Azim Premji University

A publication of Azim Premji University
together with Community Mathematics Centre,
Rishi Valley and Sahyadri School, Pune

COMMUNITY MATH CENTRE
RISHI VALLEY SCHOOL (KFI)

At Right Angles

A RESOURCE FOR SCHOOL MATHEMATICS

Volume 2, No. 1
March 2013

Features

Slicing a cube

Sum of Cubes and Square of a Sum

One equation . . . many connects

Harmonic Triples

Tech Space

Exploring geometry problems using Geogebra

In the Classroom

Connecting trigonometry, coordinate geometry, vectors and complex numbers

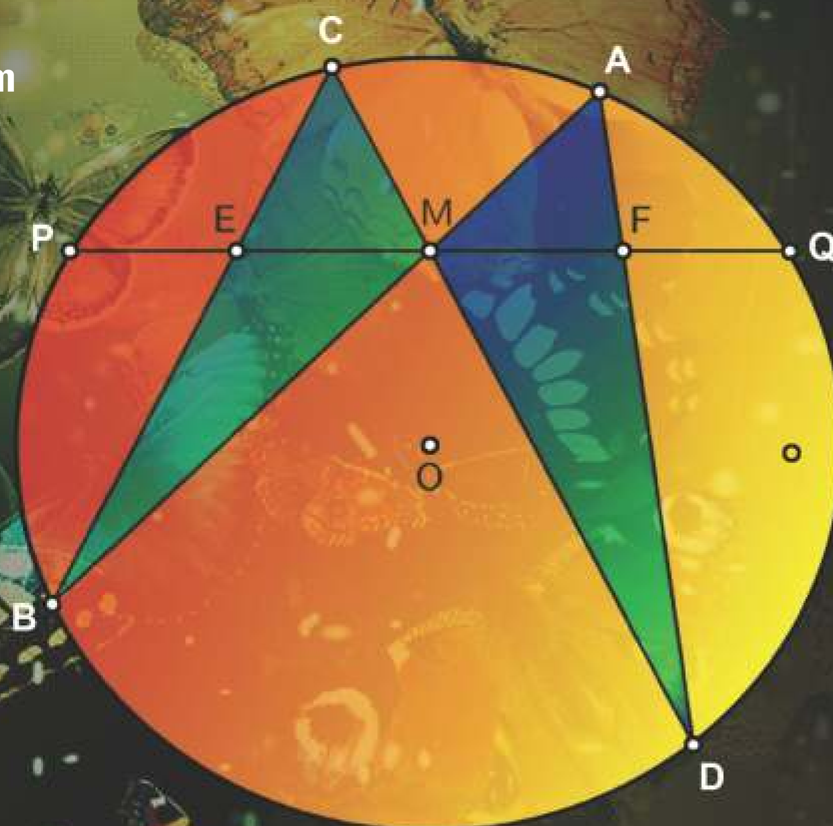
One Problem, Six Solutions

Strategies

George Pólya - In his own words

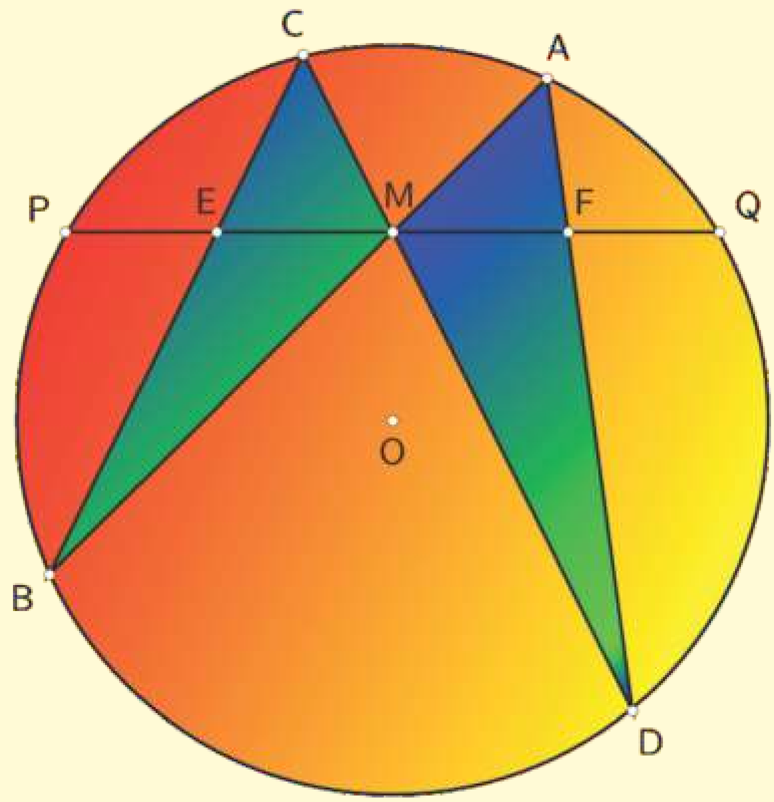
Reviews

When you don't know the solution to a problem



PULLOUT PLACE VALUE

Notes on the Cover Image



The Butterfly Theorem

The figure illustrates a beautiful result called the Butterfly Theorem: In a circle with centre O, let PQ be any chord, and let M be its midpoint. Let AB and CD be chords of the circle passing through M. Let chords BC and AD meet line PQ at points E and F, respectively. Then M is the midpoint of EF as well.

The figure reminds us of a butterfly, and that explains the name of the theorem. It is a challenge to prove the theorem! Numerous mathematicians have succumbed to its lure, and many beautiful proofs as well as extensions of the theorem have been found over the decades. The theorem dates to 1815 when it was published by a mathematician named William Horner, known for a method for solving equations ('Horner's method').

If the sentence "Let chords BC and AD meet line PQ at points E and F, respectively" is replaced by "Let chords BD and AC meet line PQ at points E and F, respectively" (so the points E and F now lie outside the circle), the result remains true! That is, the midpoint of the new segment EF is still M.

From The Chief Editor's Desk . . .

In the December 2012 issue of *At Right Angles* we talked of the 'secret garden' which mathematics possesses. In this issue we feature more offerings from this garden. Shiv Gaur kicks off, showing how to make an Origamic skeletal dodecahedron using paper, with no use being made of scissors or adhesives. Giri Kodur follows by describing how a well known and familiar identity involving the cubes, often used to illustrate proof by induction, generalises in a non-obvious way. B Sury describes a crucial result in the art and science of counting – the principle of inclusion and exclusion, nicknamed 'PIE' – and showcases some of its many implications. Following this we have an article on a lesser known cousin of the Pythagorean triples - 'Harmonic Triples', which too have a geometric origin.

This issue has many pieces featuring the theme of meaningful education. In 'Classroom', J Shashidhar explores the possibilities offered by a small school, in an environment where learning is not driven or motivated by fear, competition, reward and punishment. Following this we have a sample of the writings of George Pólya, in which he expounds in his simple and straightforward way on his 'Ten Commandments to Math Teachers'. The 'Review' section has a review, by K Subramaniam, of one of Pólya's most famous books, and a review of a successful website. Elsewhere in 'Classroom' we learn about a new way to convert from the Celsius scale to the Fahrenheit scale, and about a problem in geometry that can be solved in a half-dozen different ways. Ajit Athle continues on his 'Problem Solving in Geometry' series. In the 'Pullout' for this issue, Padmapriya Shirali offers tips and insights into the teaching of place value.

In 'Tech Space', Sneha Titus and Jonaki Ghosh describe the use of *Geogebra* (software package for Dynamic Geometry) in tackling a problem in geometry. This package which is barely ten years old has already made deep inroads into the educational world; not only is it very well designed and user friendly, it is also freely available *and* Open Source. It is clearly a package with a great future, and this country must exploit its potential to the fullest extent. It potentially has a great role to play in the mathematical education of teachers, but for this to happen, careful deliberation is required by the concerned Government departments and by the community of teacher educators.

We close the issue by describing the contents of a heart-warming letter received from Prof Michael de Villiers of South Africa, which underscores how mathematics is a subject without boundaries in either space or in time: how it can happen that the same discovery can be made in unconnected places at different points in time. It also brings to attention the great importance of encouraging exploration at the school level. The common feature between the occurrences is that both feature the use of Dynamic Geometry software: *Geometer's Sketchpad* in one case, and *GeoGebra* in the other. That, surely, is telling us something.

- Shailesh Shirali

Design & Print

SCPL
Bangalore - 560 062
+91 80 2686 0585
+91 98450 42233
www.scpl.net

At Right Angles is a publication of Azim Premji University together with Community Mathematics Centre, Rishi Valley School and Sahyadri School (KFI). It aims to reach out to teachers, teacher educators, students & those who are passionate about mathematics. It provides a platform for the expression of varied opinions & perspectives and encourages new and informed positions, thought-provoking points of view and stories of innovation. The approach is a balance between being an 'academic' and 'practitioner' oriented magazine.

At Right Angles
A Resource for School Mathematics

Azim Premji University

A Publication of Azim Premji University
together with Community Mathematics Centre,
Rishi Valley and Sahyadri School, Pune

Contents

Features

This section has articles dealing with mathematical content, in pure and applied mathematics. The scope is wide: a look at a topic through history; the life-story of some mathematician; a fresh approach to some topic; application of a topic in some area of science, engineering or medicine; an unsuspected connection between topics; a new way of solving a known problem; and so on. Paper folding is a theme we will frequently feature, for its many mathematical, aesthetic and hands-on aspects. Written by practising mathematicians, the common thread is the joy of sharing discoveries and the investigative approaches leading to them.

In the Classroom

This section gives you a 'fly on the wall' classroom experience. With articles that deal with issues of pedagogy, teaching methodology and classroom teaching, it takes you to the hot seat of mathematics education. 'In The Classroom' is meant for practising teachers and teacher educators. Articles are sometimes anecdotal; or about how to teach a topic or concept in a different way. They often take a new look at assessment or at projects; discuss how to anchor a math club or math expo; offer insights into remedial teaching etc.

At Right Angles
A Resource for School Mathematics

Azim Premji University
Publication of Azim Premji University together with Community Mathematics Centre, Rishi Valley and Sahyadri School, Pune

Contents contd.

Tech Space

'Tech Space' is generally the habitat of students, and teachers tend to enter it with trepidation. This section has articles dealing with math software and its use in mathematics teaching: how such software may be used for mathematical exploration, visualization and analysis, and how it may be incorporated into classroom transactions. It features software for computer algebra, dynamic geometry, spreadsheets, and so on. It will also include short reviews of new and emerging software.

Problem Corner

Reviews

Letter

Poem

Pullout

Paper play

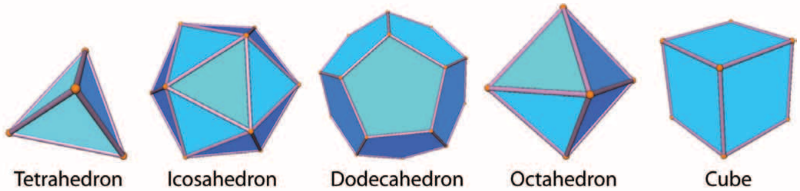
Making a Skeletal Dodecahedron

A solid geometry experience

Vertices, edges, faces, ... so much has been said about them and important connects made between them. But how can a student ever understand these relationships using a 2 dimensional sketch? Constructing your own model personalises the learning in a meaningful and unforgettable way, and nothing can beat that experience.

Shiv Gaur

A dodecahedron is one of the five Platonic or 'regular' polyhedra, and has been known since the times of the ancient Greeks; the other four such polyhedra are the tetrahedron (with 4 triangular faces), the hexahedron (with 6 square faces; better known as a cube!), the octahedron (with 8 triangular faces), and the icosahedron (with 20 triangular faces). The dodecahedron has 12 pentagonal faces. Here are some images of these five polyhedra (source: http://www.ma.utexas.edu/users/rgrizzard/M316L_SP12/platonic.jpg):



What is appealing about all these solids is their high degree of symmetry: their faces are regular polygons, congruent to each other, and at each vertex the same number of edges meet.

