his papers, as though extempore, yet there was deep preparation behind it. It is interesting that Richard Feynman who too was an outstandingly successful lecturer prepared his lectures in exactly the same way; indeed, he even prepared his jokes!

In the accounts of these great teachers one sees a common thread: an ability to put oneself in the students' place and see the world through their eyes; an ability and desire to project oneself into the space of the classroom or lecture hall; a desire to reach out and not be limited by one's natural inclinations or by one's personality; a love of play acting (for good teaching is also good theatre!); and a willingness to work very hard. And one sees what a serious business teaching is, and the quality of passion and hard work it demands.

Where are we placed in our country with regard to all this? Do we appreciate how serious an enterprise is good teaching? Is it something we value? Is it a tradition we wish to nurture, to work towards? What will we be prepared to give up in order to have it?

- Shailesh Shirali

Specific Guidelines for Authors

Prospective authors are asked to observe the following guidelines.

1. Use a readable and inviting style of writing which attempts to capture the reader's attention at the start. The first paragraph of the article should convey clearly what the article is about. For example, the opening paragraph could be a surprising conclusion, a challenge, figure with an interesting question or a relevant anecdote. Importantly, it should carry an invitation to continue reading.
2. Title the article with an appropriate and catchy phrase that captures the spirit and substance of the article.
3. Avoid a 'theorem-proof' format. Instead, integrate proofs into the article in an informal way.
4. Refrain from displaying long calculations. Strike a balance between providing too many details and making sudden jumps which depend on hidden calculations.
5. Avoid specialized jargon and notation — terms that will be familiar only to specialists. If technical terms are needed, please define them.
6. Where possible, provide a diagram or a photograph that captures the essence of a mathematical idea. Never omit a diagram if it can help clarify a concept.
7. Provide a compact list of references, with short recommendations.
8. Make available a few exercises, and some questions to ponder either in the beginning or at the end of the article.
9. Cite sources and references in their order of occurrence, at the end of the article. Avoid footnotes. If footnotes are needed, number and place them separately.
10. Explain all abbreviations and acronyms the first time they occur in an article. Make a glossary of all such terms and place it at the end of the article.
11. Number all diagrams, photos and figures included in the article. Attach them separately with the e-mail, with clear directions. (Please note, the minimum resolution for photos or scanned images should be 300dpi).
12. Refer to diagrams, photos, and figures by their numbers and avoid using references like 'here' or 'there' or 'above' or 'below'.
13. Include a high resolution photograph (author photo) and a brief bio (not more than 50 words) that gives readers an idea of your experience and areas of expertise.
14. Adhere to British spellings – organise, not organize; colour not color, neighbour not neighbor, etc.
15. Submit articles in MS Word format or in LaTeX.

Azim Premji University

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NUMBER OPERATIONS: ADDITION

A PRACTICAL APPROACH

PADMAPRIYA SHIRALI

At Right Angles
A Resource for School Mathematics

INTRODUCTION

In the various workshops that I have held with primary school teachers I have often found that teachers do not spend sufficient time on scaffolding exercises and tend to plunge straight into formal operations. Often the pace at which they proceed and the text materials they use are not built up gradually enough to allow time for internalisation. Also the prerequisite knowledge and skills are not looked into adequately. I hope the suggestions made here help teachers to fill these gaps and lay a stronger foundation.

Teaching of number operations is intrinsically linked with teaching of numbers. The number sequence 1, 2, 3, 4, … is produced by addition by 1: $2 + 1$ makes 3, $3 + 1$ makes 4. So a child who has learnt to count implicitly understands the notion of addition as coming together of a given number with 1 or increasing in quantity by 1. Before one attempts to teach children formal addition (that is, usage of the symbol $+$, methods of addition and vertically aligning numbers using place values) one needs to spend a fair amount of time in strengthening their number decomposition skills.

PRE-REQUISITE

Ability to recognise the number in a small set of objects (1 to 6) instantaneously, without resorting to counting; knowledge of the fact that a hand has 5 fingers and both hands together have 10 fingers.