So the polyhedron 'looks the same' when seen from above any face. We may associate two numbers with each such solid: m, the number of edges around each face, and n, the number of edges meeting at each vertex (this will equal the number of faces coming together at that vertex). Hence: $(m, n) = (3, 3)$ for a tetrahedron, (4, 3) for a cube, (3, 4) for an octahedron, (3, 5) for an icosahedron, and (5, 3) for a dodecahedron. A dodecahedron has twelve congruent pentagonal faces, with three edges coming together at each vertex.

The numbers indicate certain symmetries that go across the five solids: the cube and octahedron are linked with each other and are said to be duals of each other; so are the icosahedron and dodecahedron. Alone in the family is the tetrahedron, which is self-dual.

In this article we shall show how to make an elegant see-through (skeletal) dodecahedron by first making 30 identical modules which are the building blocks. Later we interlock the modules to form the dodecahedron.

The authorship for the module and design is unknown, and the URL from YouTube where I saw the video for the first time is: http://www.youtube.com/watch?v=JexZ3NIaoEw.

Materials required

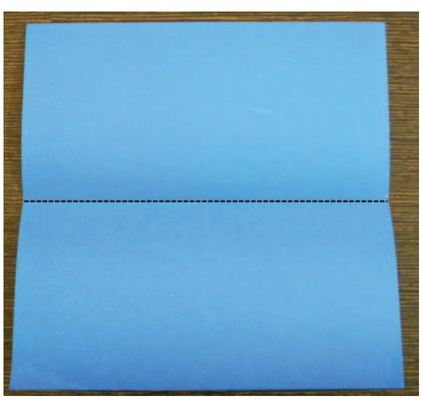
30 coloured square sheets (avoid soft paper which doesn't retain a crease), paper clips, steel ruler

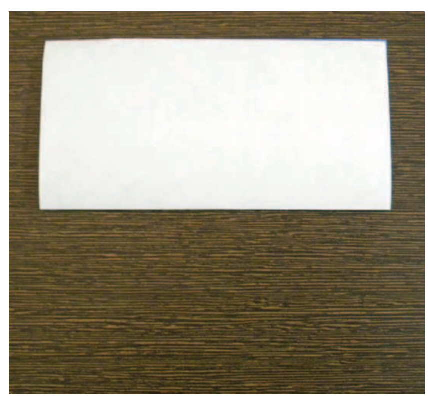
The module

The following are the steps for making a module from one square sheet:

Step 01

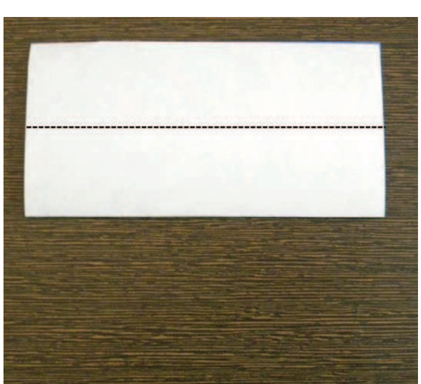
Colour side up, fold the paper in half inwards as shown on the right

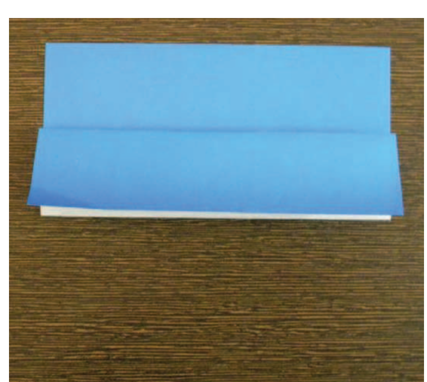




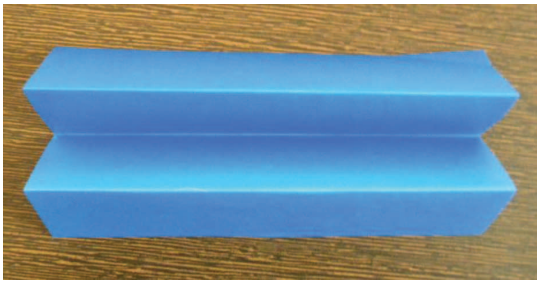
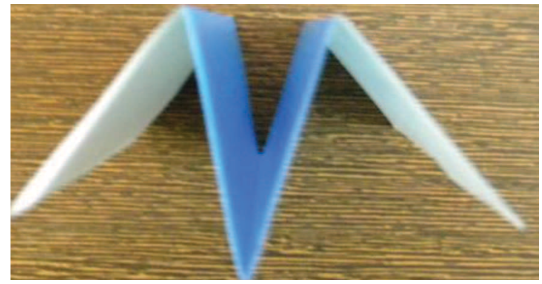
Step 02

Fold both sides in half outwards as shown on the right



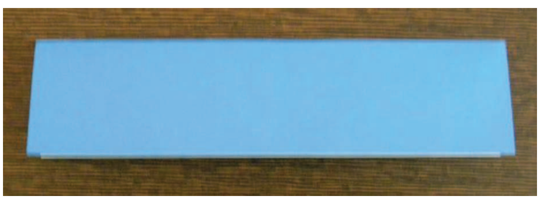
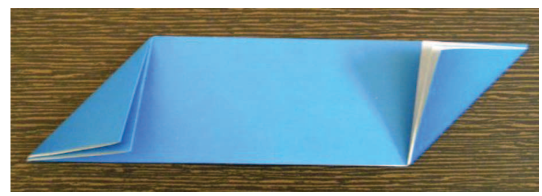


A top and a side view of what you should get at the end of Step 02
(it looks like the letter M from the side):

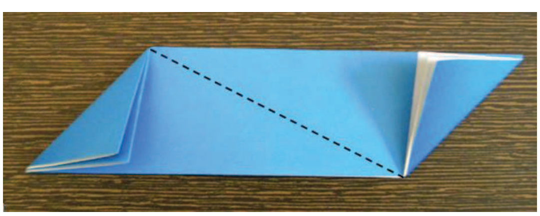
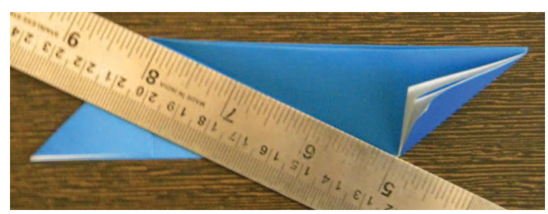
Step 03

Turn the corners inwards as shown in the picture on the right

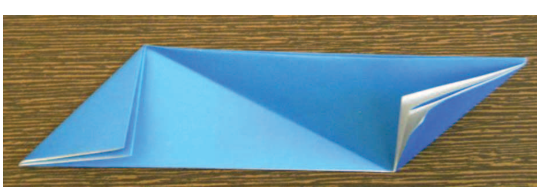
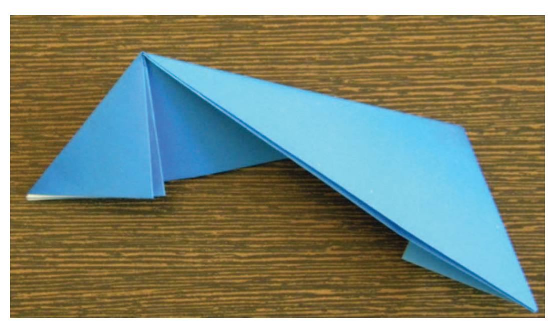



Step 04

Along the dotted diagonal, fold a valley crease (a steel ruler is helpful for accuracy and also speeds up the process).

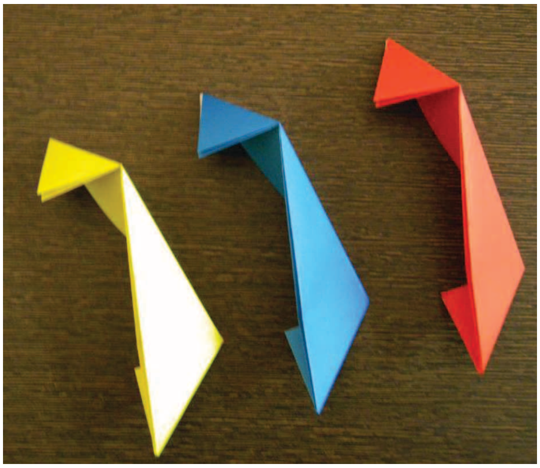



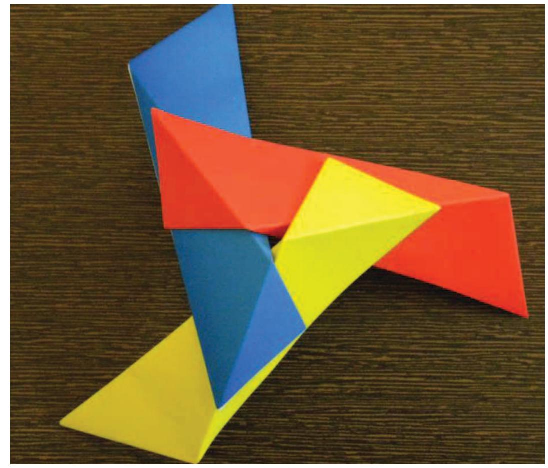
End of step 04 gives us the required module (two views of the same).

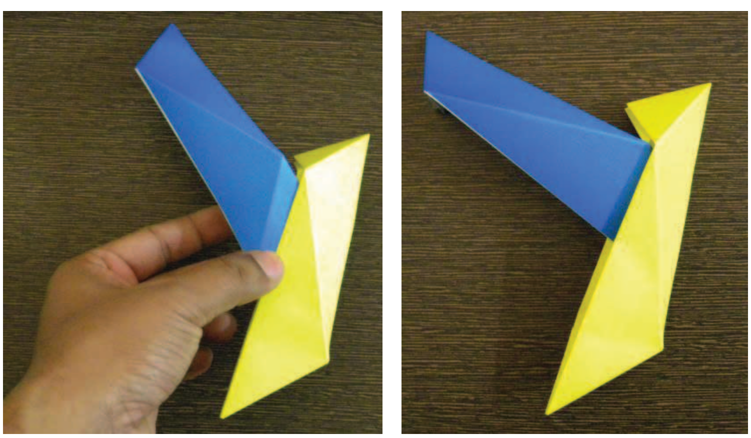
Make 30 modules of different colours as shown above.

The connecting process

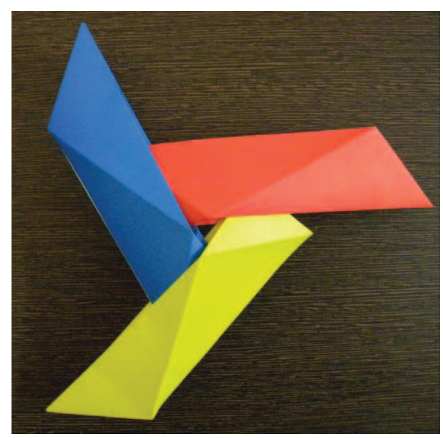




Take 3 modules and keep the corners together. This is the key idea. We need a small mountain and the 3 corners need to go into the side pockets of the neighbouring colours. So the blue corner will go into the yellow side pocket, the yellow corner into the red side pocket, and the red corner into the blue side pocket.

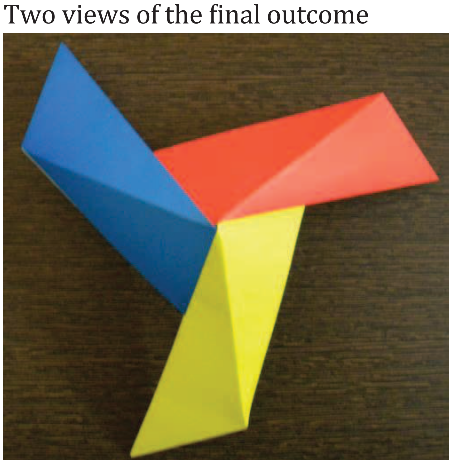


The blue corner being slid into the yellow side pocket.
Do likewise for the red module.

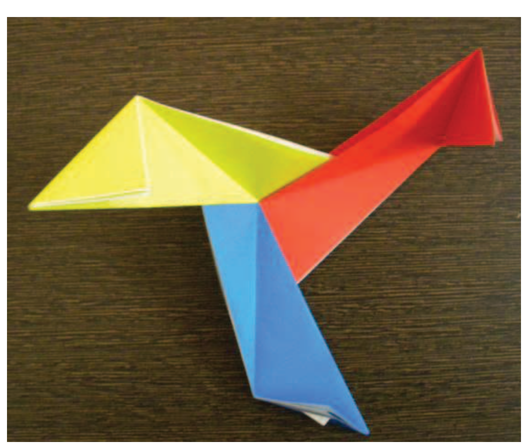


The mountain about to be completed.

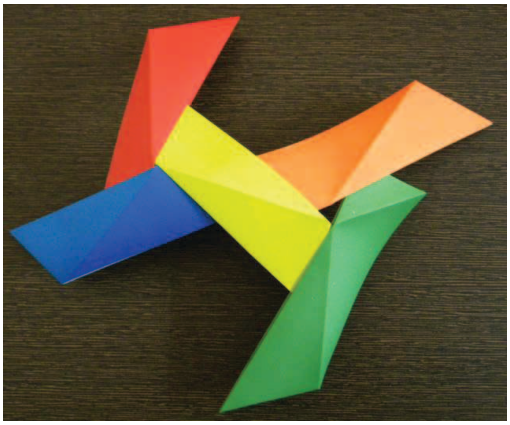
Two views of the final outcome

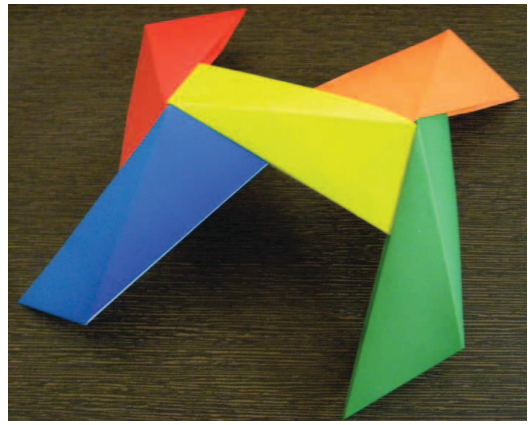


A top view

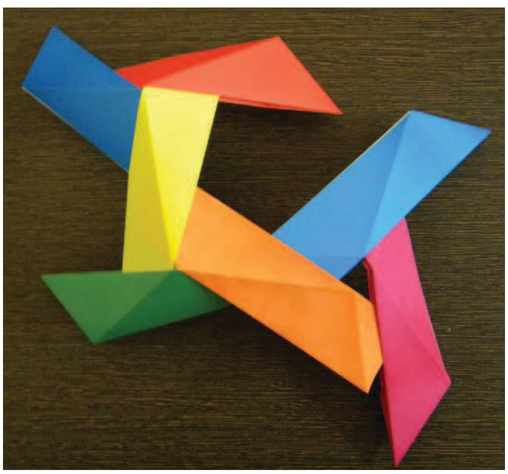


A bottom view

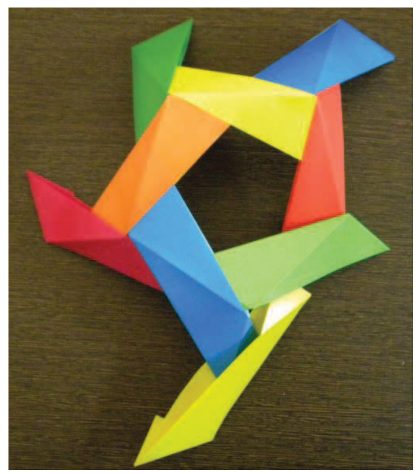




From here onwards we apply the same process to **each loose corner** we see till we get a **pentagon**.



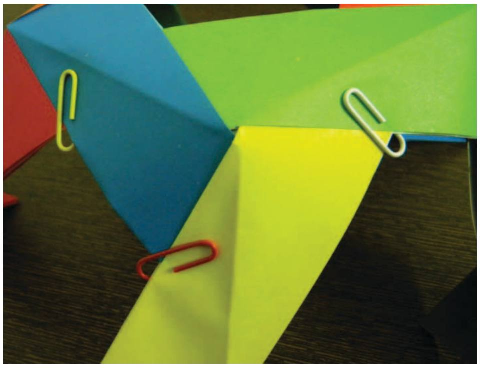
The third mountain being formed



The fifth and the final mountain being formed

The pentagon face finally(A top view)!

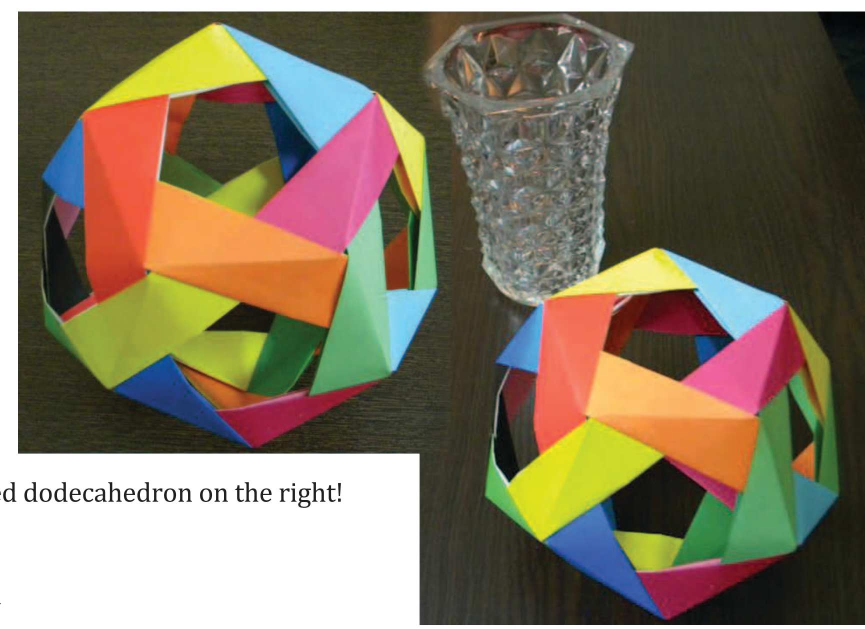
A view from the bottom



At this stage it is a good idea to clip the corners with paper clips. Continue **working pentagons along every length** and a curvature will emerge.

The halfway mark

Close to completion



The last mountain and finally, the completed dodecahedron on the right!

Reference

http://www.youtube.com/watch?v=JexZ3NIaoEw

A B.Ed. and MBA degree holder, SHIV GAUR worked in the corporate sector for 5 years and then took up teaching at the Sahyadri School (KFI). He has been teaching Math for 12 years, and is currently teaching the IGCSE and IB Math curriculum at Pathways World School, Aravali (Gurgaon). He is deeply interested in the use of technology (Dynamic Geometry, Computer Algebra) for teaching Math. His article "Origami and Mathematics" was published in the book "Ideas for the Classroom" in 2007 by East West Books (Madras) Pvt. Ltd. He was an invited guest speaker at IIT Bombay for TIME 2009. Shiv is an amateur magician and a modular origami enthusiast. He may be contacted at shivgaur@gmail.com.

feature

Slicing a cube

Sum of Cubes and Square of a Sum

Understanding your identity

Memorisation is often the primary skill exercised when learning algebraic identities. Small wonder that students tend to forget them well before their use-by date! Here, the sum of cubes identity is unpacked using a series of pictures more powerful than symbols. It doesn't stop there — the article then investigates other sets of numbers for which 'the sum of the cubes is equal to the square of the sum'.

Giri Kodur

The following identity is very well known: for all positive integers n,

(1) $$1^3 + 2^3 + 3^3 + \cdots + n^3 = (1 + 2 + 3 + \cdots + n)^2$$

For example, when $n = 2$ each side equals 9, and when $n = 3$ each side equals 36. The result is seen sufficiently often that one may not quite realize its strangeness. Just imagine: a sum of cubes equal to the square of a sum!

Identity (1) is generally proved using the method of *mathematical induction* (indeed, this is one of the standard examples used to illustrate the method of induction). The proof does what it sets out to do, but at the end we are left with no sense of *why* the result is true.

In this article we give a sense of the 'why' by means of a simple figure (so this is a 'proof without words'; see page 85 of [1]; see also [2]). Then we mention a result of Liouville's which extends this identity in a highly unexpected way.

1. A visual proof

We represent n^3 using a cube measuring $n \times n \times n$, made up of n^3 unit cubes each of which measures $1 \times 1 \times 1$. We now divide this cube into n slabs of equal thickness (1 unit each), by cuts parallel to its base; we thus get n slabs, each measuring $n \times n \times 1$ and having n^2 unit cubes.

When n is odd we retain the n slabs as they are. When n is even we further divide one of the slabs into two equal pieces; each of these measures $n/2 \times n \times 1$. Figure 1 shows the dissections for $n = 1, 2, 3, 4, 5$. Observe carefully the difference between the cases when n is odd and when n is even.

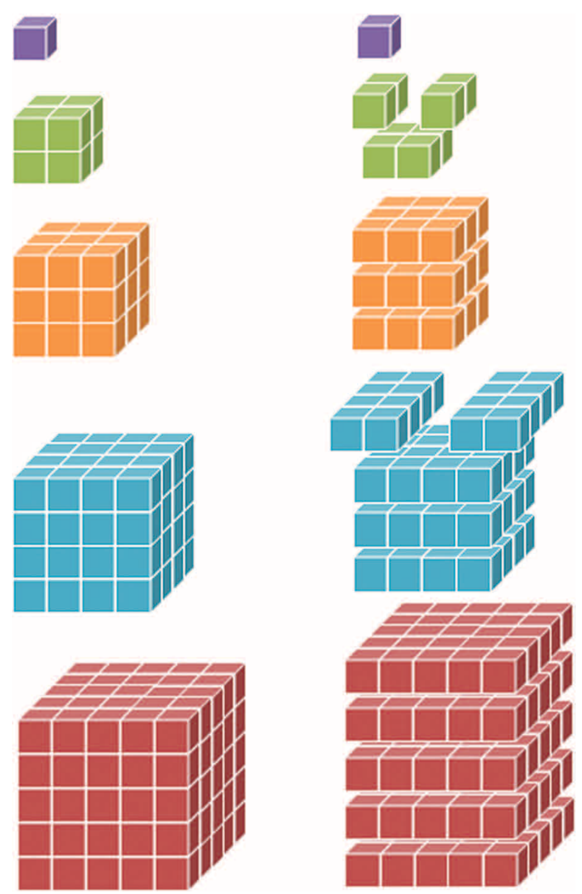


FIGURE 1. Dissecting the cubes into flat slabs (credits: Mr Rajveer Sangha)

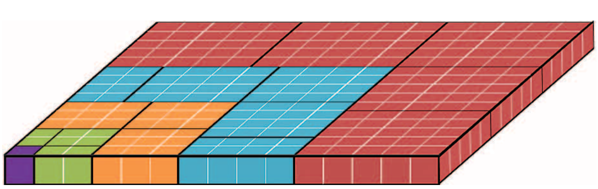


FIGURE 2. Rearranging the slabs into a square shape; note how the odd and even-sized cubes are handled differently (credits: Mr Rajveer Sangha)

Now we take one cube each of sizes $1 \times 1 \times 1$, $2 \times 2 \times 2$, $3 \times 3 \times 3, \ldots, n \times n \times n$, dissect each one in the way described above, and rearrange the slabs into a square shape as shown in Figure 2. (We have shown a slant view to retain the 3-D effect.) Note carefully how the slabs have been placed; in particular, the difference between how the even and odd cases have been handled.