ACTIVITY *ONE* | DECOMPOSITION OF THE NUMBER 5

A simple activity can be done to teach and strengthen decomposition of 5 through the usage of the five fingers on the hand. Hold one finger apart from the other four. Let the children say, '1 and 4 make 5'. Now hold 2 fingers together away from the rest. Let children say, '2 and 3 make 5'. Help them to discover that '5 and 0' is also a decomposition of 5. Let children realize on their own that '1 and 4' could also have been looked at as '4 and 1'. Encourage them to experiment and show different ways of 4 and 1 on one hand. It will help their motor coordination skills and help them to internalize the decomposition facts of 5 which are crucial in the skills needed for mental arithmetic.

ACTIVITY *TWO* | DECOMPOSITION OF THE NUMBER 10

Decomposition of 10 or number complements of 10 can now be introduced by using both the hands. Hold 1 finger up and keep the others down. Let the children say, '1 and 9 make 10'. Now hold 2 fingers up and keep the others down. Let children say '2 and 8 make 10'. Continue in sequence till all combinations are completed (including '10 and 0 make 10').

ACTIVITY *THREE*

Place 10 clips of one colour in a line as shown. Place the straw between the first and second clip. Read off the addition facts: 1 + 9 = 10. Also record it on the board. Now place the straw between the second and third clip. Read off the addition fact: 2 + 8 = 10. Record it under the previously written fact. Continue in this manner till you reach 9 + 1 = 10. It would be interesting to see if any of the children ask what would be recorded if the straw is placed after the last clip or before the first one. Tell them that it would be read as 10 + 0 = 10 or 0 + 10 = 10 as the case may be.

Now show the pattern as recorded on the board.

0+10=10	3+7=10	6+4=10	9+1=10
1+9=10	4+6=10	7+3=10	10+0=10
2+8=10	5+5=10	8+2=10	

Various questions can be posed to bring out properties.

1. What is happening to the numbers on the left side?
2. What is happening to the numbers on the right side?
3. Is 4 + 6 the same as 6 + 4? Are there any other pairs like that?

GAME

Game 1: MAKE 10
Objective: Reinforce complements of 10.

Materials required:
- Two sets of Number Flashcards

Children play this game in pairs. Each child has a number set 1 to 9. One child shows a number (say 6) and the other child must quickly show its complement (4).

GAME

Game 2: SNAP 10

Objective: Instant recognition of complements of 10.

Materials required:

- Three sets of Number Flashcards, 1 to 9 (three extra flash cards of 5 will be needed)

Children play this in fours. The flash cards are mixed up and placed in one pile upside down so that the numbers are not visible. By turns, each child takes the top card off the pile and places it open in front of everyone. If the next card is not a complement of the previous one, the new card is placed on the open card. If the next card removed from the closed pile is a complement of the card on top of the opened pile, whoever says 10 first gets the set of 2 cards. The game continues till all sets are removed. Whoever gets the maximum number of sets is the winner.

ACTIVITY *FOUR* | MASTERING ADDITION OF 5 TO NUMBERS FROM 1 TO 5

Fingers on the hand become an aid again to help children master adding 5 + 1, 5 + 2, 5 + 3, 5 + 4, 5 + 5. The teacher again needs to help the child use his knowledge of the fact that a hand has 5 fingers (so he doesn't count them all over again) and do forward counting from 6 onwards. In fact the child should be in a position to quickly recognise combinations of the 5 fingers on one hand along with fingers on the other hand as 6, 7, 8, 9 and 10.

The usage of hands and fingers is an excellent aid in teaching number decomposition of 5, 10 and addition facts up to 10. But I often see children extending this one to one correspondence for additions like 9+8 and going over fingers repeatedly and thus getting muddled. Sometimes teachers also encourage students in using the finger segments for adding bigger numbers. While it helps in solving the problem this does not lead to enhancement of further learning strategies. It is imperative that the teacher helps the child to use number decomposition skills and rounding numbers to 10 to arrive at the answer. Through repeated exercises, by discovering patterns in numbers addition rules can be discovered and internalised. Addition facts (1+1 to 9+9) can be thus committed to memory.

ACTIVITY *FIVE*

CREATION OF 1 TO 10 ADDITION TABLE TO HELP LEARN ADDITION FACTS

This is an important chart which every child should be encouraged to make and a larger one should be displayed in the class while children are learning additions.

Let the children write all the numbers 1 to 10 in the top row of the chart and all the numbers 1 to 10 in the first vertical column. Write + sign in the left most top corner. Tell the children to fill the addition table row wise with the corresponding sum.