Figure 2 makes it clear 'why' identity (1) is true. For, the side of the square is simply $1 + 2 + 3 + \cdots + n$, and hence it must be that $1^3 + 2^3 + 3^3 + \cdots + n^3 = (1 + 2 + 3 + \cdots + n)^2$.

It is common to imagine after solving a problem that the matter has now been 'closed'. But mathematics is not just about 'closing' problems! Often, it is more about showing linkages or building bridges. We build one such 'extension-bridge' here: a link between the above identity and divisors of integers.

2. A generalization of the identity

First we restate identity (1) in a verbal way: *The list of numbers* 1, 2, 3, . . . , *n has the property that the sum of the cubes of the numbers equals the square of the sum of the numbers*. The wording immediately prompts us to ask the following:

Query. *Are there other lists of numbers with the property that "the sum of the cubes equals the square of the sum"?*

It turns out that there are lists with the SCSS (short for 'sum of cubes equals square of sum') property. Here is a recipé to find them. It is due to the great French mathematician Joseph Liouville (1809–1882), so 'L' stands for Liouville.

L1: Select any positive integer, N.

L2: List all the divisors d of N, starting with 1 and ending with N.

L3: For each such divisor d, compute the number of divisors that d has.

L4: This gives a new list of numbers which has the SCSS property!

The recipé may sound confusing (divisors of divisors! What next, you may ask) so we give a few examples. (In the table, '# divisors' is a short form for 'number of divisors')

Example 1. Let N = 10. Its divisors are 1,2,5,10 (four divisors in all). How many divisors do these numbers have? Here are the relevant data, exhibited in a table:

d	1	2	5	10
Divisors of d	{1}	{1, 2}	{1, 5}	{1, 2, 5, 10}
# divisors of d	1	2	2	4

We get this list: 1, 2, 2, 4. Let us check whether this has the SCSS property; it does:

- The sum of the cubes is $1^3 + 2^3 + 2^3 + 4^3 = 1 + 8 + 8 + 64 = 81$.
- The square of the sum is $(1 + 2 + 2 + 4)^2 = 9^2 = 81$.

Example 2. Let $N = 12$. Its divisors are 1, 2, 3, 4, 6, 12 (six divisors in all). How many divisors do these numbers have? We display the data in a table:

d	1	2	3	4
Divisors of d	{1}	{1, 2}	{1, 3}	{1, 2, 4}
# divisors of d	1	2	2	3

d	6	12
Divisors of d	{1, 2, 3, 6}	{1, 2, 3, 4, 6, 12}
# divisors of d	4	6

This time we get the list 1, 2, 2, 3, 4, 6. And the SCSS property holds:

- The sum of the cubes is $1^3 + 2^3 + 2^3 + 3^3 + 4^3 + 6^3 = 1 + 8 + 8 + 27 + 64 + 216 = 324$.
- The square of the sum is $(1 + 2 + 2 + 3 + 4 + 6)^2 = 18^2 = 324$.

Example 3. Let $N = 36$. Its divisors are 1, 2, 3, 4, 6, 9, 12, 18, 36 (nine divisors). Counting the divisors of these numbers (this time we have not displayed the data in a table) we get the list 1, 2, 2, 3, 4, 3, 6, 6, 9. Yet again the SCSS property holds true:

- The sum of the cubes is $1^3 + 2^3 + 2^3 + 3^3 + 4^3 + 3^3 + 6^3 + 6^3 + 9^3 = 1296$.
- The square of the sum is $(1 + 2 + 2 + 3 + 4 + 3 + 6 + 6 + 9)^2 = 362 = 1296$.

Now we must show that equality holds for each N. The full justification involves a fair bit of algebra; we shall do only the initial part, leaving the rest for you. It turns out that a critical role is played by the prime factorization of N. We consider two cases: (i) N is divisible by just one prime number; (ii) N is divisible by two or more distinct prime numbers.

A key observation which we shall use repeatedly is the following: **A divisor of a positive integer N has for its prime factors only those primes which divide N.** For example, the divisors of a power of 2 can only be powers of 2. If N is divisible by only two primes p and q, then every divisor of N must be made up of the very same two primes.

The case when N is divisible by just one prime number. Rather conveniently, this case turns out to reduce to the very identity with which we started! Suppose that $N = p^a$ where p is a prime number and a is a positive integer. Since the divisors of a prime power can only be powers of that same prime number, the divisors of p^a are the following $a + 1$ numbers:

$$1, p, p^2, p^3, \dots, p^a.$$

How many divisors do *these* numbers have? 1 has just 1 divisor; p has 2 divisors (1 and p); p^2 has 3 divisors (1, p and p^2); p^3 has 4 divisors (1, p, p^2 and p^3); . . . ; and p^a has $a + 1$ divisors. So after **carrying out** Liouville's recipé we get the following list of numbers:

$$1, 2, 3, \dots, a + 1.$$

Does this have the SCSS property? That is, is it true that

$$1^3 + 2^3 + 3^3 + \cdots + (a + 1)^3 = (1 + 2 + 3 + \cdots + (a + 1))^2?$$

Yes, of course it is true! — it is simply identity (1) with $n = a + 1$. And we know that the identity is true. So Liouville's recipé works when $N = p^a$.

We have thus found an infinite class of integers for which the recipé works: all prime powers.

Note one curious fact: the choice of prime p does not matter, we get the same sum-of-cubes relation whichever prime we choose.

The case when N is divisible by just two primes. Let us go step by step, moving from the simplest of cases. Suppose that the only primes dividing N are p and q (where $p \neq q$). We look at a few possibilities.

- $N = pq$: In this case N has four divisors: $1, p, q, pq$. The numbers of divisors that these divisors have are: 1, 2, 2, 4. This list has the SCSS property:

  $1^3 + 2^3 + 2^3 + 4^3 = 81 = (1 + 2 + 2 + 4)^2$.

- $N = pq^2$: In this case N has six divisors: $1, p, q, pq, q^2, pq^2$. The numbers of divisors that these divisors have are: 1, 2, 2, 4, 3, 6. This list too has the SCSS property:

  $1^3 + 2^3 + 2^3 + 4^3 + 3^3 + 6^3 = 324$
  $= (1 + 2 + 2 + 4 + 3 + 6)^2$.

- $N = pq^3$: In this case N has eight divisors: $1, p, q, pq, q^2, pq^2, q^3, pq^3$. The numbers of divisors are: 1, 2, 2, 4, 3, 6, 4, 8. This list has the SCSS property:

  $1^3 + 2^3 + 2^3 + 4^3 + 3^3 + 6^3 + 4^3 + 8^3 = 900$
  $= (1 + 2 + 2 + 4 + 3 + 6 + 4 + 8)^2$.

- $N = p^2q^2$: In this case N has nine divisors: $1, p, p^2, q, pq, p^2q, q^2, pq^2, p^2q^2$. The numbers of divisors are: 1, 2, 3, 2, 4, 6, 3, 6, 9. Yet again the list has the SCSS property:

  $1^3 + 2^3 + 3^3 + 2^3 + 4^3 + 6^3 + 3^3 + 6^3 + 9^3 = 1296$
  $= (1 + 2 + 3 + 2 + 4 + 6 + 3 + 6 + 9)^2$.

We see that the Liouville recipé works in each instance. (As earlier, note that the relations we get do not depend on the choice of p and q. It only matters that they are distinct primes.)

How do we handle all such cases in one clean sweep (i.e., $N = p^a \times q^b \times r^c \times \cdots$ where $p, q, r, \ldots$ are distinct prime numbers, and $a, b, c, \ldots$ are positive integers)? We indicate a possible strategy in the following sequence of problems, leaving the solutions to you.

3. Outline of a general proof

Problem 1: *Suppose that M and N are coprime positive integers. Show that every divisor of MN can be written in a unique way as a product of a divisor of M and a divisor of N.* (Note. This statement is not true if the word 'coprime' is removed.)

For example, take $M = 4$, $N = 15$; then $MN = 60$. Take any divisor of 60, say 10. We can write $10 = 2 \times 5$ where 2 is a divisor of M and 5 is a divisor of N, and this is the only way we can write 10 as such a product.

Problem 2: *Show that if M and N are coprime positive integers, and the divisors of M are $a_1, a_2, a_3, \ldots$ while the divisors of N are $b_1, b_2, b_3, \ldots$, then every divisor of MN is enumerated just once when we multiply out the following product, term by term:*

$$(a_1 + a_2 + a_3 + \cdots) \times (b_1 + b_2 + b_3 + \cdots).$$

For example, to enumerate the divisors of $60 = 4 \times 15$ we multiply out, term by term: $(1 + 2 + 4) \times (1 + 3 + 5 + 15)$, giving us the divisors $1 \times 1 = 1$, $1 \times 3 = 3$, $1 \times 5 = 5$, $1 \times 15 = 15$, $2 \times 1 = 2$, $2 \times 3 = 6$, $2 \times 5 = 10$, $2 \times 15 = 30$, $4 \times 1 = 4$, $4 \times 3 = 12$, $4 \times 5 = 20$ and $4 \times 15 = 60$. Check that we have got all the divisors of 60, once each.

Problem 3: *Show that if M and N are coprime positive integers, and the Liouville recipé works for M and N separately, then it also works for the product MN.*

We invite you to supply proofs of these three assertions. With that the proof is complete; for, the prescription works for prime powers (numbers of the form p^a). Hence it works for numbers of the form $p^a \times q^b$ (where the primes p, q are distinct). Hence also it works for numbers of the form $p^a \times q^b \times r^c$ (where the primes p, q, r are distinct). And so on.

You may wonder: Does Liouville's recipé generate all possible lists of numbers with the SCSS property? Think about it.

Acknowledgement

The author thanks Mr. Rajveer Sangha (Research Associate, Azim Premji University, Bangalore) for the use of figures 1 and 2.

References

[1] Nelsen, Roger B. *Proofs without Words: Exercises in Visual Thinking, Volume 1.* MAA, 1993.

[2] http://mathoverflow.net/questions/8846/proofs-without-words

[3] V Balakrishnan, *Combinatorics: Including Concepts Of Graph Theory* (Schaum Series)

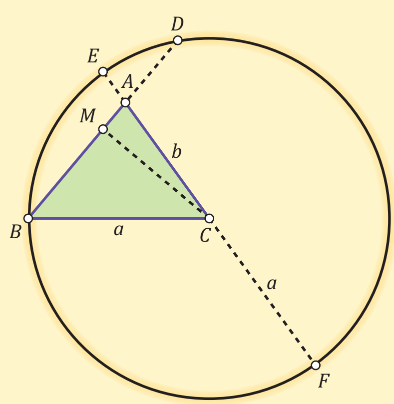
GIRI V KODUR graduated from SATI Vidisha as an Electrical Engineer. He has trained students for various graduate-level competitive examinations but likes to call himself an amateur mathematician. His interest lies in making math education more interactive, interesting and less frightening. He enjoys reading, teaching, learning, travelling, studying nature and photography. He may be contacted at girivkodur@gmail.com.

THE COSINE RULE

by C⊗MαC

In the last issue of this magazine, we saw a proof of the theorem of Pythagoras based on the intersecting chords theorem ("If chords *AB* and *CD* of a circle intersect at a point *P*, then $PA \bullet PB = PC \bullet PD$"). It turns out that the same approach (and very nearly the same diagram) will yield a proof of the cosine rule as well.

Let ΔABC be given; for convenience we take it to be acute angled. Draw a circle with centre *C* and radius *a*; it passes through *B*. Next, extend *BA* to *D*, and *AC* to *E* and *F*, with *D*, *E* and *F* on the circle, as shown. (We have drawn the figure under the assumption that $a > b$.) Let *M* be the midpoint of chord *BD*; then $CM \perp BD$. We now reason as shown.