+	1	2	3	4	5	6	7	8	9	10
1	2	3	4	5	6	7	8	9	10	11
2	3	4	5	6	7	8	9	10	11	12
3	4	5	6	7	8	9	10	11	12	13
4	5	6	7	8	9	10	11	12	13	14
5	6	7	8	9	10	11	12	13	14	15
6	7	8	9	10	11	12	13	14	15	16
7	8	9	10	11	12	13	14	15	16	17
8	9	10	11	12	13	14	15	16	17	18
9	10	11	12	13	14	15	16	17	18	19
10	11	12	13	14	15	16	17	18	19	20

What are the ways in which we can break down these facts into manageable sub-goals?

1. Addition by 1 is simple (the succeeding number), so is addition by 2 (skip one number)
2. Addition of numbers 1 to 5 to 5 (i.e., 5+1, 5+2, ..., 5+5)
3. Addition of numbers 1 to 10 to 10 (i.e., 10+1, 10+2, ..., 10+10)
4. Addition of doubles (i.e., 1+1, 2+2, 3+3, ..., 9+9)
5. Addition of numbers differing by 1 (i.e., 7+6 can be done as 6+6+1 or 7+7-1)
6. Number pairs which add up to 10 (complements of 10)
7. Adding numbers to 9 by regrouping (using the fact that 9 is 1 less than 10, so 9+7 to be viewed as 9+1+6, i.e.,10+6)
8. Adding numbers to 8 by regrouping (using the fact that 8 is 2 less than 10, so 8+6 to be viewed as 8+2+4, i.e., 10+4).

ACTIVITY *SIX* | REGROUPING

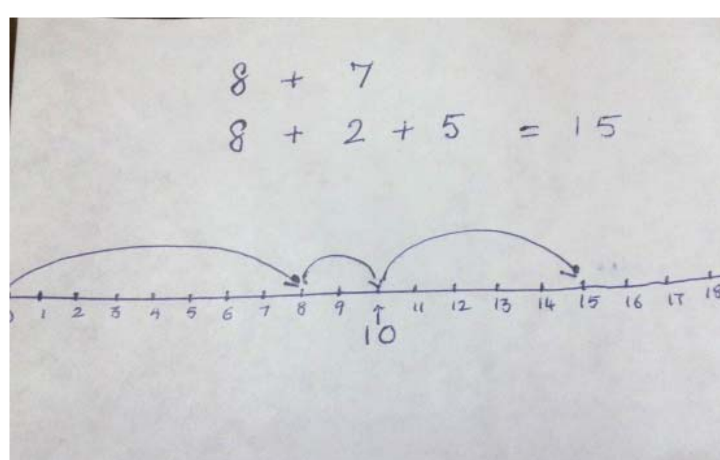
Give the child two number cards, say 8 and 7. Let the child pick up 8 straws and 7 straws and place them separately as two piles, lined up. Ask the child to count the number of straws in the left pile and move a few from the right pile to have 10 in the left pile. Now let the child say '8 and 7 is now 10 and 5', so '8 and 7 is 15'. Repeat this activity with various other number pairs to gain mastery over regrouping to 10 and adding 10 and some other number.

While the child is doing this activity the teacher needs to keep a watch on the following aspects.

1. Does the child recognise instantly the number of straws he has to move from the right side? (That is, check for mastery of decomposition facts of 10.)
2. Once he moves the straws does he recount the pile of 10? (That is, check for mastery of conservation principle.)
3. Is he able to quickly combine 10 and a number and give the right answer?

ACTIVITY *SEVEN* | REGROUPING ON A NUMBER LINE

Additions by regrouping should also be shown using a number line to help the child in building visualisation skills (10 and multiples of 10 are to be prominently displayed on the number line).