- $BC = a, CA = b, AB = c$
- $CF = a, EA = a - b$
- $CM \perp BD, \therefore BM = DM$
- $AM = b \cos A$
- $BM = c - b \cos A$
- $DM = c - b \cos A$
- $DA = c - 2b \cos A$
- $FA = a + b, EA = a - b$

Apply the intersecting chord theorem to chords *BD* and *EF*; we get:

$$c(c - 2b\cos A) = (a - b)(a + b),$$

$$\therefore a^2 = b^2 + c^2 - 2bc\cos A,$$

which is the cosine rule applied to side *a* of ΔABC.

We had drawn the figure under the assumption that $a > b$. Please find out for yourself what changes we need to make if instead we have $a < b$ or $a = b$.

feature

Set Theory Revisited

As easy as PIE

The Principle of Inclusion and Exclusion – Part 1

Recall the old story of two frogs from Osaka and Kyoto which meet during their travels. They want to share a pie. An opportunistic cat offers to help and divides the pie into two pieces. On finding one piece to be larger, she breaks off a bit from the larger one and gobbles it up. Now, she finds that the other piece is slightly larger; so, she proceeds to break off a bit from that piece and gobbles that up, only to find that the first piece is now bigger. And so on; you can guess the rest. The frogs are left flat!

B Sury

We are going to discuss a simple but basic guiding principle which goes under the name *principle of inclusion and exclusion*, or PIE for short. Was it inspired by the above tale? Who knows . . . . The principle is very useful indeed, because counting precisely, contrary to intuition, can be very challenging!

An old formula recalled

Here is a formula which you surely would have seen many times: If A and B are two finite, overlapping sets, then

$$|A \cup B| = |A| + |B| - |A \cap B|. \qquad (1)$$

Here, of course, the vertical bars indicate *cardinality*: $|A|$ is the cardinality of (or number of elements in) A, and so on. The formula is rather obvious but may be justified by appealing to the Venn diagram (see Figure 1).

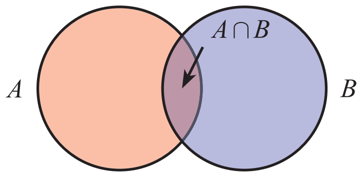


Figure 1

Once one has the basic idea, it is easy to generalize the formula to *three* overlapping finite sets A, B, C. In order to find the cardinality of $A \cup B \cup C$ we start naturally enough with an addition: $|A| + |B| + |C|$. But now several items have been counted twice, and some have even been counted thrice (those that lie in all three sets). So we compensate by subtracting the quantities $|A \cap B|$, $|B \cap C|$ and $|C \cap A|$. But now we have bitten off too much: the items originally in $A \cap B \cap C$ have been left out entirely (see Figure 2). So we compensate by putting these items back in, and now we have the correct formula:

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |C \cap A| + |A \cap B \cap C|. \tag{2}$$

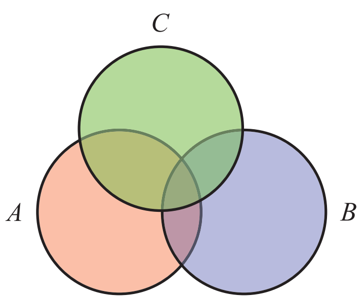


Figure 2

Generalizing the formula

How shall we generalize these formulas? We do so by considering the following problem. Suppose there are N students in a class and a fixed, finite number of subjects which they all study. Denote by N_1 the number of students who like subject #1, by N_2 the number of students who like subject #2, and so on. Likewise, denote by $N_{1,2}$ the number of students who simultaneously like the subjects 1 and 2, by $N_{2,3}$ the number of students who simultaneously like subjects 2 and 3, and so on. Similarly, denote by $N_{1,2,3}$ the number of students who simultaneously like subjects 1, 2, 3; and so on. Now we ask: Can we express, in terms of these symbols, the number of students who do not like *any* of the subjects? (There may well be a few students in this category!) We shall show that this number is given by

$$N - (N_1 + N_2 + \cdots) + \left(N_{1,2} + N_{2,3} + \cdots\right) - \left(N_{1,2,3} + \cdots\right) + \cdots . \tag{3}$$

Note the minus-plus-minus pattern of signs: we alternately subtract to avoid over counting, then add to compensate as we have taken away too much, then again subtract, and so on. The formula follows from a reasoning known as the *principle of inclusion and exclusion*, commonly abbreviated to 'PIE'.

Here is how we justify the formula. We start, naturally, by subtracting $N_1 + N_2 + \cdots$ from N. Now study the expression $N - (N_1 + N_2 + \cdots)$. The subtraction of $N_1 + N_2$ means that we have *twice* subtracted the number of students who like the 1st and 2nd subjects. To compensate for this, we must add $N_{1,2}$. Similarly we must add $N_{1,3}$, $N_{2,3}$, and so on.

However, when we add $N_{1,2} + N_{2,3} + N_{1,3} + \cdots$, we have included those who like the first three subjects (numbering $N_{1,2,3}$) *twice*. So we must subtract $N_{1,2,3}$. Similarly for other such terms. Proceeding this way, we get the right number by alternately adding and subtracting.

Divide and conquer counting

The PIE allows us to solve the following problem in which N is any positive integer. *Among the numbers* 1, 2, 3, . . . , N*, how many are not divisible by either* 2 *or by* 3*?*

Here's how we solve this problem. Among the given numbers the number of multiples of 2 is $[N/2]$. Here the square brackets indicate the *greatest integer function*, also called the *floor function*. The meaning is this: if x is a real number, then $[x]$ is the largest integer not greater than x. For example: $[5] = 5$, $[2.3] = 2$, $[10.7] = 10$, $[\sqrt{10}] = 3$, $[-2.3] = -3$, and so on. (Note the way the definition applies to negative numbers.)

Similarly, the number of multiples of 3 in the set $\{1, 2, 3, \ldots, N\}$ is $[N/3]$. So we subtract both these quantities from N. But the numbers divisible by both 2 and 3 (i.e., the numbers divisible by 6) have been subtracted twice, so we add back the number of multiples of 6, which is $[N/6]$. Hence the answer to the question is:

$$N - \left[\frac{N}{2}\right] - \left[\frac{N}{3}\right] + \left[\frac{N}{6}\right].$$

We solve the following in the same way: *Let* N *be any positive integer. Among the numbers*

$1, 2, 3, \ldots, N$, *how many are not divisible by any of the numbers* $2, 3, 5$?

By alternately "biting away" too much, then compensating, we see that the answer is

$$N - \left[\frac{N}{2}\right] - \left[\frac{N}{3}\right] - \left[\frac{N}{5}\right] + \left[\frac{N}{6}\right] + \left[\frac{N}{10}\right] + \left[\frac{N}{15}\right] - \left[\frac{N}{30}\right].$$

Here 30 is the LCM of 2, 3, 5 (if a number is divisible by 2, 3 and 5 then it must be divisible by 30; and conversely).

The general formula. From this reasoning we arrive at the following general formula. *If* N *is a positive integer, and* $n_1, n_2, \ldots$ *are finitely many positive integers, every two of which are relatively prime, then the number of elements of* $\{1, 2, 3, \ldots, N\}$ *which are not divisible by any of the numbers* $n_1, n_2, \ldots$ *is*

$$N - \left(\left[\frac{N}{n_1}\right] + \left[\frac{N}{n_2}\right] + \cdots\right) + \left(\left[\frac{N}{n_1 n_2}\right] + \left[\frac{N}{n_1 n_3}\right] + \left[\frac{N}{n_2 n_3}\right] + \cdots\right) - \cdots. \qquad (4)$$

You should now be able to provide the formal justification for the formula on your own.

Euler's totient function

There is a special case of the above formula which is of great interest in number theory. We consider the following problem.

For a given positive integer N, *what is the number of positive integers not exceeding* N *which are relatively prime to* N?

The numbers which are relatively prime to N are exactly those which are not divisible by any of the prime divisors of N. Let us denote the primes dividing N by $p, q, r, \ldots$. Now we apply the idea described in the last section. We conclude that the required number is:

$$N - \left(\frac{N}{p} + \frac{N}{q} + \frac{N}{r} + \cdots\right) + \left(\frac{N}{pq} + \frac{N}{qr} + \frac{N}{pr} + \cdots\right) - \cdots. \qquad (5)$$

By factoring out N we find that the resulting expression can be factorized in a convenient manner; we get the following:

$$N\left(1 - \frac{1}{p}\right)\left(1 - \frac{1}{q}\right)\left(1 - \frac{1}{r}\right)\cdots. \qquad (6)$$

For example, take $N = 30$. Since $30 = 2 \times 3 \times 5$, we see that the number of positive integers not exceeding 30 and relatively prime to 30 is

$$30\left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right)\left(1 - \frac{1}{5}\right) = 30 \cdot \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{4}{5} = 8.$$

This is easily checked. (The positive integers less than 30 and relatively prime to 30 are 1, 7, 11, 13, 17, 19, 23 and 29.)

Formula (6) defines the famous *totient function* which we associate with the name of Euler. The symbol reserved for this function is $\varphi(N)$. So we may write:

$$\varphi(N) = N \prod_{p|N} \left(1 - \frac{1}{p}\right), \qquad (7)$$

the product being taken over all the primes p that divide N; that is why we have written '$p \mid N$' below the product symbol. (The symbol $\prod$ is used for products in the same way that $\sum$ is used for sums.)

Corollary: a multiplicative property

The formula for $\varphi(n)$ gives us another property as a bonus — the property that Euler's totient function is *multiplicative*: if m and n are relatively prime positive integers, then $\varphi(mn) = \varphi(m)\varphi(n)$.

Example: Take $m = 4, n = 5, mn = 20$. We have: $\varphi(4) = 2, \varphi(5) = 4$; next, by applying formula (6) we get: $\varphi(20) = 20 \times 1/2 \times 4/5 = 8$. Hence we have $\varphi(20) = \varphi(4) \cdot \varphi(5)$.

It is an interesting exercise to prove this multiplicative property without using formula (6). (It *can* be done, by looking closely at the definition of the function.)

In closing: relation between GCD and LCM

To demonstrate how unexpectedly useful the PIE formula can be, we mention here a nice application of the formula. However we shall leave it as a question without stating the actual result, and discuss the problem in detail in a sequel to this article.

Here is the context. We all know the pleasing formula that relates the GCD ("greatest common divisor", also known as "highest common factor") and the LCM ("lowest common multiple") of any two positive integers a and b:

$$\text{GCD}\,(a, b) \times \text{LCM}\,(a, b) = ab. \tag{8}$$

You may have wondered: The above formula relates the GCD and LCM of *two* integers a, b. What would be the corresponding formula for *three* integers a, b, c? For *four* integers a, b, c, d? ...

In Part II of this article we use the PIE to find a generalization of formula (8). Alongside we discuss a problem about a seemingly absent-minded but actually mischievous secretary who loves mixing up job offers sent to applicants so that every person gets a wrong job offer (for which he had not even applied!), and another problem concerning placement of rooks on a chessboard. And, venturing into deeper waters, we also mention a famous currently unsolved problem concerning prime numbers.

Exercises

(1) Show how the factorization in formula (6) follows from formula (5).
(2) Explain how formula (7) implies that the totient function $\varphi(N)$ is multiplicative.
(3) Let N be an odd positive integer. Prove directly, using the definition of the totient function (i.e., with invoking the property of multiplicativity), that $\varphi(2N) = \varphi(N)$.
(4) What can you say about the family of positive integers N for which $\varphi(N) = N/2$? For which $\varphi(N) = N/3$?
(5) Try to find a relation connecting LCM (a, b, c) and GCD (a, b, c).

Further reading

- V Balakrishnan, *Combinatorics: Including Concepts Of Graph Theory* (Schaum Series)
- Miklos Bona, *Introduction to Enumerative Combinatorics* (McGraw-Hill)

B. SURY got his Ph.D. in Mathematics from TIFR (Mumbai). He has been with the Indian Statistical Institute (Bangalore) since 1999. He has been interested in expository writing at school and college level and in interacting with mathematically talented students. He is the regional co-ordinator for the Math Olympiad in Karnataka and a member of the editorial committee of the newsletter of the Ramanujan Mathematical Society. His research interests are in algebra and number theory. Mathematical limericks are an abiding interest. He may be contacted at sury@isibang.ac.in. His professional webpage is www.isibang.ac.in/~sury.

The quadratic was solved with ease.
The cubic and biquadratic did tease
but were solved not long ere.
It was the quintic which made it clear
that algebra developed by degrees !

– B. Sury

feature

One Equation . . . Many Connects

Harmonic Triples

Part–1

Can the same simple equation be hidden in the relationships between the side of a rhombus and the sides of the triangle in which it is inscribed, the width of a street and the lengths of two ladders crossed over it, and the lengths of the diagonals of a regular heptagon? Read on to find the magic.

SHAILESH A SHIRALI

We are all familiar with the notion of a primitive Pythagorean triple, which is the name given to a triple (a, b, c) of coprime positive integers satisfying the equation $a^2 + b^2 = c^2$; we studied this equation in Issue-I-1 and Issue-I-2 of this magazine. What is pleasing about this equation is its rich connections in both geometry and number theory.

Now there are other equations of this kind which too have nice connections in geometry and number theory. (Not as rich as the Pythagorean equation, but to compare any theorem with the theorem of Pythagoras seems unfair, like comparing a batsman with Bradman . . . .) Here are three such equations: $1/a + 1/b = 1/c$, $1/a^2 + 1/b^2 = 1/c^2$ and $(1/\sqrt{a}) + (1/\sqrt{b}) = (1/\sqrt{c})$.

Remarkably, each of these equations surfaces in some geometric context, and each has something number theoretically interesting about it.

In this three part article we focus on the first of these: $1/a + 1/b = 1/c$, called the *harmonic relation* because of its occurrence in the study of harmonic progressions. (It implies that c is twice the harmonic mean of a and b.) You may recall seeing such relations in physics:

- The relation $1/u + 1/v = 1/f$ for concave and convex mirrors, where u, v, f denote distance of object, image and focus (respectively) from the mirror;
- The relation $1/R_1 + 1/R_2 = 1/R$ for the effective resistance (R) when resistances R_1 and R_2 are in parallel.

There are other occurrences of the harmonic relation in physics. See [1] for a list of more such instances.

If a triple (a, b, c) of positive integers satisfies the equation $1/a + 1/b = 1/c$, we call it a *Harmonic Triple*. Two examples: the triples $(3, 6, 2)$ and $(20, 30, 12)$. As with Pythagorean triples, our interest will be on harmonic triples which have no common factor exceeding 1; we shall call them *primitive harmonic triples*, 'PHT' for short. So $(20, 30, 12)$ is harmonic but not primitive, and $(10, 15, 6)$ is a PHT. (Note one curious feature of this triple: 10 and 15 are not coprime, nor 15 and 6, nor 6 and 10; but 10, 15 and 6 are coprime.)

In Part I of this article we showcase the occurrence of this equation in geometry; we dwell on four such contexts. In Parts II and III (in later issues of *At Right Angles*), we explore the number theoretic aspects of the harmonic relation: how to find such triples, discovering some of their properties, and so on.

1. Triangle with a 120 degree angle

Let $\triangle PQR$ have $\angle P = 120^\circ$. Let PS be the bisector of $\angle QPR$, and let a, b, c be the lengths of PQ, PR, PS respectively (Figure 1). We shall show that $1/a + 1/b = 1/c$.

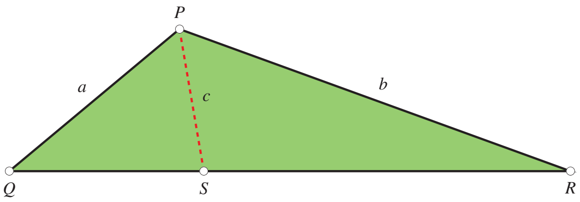


FIGURE 1. Angle bisector in a 120° triangle

The proof involves a computation of areas, using the trigonometric formula for area of a triangle ("half the product of the sides and the sine of the included angle"). Since the area of $\triangle PQR$ equals the sum of the areas of $\triangle PQS$ and $\triangle PSR$, and $\angle QPS = 60^\circ$, $\angle SPR = 60^\circ$, $\angle QPR = 120^\circ$, we have:

$$\frac{1}{2}ac \sin 60^\circ + \frac{1}{2}bc \sin 60^\circ = \frac{1}{2}ab \sin 120^\circ.$$

Now $\sin 60^\circ = \sin 120^\circ$. On cancelling the common factors in the above relation we get $ac + bc = ab$. Dividing through by abc, we get the relation we want right away:

$$\frac{1}{a} + \frac{1}{b} = \frac{1}{c},$$

It is interesting to note the key role played by the equality $\sin 60^\circ = \sin 120^\circ$. (This is just one of *many* results in geometry which depend on this simple equality. In some results a similar role is played by the equality $\cos 60^\circ = -\cos 120^\circ$, or by the equality $\cos 60^\circ = 1/2$.)

You may prefer to see a proof that avoids trigonometry; but we shall turn this question back on you. Try to find such a proof for yourself!

2. Rhombus in a triangle

Given any $\triangle ADB$, we wish to inscribe a rhombus $DPQR$ in the triangle, with P on DB, Q on AB, and R on DA (see Figure 2). It turns out that precisely one such rhombus can be drawn. For now, we shall not say how we can be so sure of this. Instead we ask *you* to prove it and figure out how to construct the rhombus.

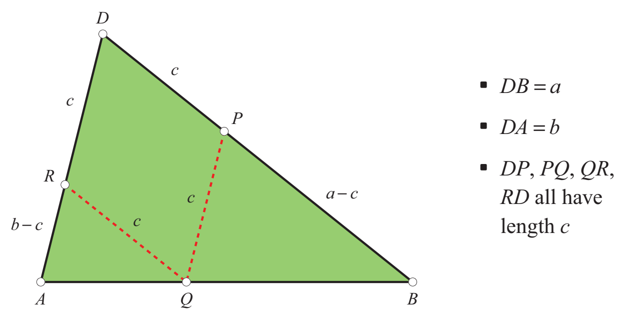


FIGURE 2. Rhombus inscribed in a triangle

In this configuration let the lengths of DB and DA be a and b, and let c be the side of the rhombus (as in the diagram); we shall show that $1/a + 1/b = 1/c$. The proof is quickly found once one notices the similarities $\triangle BPQ \sim \triangle QRA \sim \triangle BDA$, which follow from the relations $PQ \parallel DA$ and $RQ \parallel DB$. These yield the following proportionality relations among the sides:

$$\frac{a-c}{c} = \frac{c}{b-c} = \frac{a}{b}.$$

The second equality yields, after cross-multiplication, $bc = ab - ac$, hence $ac + bc = ab$. On dividing the last relation by abc, we get $1/a + 1/b = 1/c$ as claimed.

3. The crossed ladders

The 'crossed ladders problem' is a famous one. In Figure 3 we see two ladders PQ and RS placed across a street SQ, in opposite ways; they cross each other at a point T, and U is the point directly below T. The problem usually posed is: *Given the lengths of the two ladders, and the height of their point of crossing above the street, find the width of the street.* In one typical formulation we have $PQ = 40$, $RS = 30$, $TU = 12$, and we must find QS. The problem has a deceptive appearance: it looks simple but in fact presents quite a challenge, involving a lot of algebra. For example, see [2] and [3].

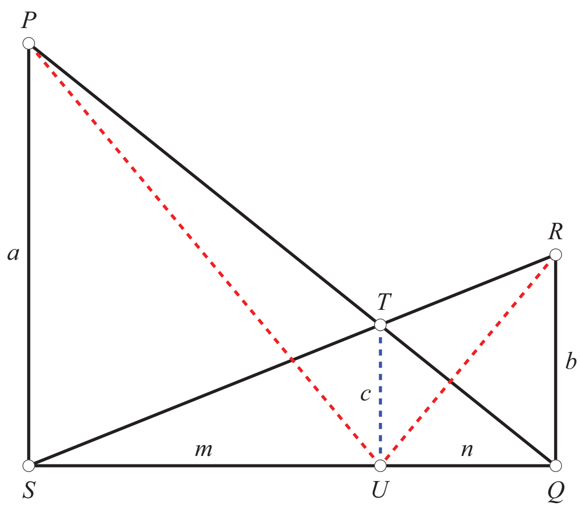


FIGURE 3. The crossed ladders

Our interest here is in something much simpler. Let the lengths of PS, RQ, TU be a, b, c, respectively. Then we claim that $1/a + 1/b = 1/c$.

For the proof we introduce two additional lengths: $SU = m$ and $QU = n$. There are many pairs of similar triangles in the diagram. From the similarity $\triangle PSQ \sim \triangle TUQ$ we get:

$$\frac{a}{m+n} = \frac{c}{n}, \qquad \therefore \ \frac{c}{a} = \frac{n}{m+n}.$$

Next, from the similarity $\triangle RQS \sim \triangle TUS$ we get:

$$\frac{b}{m+n} = \frac{c}{m}, \qquad \therefore \ \frac{b}{c} = \frac{m}{m+n}.$$

Since $m/(m+n) + n/(m+n) = 1$ it follows that $c/a + c/b = 1$, and hence:

$$\frac{1}{a} + \frac{1}{b} = \frac{1}{c}.$$

Remark. There are some unexpected features of interest in Figure 3. For example, the similarity $\triangle PTS \sim \triangle QTR$ yields $a/m = b/n$ (for the ratio of base to altitude must be the same in both the triangles), which implies that $\angle PUS = \angle RUQ$ and hence that $\angle PUT = \angle RUT$. Thus a ray of light proceeding from P to U will be reflected off the street at U towards R.

4. Diagonals of a regular heptagon

The last occurrence of the harmonic relation we shall feature concerns a regular heptagon; i.e., a regular 7-sided polygon. If you examine such a heptagon carefully, you will find just three different lengths within it! — its various diagonals come in just two different lengths, and there is the side of the heptagon. (See Figure 4.)

Let a, b, c be (respectively) the lengths of the longer diagonal, the shorter diagonal, and the side of the heptagon, so that $a > b > c$. Then we find that $1/a + 1/b = 1/c$. For this reason, a triangle with sides proportional to a, b, c (and therefore with angles $720^\circ/7, 360^\circ/7, 180^\circ/7$) is called a *harmonic triangle*. But this time we shall leave the task of proving the harmonic relation to you.

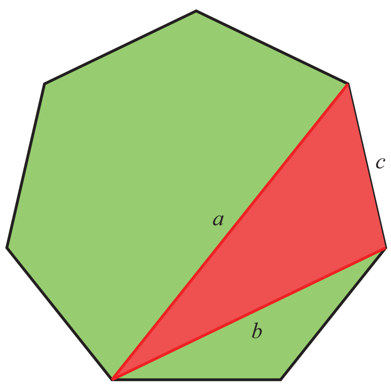


FIGURE 4. A regular heptagon and an inscribed harmonic triangle

In Part II of this article we provide the proof of the above claim and then study ways of generating primitive harmonic triples.