ACTIVITY *EIGHT* | ADDITIONS OF MULTIPLES OF TENS

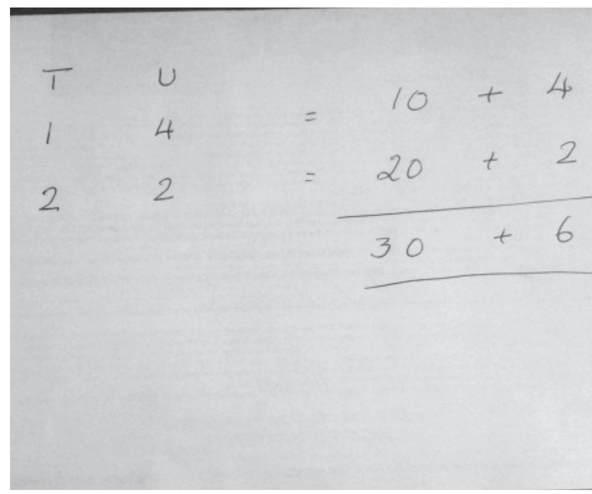
Use straw bundles, tens strips and a number line to show additions of different multiples of tens (20+30, 50+20, etc). Ask one child to pick up 2 tens or 20, and another to pick up 4 tens or 40, and ask for the sum. Use the words 2 tens and 20 interchangeably so that children recognise their equivalence. It must be clearly established that only like ones can be grouped together. While doing additions of multiples of ten, intersperse with questions which require summing of tens and ones. Ex. Take 2 tens and 3 ones. How much is that? Children must learn to pay close attention to the words 'tens' and 'ones'. Place value needs to be emphasised while teaching all the operations.

Complements of 100 in terms of tens: At this point it is also useful to focus on the combinations of multiples of 10 which sum up to 100. Questions can be framed in terms of "I have 3 tens, how many more tens to make 100?" Child must have a clear understanding at this point that 100 is the same as 10 tens. Their knowledge of complements of 10 is applied here as well and will be applied later in finding complements of 1000 in terms of multiples of 100. (Ex. 1000 = 8 hundreds and 2 hundreds.)

ACTIVITY *NINE* | ADDITION OF 3 SINGLE DIGIT NUMBERS

While adding three single digit numbers (7+5+3), the usual method to follow is the left to right sequential approach. Add the first two numbers and then add the third number to the sum. It will be good to also encourage children to look for a pair which sums easily (complements of 10 or doubles etc, and then add the third). This helps children to look at numbers in a flexible manner and use their understanding of arithmetic laws and properties.

ACTIVITY TEN | ADDITIONS OF TENS AND ONES WITH TENS AND ONES WITHOUT REGROUPING



This can be easily demonstrated using tens and ones materials placing them under one another as shown and recording it with place value. As mentioned, the teacher needs to constantly bring it to the child's attention that ones are being added to ones and tens are being added to tens. It is important that in a problem like this, the child does not read it as 4 + 2 = 6 and 1 + 2 =3:

It should be read as 4 ones and 2 ones make 6 ones; 1 ten and 2 tens make 3 tens.

To reinforce this it is also good to write a few problems initially in expanded form.

T U

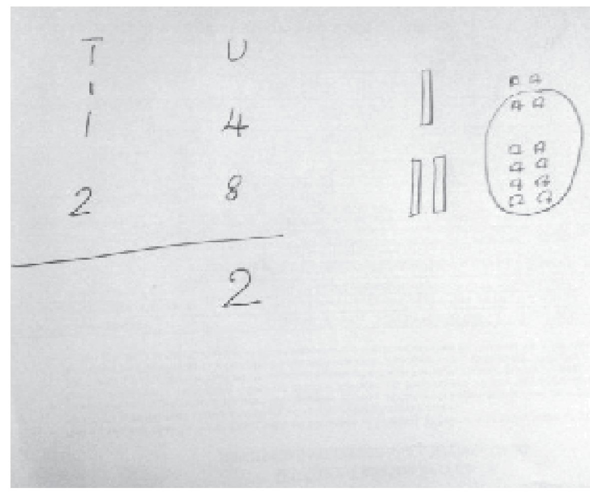
2 4 = 20 + 4

3 5 = 30 + 5

Problem exercises should initially contain visuals. Also, let children use materials (tens and ones) till they gain confidence and drop usage of aids on their own.

Oral arithmetic: While doing additions mentally we often work out left to right, that is add tens and then ones. It is important that we allow children to use their own methods and not restrict them in any way as long as the methods being used are logical.

ACTIVITY ELEVEN | ADDITIONS OF TENS AND ONES WITH REGROUPING



Additions of the kind 14+28 require regrouping of ones into a ten and recording the result appropriately. Let the child pick up tens and ones corresponding to 14 and place it on the place value card. The number should also be recorded using tens and ones as headers. The child then picks up material corresponding to 28 and places it in the second row of the place value card. The second number is also recorded now. Now the child counts the ones and exchanges 10 ones for a ten and the teacher places it on the top in the place value card as shown. At this point the teacher needs to show the correspondence between the recording norms that are followed for addition. In the regrouped ones the ten is indicated by 1 on the top in tens place, and the ones are written underneath.

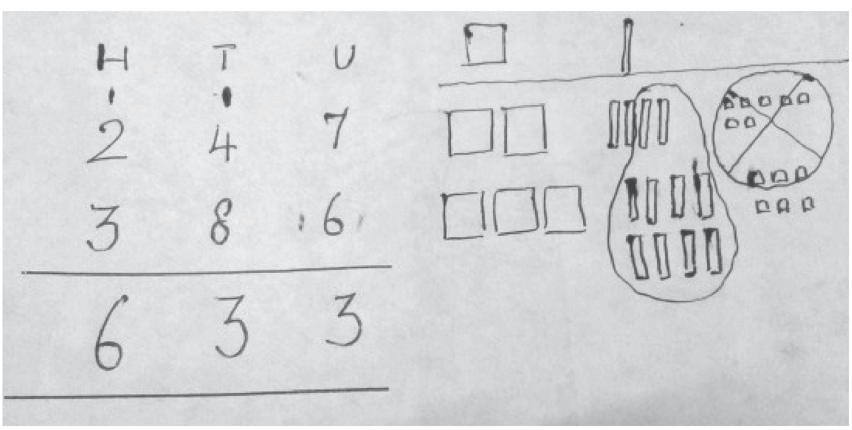
This whole sequence has been spelled out to make the reader see the importance of matching the activity to the writing process. Often I find that activities are performed independently and writing is taken up later. The child does not necessarily see the connection between actions performed in the activity and the process being followed while recording. Activities should be made explicit by verbalizing every action and matching it with writing, it is like a running commentary.

ACTIVITY *TWELVE* | ADDITIONS OF 3 DIGIT NUMBERS WITH REGROUPING

A similar procedure needs to be followed while teaching additions of 3 digit numbers. The place value kit (hundreds, tens and units) needs to be used. However it is important to initially teach regrouping in one place only (say units to tens) e.g., 135+248, and the second stage will then be regrouping tens as hundreds, e.g., 246+172. At the third stage one must carefully introduce regrouping in both the places, e.g., 247+386.

Adding 7 and 6 units gives 13 units, which is regrouped as 1 ten and 3 units. 3 units is recorded under the units place and the regrouped 1 ten is recorded in the tens place, at the top.

Adding 1 ten, 4 tens and 8 tens gives 13 tens, which is regrouped as 1 hundred and 3 tens. 3 tens is recorded under the tens place and the regrouped 1 hundred is recorded in hundreds place, at the top.



In a similar manner additions with regrouping can be done for four-digit numbers. Here too it is advisable to first practice with problems which involve regrouping in just one place, followed by regrouping in two places, and then by regrouping in all the three places.

WORD PROBLEMS AND CHALLENGES

A conscious effort must be made to discuss all the three addition situations:

- *Combining two groups.* Example: 20 children in class 1; 25 children in class 2; how many children in class 1 and 2 together?
- *Increasing a number by another quantity.* Example: 15 children in the bus; 4 more get in; how many children are there now?
- *Finding the required amount to raise a given number to a higher number.* Example: 12 children in the group; 8 pencils; how many more pencils are needed so that each child has one?

Also, one needs to use different contexts and different words which require the numbers to be totalled. Children should also become familiar with the word *sum*, that it denotes the answer to the numbers which have been summed.

It is good to raise the level of challenge for students by giving 'missing number' additions. One can build up problems of this kind in a graded manner as below, where the blanks (or empty squares) need to be filled by digits appropriately.

- 1_ _ + 403 = 529
- 267 + _3_ = 400
- 4_ _ _ + 1257 = 6032

A wide variety of such problems can be devised.

GAME

Game 3: ADDITION MATRIX BOARD

Make a 5 x 5 matrix board with squares of size 2 x 2 cm. Enter the numbers 6 to 10 in random order along both axes as shown. Also make 25 square cards (size 2 x 2 cm) and write the sums on these cards.

Before the game can begin, all the small cards are laid face downward and the time noted. Each player in turn turns the small cards face up one at a time and places them in the correct square as quickly as possible. The time taken to fill the board is noted. The other players can challenge any error and each wrongly placed card earns a five second penalty. The player completing the matrix in the shortest time is the winner.

	8	7	10	9	6
7					
6					
9					
10					
8					

Padmapriya Shirali

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