References

[1] James Mertz, *The Ubiquitous Harmonic Relation*, Universities Press, Hyderabad.
[2] http://en.wikipedia.org/wiki/Crossed_ladders_problem
[3] http://mathworld.wolfram.com/CrossedLaddersProblem.html

SHAILESH SHIRALI is Head of the Community Mathematics Centre in Rishi Valley School (AP) and Director of Sahyadri School (KFI), Pune. He has been in the field of mathematics education for three decades, and has been closely involved with the Math Olympiad movement in India. He is the author of many mathematics books addressed to students; serves as an editor for the undergraduate science magazine *Resonance*; and engages in outreach projects in teacher education. He is a keen nature enthusiast and enjoys trekking and looking after animals. He may be contacted at shailesh.shirali@gmail.com

SOLUTION FOR

number crossword-2

by D.D. Karopady

		(1) 2				(2) 8		
	(3) 1	1	(4) 9		(5) 2	1	(6) 7	
(7) 6	0		(8) 9	0	1		(9) 6	4
	(10) 8	(11) 1	9		(12) 2	(13) 3	5	
		2				6		
	(14) 3	1	(15) 2		(16) 1	0	(17) 2	
(18) 4	6		(19) 4	5	9		(20) 4	4
	(21) 2	(22) 2	1		(23) 6	(24) 6	6	
		4				3		

in the classroom

Towards mathematical disposition

Notes from a small school

Supportive learning spaces

Shashidhar Jagadeeshan

If there is one thing mathematicians or math educators are agreed upon, it is that the state of math education the world over is very unsatisfactory. Many schools have poor infrastructure and often an absent teacher. Even in schools with good infrastructure and teachers present, the curriculum is often dry and unimaginative and the textbooks more so. Significant numbers of teachers have a poor understanding of their subject, and are often burdened with large or mixed classes, and administrative duties. Not surprisingly, they seem unmotivated and uninterested in creative solutions. Vast numbers of students dread mathematics. The fear associated with learning mathematics often persists into adulthood. Moreover, many students do not achieve minimum learning standards.

What is more disappointing is that the motivation behind most attempts to reform math education is to create mathematically competent humans who will become part of a 'knowledge society' whose goal is to compete economically with other knowledgeable societies! This approach has not solved anything.

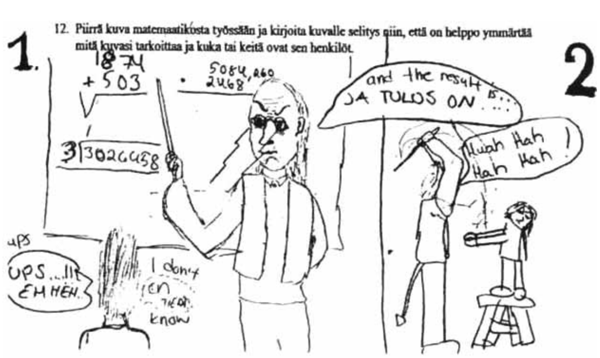
Though the global picture is depressing, at a local scale the situation can be completely different! In this article, I would like to share with you our experience at Centre For Learning in creating an environment where children enjoy learning mathematics.

Centre For Learning (www.cfl.in) is a small school in Bangalore started in 1990 by a group of educators interested in the nature of true learning in all its aspects. Based on my experience as a teacher of mathematics for the last 26 years, 17 of them at CFL, I can say with some certainty that it is possible to create a learning environment where children develop a love for mathematics, and a conceptual understanding that goes beyond textbook problems. Don't misunderstand me – we are not churning out mathematicians by the dozen! My point is the following: I firmly believe that if we are going to make mathematics a core subject in primary education, then we owe it to our students that they find their experience of learning both meaningful and enjoyable.

What is an enabling environment for the learning of mathematics?

At CFL we believe that for meaningful education it is important that we create a space where learning is not motivated by fear, competition, reward and punishment.

Unfortunately the word 'fear' has become indelibly linked with math learning, and the term 'math phobia' has become part of common parlance. A study published in 2000 by Susan Picker and John Berry entitled "Investigating Pupils' Images of Mathematicians" brings home the urgency of the problem. The researchers asked 12- to 13-year old children from the US, UK, Finland, Sweden and Romania to draw a picture of "a mathematician at work". The images are graphic. It comes across clearly that the children have no clue what mathematicians do for a living, but their stereotypical images of mathematics teachers are more damning! They experience themselves as helpless, their mathematics teachers as authoritarian and intimidating, and the learning process as highly coercive.



Source: [1]. Reproduced by kind permission from Dr Susan Picker and Dr John Berry

I am sure we can find countless stories from our contexts where children experience learning mathematics as traumatic.

Given that fear and learning do not and must not go together, we feel that there has to be an environment of learning where the relationship between teacher and student is based on mutual trust and affection. It is important that the child's self-worth is not linked to intellectual ability. These conditions are absolutely necessary because when the opposite prevails—when fear and competition are the main tools of motivation—they do great harm to children and create uncaring and dysfunctional societies.

Source: [1]. Reproduced by kind permission from Dr Susan Picker and Dr John Berry.

Creating a space where fear is not a motivating factor does not automatically eliminate fear in the child. Far from ignoring it, we tackle fear head-on. Students are encouraged to be aware of their fears, to express them and to observe how they may be impacting their learning. In looking for the roots of this insecurity, one may find that it arises because the student has linked his self-worth with the ability to perform. Often it arises because the student isn't confident about his understanding. This can be addressed by the teacher putting more energy and imagination into explaining, and also the student himself working at it.

Doing mathematics can make us acutely conscious of ourselves. It gives us constant feedback about how 'intelligent' we are. This is heightened in a society where ability to calculate quickly is equated with intelligence! Therefore, we need to dialogue not only about fear but also one's images of oneself as a learner. In a supportive environment, a student can recognise the reactions and emotions that block her learning, while at the same time coming to terms with her own strengths and weaknesses. The emphasis thus shifts from performance and self-worth to learning and self-understanding. This allows children to acquire meta-cognitive and self-regulatory skills, two important ingredients in an educational programme, which I will return to later in the article.

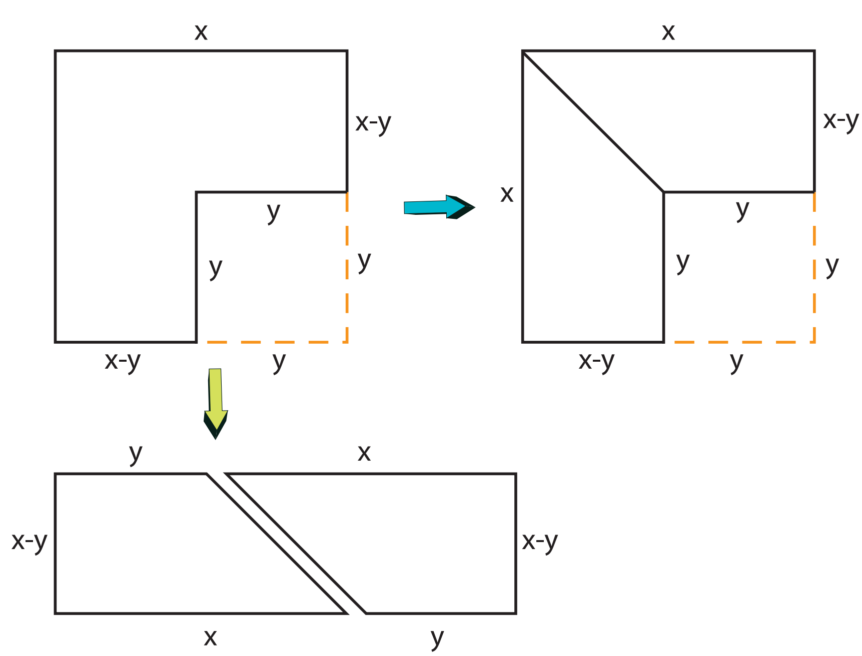
Creating the right environment for learning is necessary, but not sufficient, in meeting the challenges that the learning of mathematics throws up. We have to understand the underlying beliefs and attitudes that teachers and children have about mathematics, and what it takes to become mathematically competent. Let us look at beliefs to begin with.

Epistemological beliefs

Since the 1980s (see [2]) many researchers have studied the link between beliefs and competence in mathematics, so much so that positive beliefs are listed as a criterion for mathematical competence. It would be a very interesting exercise for teachers to jot down their own beliefs about the nature of the subject and why they are teaching it. Let me state the overarching ideas at CFL about the nature and teaching of mathematics.

Mathematics is deep and beautiful, and children should get a taste of this and experience the joy of understanding concepts and the pleasure of making connections. Mathematics can be viewed in many ways: as an art form, as the language of nature, as a tool to model our environment, as a tool for bookkeeping in the world of commerce. Children should be exposed to these different aspects, and no one view should dominate. While problem solving is an important part, there is more to it than that. Children should be exposed to theory building alongside problem solving. Mathematics is not just a body of knowledge but a lively activity consisting of recognizing patterns, making conjectures, and proving the conjectures. Children should learn how to play with ideas and patterns and learn to represent their recognition of these patterns using mathematical notation.

Let me illustrate this with an example.
When teaching the identity $x^2 - y^2 = (x-y)(x+y)$, we can conduct the following simple investigation. Start by asking them to compute the difference of squares $x^2 - y^2$, with $x - y = 1$. Under these conditions, students will soon see that $x^2 - y^2 = x + y$. Do not forget to ask them to state the conditions on x and y! Then ask them to compute the difference of squares $x^2 - y^2$, with $x - y = 2$. Students will soon see that $x^2 - y^2$ in this case is $2(x + y)$. Again, ask them to state the conditions on x and y. Continue to compute difference of squares by changing the value of $x - y$. They will then discover in general that $x^2 - y^2 = (x - y)(x + y)$. One can further reinforce this concept with the following geometric 'proof' (students can be asked to come up their own geometric 'proof').



There are many myths about mathematics that need to be constantly dispelled: the teacher knows everything, mathematics is all about 'calculating', there is only one way to do a problem, if I am not good at mathematics then I must be stupid, and mathematics does not afford experimentation and exploration.

Mathematical Disposition

Once we are clear about an enabling environment and epistemological beliefs, we must be clear what it means when we want our students to be mathematically competent. It is best to look at what math education research has to say in this regard. I have taken the liberty to edit an excerpt from [2] for brevity.

There is currently a consensus among scholars in the field of mathematics education that becoming competent in mathematics can be conceived of as acquiring a ***mathematical disposition****. Building up and mastering such a disposition requires the acquisition of five abilities:*

1. *A well-organized knowledge base involving the facts, symbols, algorithms, concepts, and rules of mathematics*
2. *Heuristic methods, i.e., search strategies for problem solving, which increase the probability of finding the correct solution: for instance, decomposing a problem into sub goals.*
3. *Meta-knowledge, about one's cognitive functioning on the one hand, and about one's motivation and emotions on the other hand (e.g., becoming aware of one's fear of failure when confronted with a complex mathematical task or problem).*
4. *Positive mathematics-related beliefs, about mathematics education, about the self as a learner of mathematics, and about the social context of the mathematics classroom.*
5. *Self-regulatory skills, i.e., one's cognitive processes (planning and monitoring one's problem-solving processes) on the one hand, and skills for regulating one's volitional processes/activities on the other hand (keeping up one's attention and motivation to solve a given problem).*

Categories 1 and 2 have to do with curricular content and delivery; the rest have more to do with attitude and a culture of learning. Experience tells us that some students will achieve much of this mathematical disposition *in spite of* their learning environment! However, our goal, as I mentioned in the introduction, is to help *all* children enjoy the process of acquiring competence in mathematics, and to impart an education concerned with more than acquisition of skills. For this to happen, we consider it vitally important that we create the right learning environment, understand our belief systems, have a coherent curriculum, choose appropriate teaching materials, and pay attention to the process involved in acquiring mathematical competence.

Acquiring Mathematical Disposition

Some of you may say, "All this theory is fine; tell me what happens in the classroom". In our classroom practice, every attempt is made to demonstrate that mathematics is a human endeavour. This is done by talking about the history of mathematics and stories of mathematicians, trying to discover why humans might have needed/developed the mathematics being taught. The classroom environment is kept light, yet rigour is not sacrificed for informality. The students' collective attention has to be steadily focused on what is being learned, the teacher keeping track of each child as the lesson progresses. As and when possible, the teacher will try and connect what appear to be different parts of mathematics, so that the child's learning is not compartmentalised. Teachers spend a significant amount of class time explaining concepts, and children are often called upon to articulate what they have learnt, as far as possible in precise language.

"*... to achieve successful mathematical understanding, we must go beyond telling children how to solve mathematical problems; we must reach a point where children are not only successfully producing mathematical solutions but also understanding why the procedures work and when the procedures are and are not applicable. This point may be reached by providing children with, and requiring that they contribute, to adequate explanations in their mathematics classrooms.*" - Michelle Perry [3]

Students speak as much as, if not more than, the teacher. They spontaneously explain to each other what they have learned, and answer each other's questions. Some comments may seem tangential or even irrelevant to a discussion, but, if followed up, often yield unexpected connections and ways of understanding. The student who finds math easiest is not the star of the math class! Everyone feels equally important in class, in terms of attention, appreciation and affection. Students often work in groups and learn cooperatively, making mathematics a social activity. They engage in thinking together in solving problems and help each other to build the solution without a sense of competition. Along with written work, students do projects, engage in 'thinking stories' and play mathematical games. Mathematics is also part of the overall consciousness of the school, with whole school presentations in mathematics by students and experts..

A teacher burdened with the pressure to complete a syllabus may wonder how this can be done. I think the key here is that, when the emphasis is on understanding material and helping children articulate their learning, though the process in the beginning may take time, once such a culture is established, teachers find that children master many concepts quite easily and so called 'lost' time can easily be made up. In fact, some topics (the more formula oriented or algorithmic ones) can be mastered by students on their own once they are confident about their learning.

Not everything that students do in the mathematics class is formally assessed. In the next section I discuss assessment at CFL in more detail.

Assessment at CFL

At CFL we have children from age 6 to about 18. The teacher to student ratio is 1: 8 at younger ages and in the senior school 1: 4. During the course of their school year they are not subject to exams, quizzes, surprise tests or terminal exams, except at the end of the 10th and 12th standards, when they appear for the IGCSE and A-levels conducted by Cambridge International Examinations.

So then, how does assessment happen at CFL?

First of all we save a tremendous amount of learning time because we don't spend it in preparing, administering and correcting exams. In a small class, teachers are aware of the level of understanding of each student as well as other markers of learning. What has she mastered well, what does she need to work on? What are her study habits, what does she resist? Teachers have also learned how to break down concepts into various components and to figure out difficulties that students have with them. As mentioned earlier, a lot of the teaching consists of discussion, so students receive feedback from each other too. Discussions are of great value, since a teacher's greatest challenge is to enter into and understand a student's world.

We do have written work in the form of assignments; these are corrected but not graded with marks. So students are more focussed on what they have and have not learnt. They are not bothered if their peers are doing better; it is difficult for them to make such a judgment! In correcting and giving detailed feedback, both teacher and student can take corrective action in real time during the course of teaching rather than waiting for the end of a term or a year. So-called mistakes are of tremendous value, because they help understand how a student is thinking, what are the mistaken assumptions, what are the gaps in basic skills and so on.

One area we are working on is to come up with a more quantitative way of recording our observations to pass on to parents, students and teachers. Currently these are communicated in detailed descriptive reports, and face-to-face conversations with parents. We see that we can improve on this. One reason for a lack of urgency in this regard is that we have been able to demonstrate that students can and do learn without comparative and summative assessment and actually do quite well in the school-leaving exams (for example, at the IGCSE so far 83% of the students have a B or higher and at the A-levels 45% get a B or higher)!

Our Challenges

Like all educational environments we face challenges. In fact, when external motivators such as fear, competition, reward and punishment are removed, educators confront the real issues of education. Despite all the thought put into our learning environment, we still encounter resistance to learning. This problem is a human predicament (all of us face it), and our question is how to address it without resorting to the usual tricks.

One question we often ask is: are we adequately challenging the student who is 'gifted' in mathematics? These students enjoy our basic mathematics programme and retain their love and sharp mind for mathematics in high school and beyond. They especially enjoy the projects where they feel stretched, since there is the scope to take on harder challenges. But there is no doubt that such students could have done more or gone much further.

Finally, the biggest challenge: the learner is not a blank disc on which all knowledge can be burned! The learner influences his own learning. Despite a conducive environment and the best of efforts, the learner's complexities (attitudinal, motivational, emotional) can limit his or her learning.

References

[1] 'Investigating Pupils' Images of Mathematicians' by Susan Picker and John Berry, Educational Studies in Mathematics 43, 65-94, 2000, Kluwer Academic Publishers

[2] Unravelling the Relationship Between Students' Mathematics-Related Beliefs and the Classroom Culture. Erik De Corte, Lieven Verschaffel, and Fien Depaepe Center for Instructional Psychology and Technology (CIP&T), University of Leuven, Belgium

[3] Explanations of Mathematical Concepts in Japanese, Chinese, and U.S. First and Fifth-Grade Classrooms. Michele Perry, COGNITION AND INSTRUCTION. J8(2). 181-207 Copyright © 2000. Lawrence Erlbaum Associates, Inc.

[4] Knowing and Teaching Elementary Mathematics: Teachers' Understanding of Fundamental Mathematics in China and the United States, Liping Ma, 1999,Lawrence Erlbaum Associates, Inc.

SHASHIDHAR JAGADEESHAN received his PhD from Syracuse University in 1994. He has been teaching mathematics for the last 25 years. He is a firm believer that mathematics is a very human endeavour and his interest lies in conveying the beauty of mathematics to students and also demonstrating that it is possible to create learning environments where children enjoy learning mathematics. He is the author of Math Alive!, a resource book for teachers, and has written articles in education journals sharing his interests and insights. He may be contacted at jshashidhar@gmail.com

George Pólya -

In his own words

George Pólya was a highly influential mathematician of the 20th century. His research contributions span vast areas of mathematics —complex analysis, mathematical physics, probability theory, geometry, and combinatorics. He was at the same time a teacher par excellence who maintained a strong interest in matters of pedagogy right through his long and richly productive career. Among his widely read books are How To Solve It, Mathematical Discovery, and Mathematics and Plausible Reasoning (two volumes). He also wrote (with Gábor Szegö) the influential two volume series Problems and Theorems in Analysis. We give below a sampler of Pólya's writings on teaching. (In some cases we have taken the editorial liberty to modify the sentences very lightly.)

"What the teacher says in the classroom is not unimportant, but what the students think is a thousand times more important".

George Pólya, 1888–1985

I. Teaching is not a science

Teaching is not a science; it is an art. If teaching were a science there would be a best way of teaching and everyone would have to teach like that. Since teaching is not a science, there is great latitude and much possibility for personal differences. There are as many good ways of teaching as there are good teachers.

In an old British manual there was the following sentence, *Whatever the subject, what the teacher really teaches is himself*. So therefore when I am telling you to teach so or so, please take it in the right spirit. Take as much of my advice as it fits you personally.

II. The aims of teaching

My opinions are the result of long experience. . . . Personal opinions may be irrelevant and I would not dare to waste your time by telling them if teaching could be fully regulated by scientific facts and theories. This, however, is not the case. Teaching is not just a branch of applied psychology.

We cannot judge the teacher's performance if we do not know the teacher's aim. We cannot meaningfully discuss teaching, if we do not agree to some extent about the aim of teaching. I have an old fashioned idea about [the aim of teaching]: first and foremost, it should teach young people to THINK. This is my firm conviction.

If you do not regard "teaching to think" as a primary aim, you may regard it as a secondary aim —then we have enough common ground for the following discussion.

"Teaching to think" means that the teacher should not merely impart information, but should try also to develop the ability of the students to use the information imparted: he should stress knowledge, useful attitudes, desirable habits of mind.

III. The art of teaching

Teaching is not a science, but an art. This opinion has been expressed by so many people so many times that I feel a little embarrassed repeating it. If, however, we leave a somewhat hackneyed generality and get down to appropriate particulars, we may see a few tricks of our trade in an instructive sidelight.

Teaching obviously has much in common with the theatrical art. For instance, you have to present to your class a proof which you know thoroughly having presented it already so many times in former years in the same course. You really cannot be excited about the proof — but, please, do not show that to your class; if you appear bored, the whole class will be bored. Pretend to be excited about the proof when you start it, pretend to have bright ideas when you proceed, pretend to be surprised and elated when the proof ends. You should do a little acting for the sake of your students who may learn, occasionally, more from your attitudes than from the subject matter presented.

Less obviously, teaching has something in common also with music. You know, of course, that the teacher should not say things just once or twice, but three or four times. Yet, repeating the same sentence several times without pause and change may be terribly boring and defeat its own purpose. Well, you can learn from the composers how to do it better. One of the principal art forms of music is "air with variations." Transposing this art form from music into teaching, you begin by saying your sentence in its simplest form; then you repeat it again with a little more colour, and so on; you may wind up by returning to the original simple formulation. Another musical art form is the "rondo." Transposing the rondo from music into teaching, you repeat the same essential sentence several times with little or no change, but you insert between two repetitions some appropriately contrasting illustrative material. I hope that when you listen the next time to a theme with variations by Beethoven or to a rondo by Mozart, you will give a little thought to improving your teaching.

Now and then, teaching may approach poetry, and now and then it may approach profanity . . . . Nothing is too good or too bad, too poetical or too trivial to clarify your abstractions. As Montaigne put it: *The truth is such a great thing that we should not disdain any means that could lead to it.* Therefore, if the spirit moves you to be a little poetical or a little profane in your class, do not have the wrong kind of inhibition.

IV. The nature of the learning process: three principles of learning

Any efficient teaching device must be correlated somehow with the nature of the learning process. We do not know too much about the learning process, but even a rough outline of some of its more obvious features may shed some welcome light upon the tricks of our trade. [Here are] three "principles" of learning.

- **Active learning.** It has been said by many people in many ways that learning should be active, not merely passive or receptive; merely by reading books or listening to lectures or looking at moving pictures without adding some action from your own mind you can hardly learn anything and certainly you can not learn much. There is another often expressed opinion: *The best way to learn anything is to discover it by yourself*. Here is another related quote: *What you have been obliged to discover by yourself leaves a path in your mind which you can use again when the need arises*. Less colourful but perhaps more widely applicable is the following statement: *For efficient learning, the learner should discover by himself as large a fraction of the material to be learned as is feasible under the given circumstances.*
- **Principle of best motivation.** Learning should be active, we have said. Yet the learner will not act if he has no motive to act. He must be induced to act by some stimulus, by the hope of some reward, for instance. The interest of the material to be learned should be the best stimulus to learning and the pleasure of intensive mental activity should be the best reward for such activity. Yet, where we cannot obtain the best we should try to get the second best, or the third best, and less intrinsic motives of learning should not be forgotten.
- **Consecutive phases.** Here is an oft quoted piece from Kant: *Thus all human cognition begins with intuitions, proceeds from thence to cognitions, and ends with ideas.*

  I am not able (who is?) to tell you in what exact sense Kant intended to use these terms. [So] I beg your permission to present my reading of Kant's dictum: *Learning begins with action and perception, proceeds from thence to words and concepts, and should end in desirable mental habits.*

  So for efficient learning, an exploratory phase should precede the phase of verbalization and concept formation and, eventually, the material learned should be merged in, and contribute to, the integral mental attitude of the learner.

I think that these three principles can penetrate the details of the teacher's daily work and make him a better teacher. I think too that these principles should also penetrate the planning of the whole curriculum, the planning of each course of the curriculum, and the planning of each chapter of each course.

Yet it is far from me to say that you must accept these principles. These principles proceed from a certain general outlook, from a certain philosophy, and you may have a different philosophy. Now, in teaching as in several other things, it does not matter what your philosophy is or is not. It matters more whether you have a philosophy or not. And it matters very much whether you try to live up to your philosophy or not. The only principles of teaching which I thoroughly dislike are those to which people pay only lip service.

V. On problem solving

A great discovery solves a great problem but there is a grain of discovery in the solution of any problem. Your problem may be modest; but if it challenges your curiosity and brings into play your inventive faculties, and if you solve it by your own means, you may experience the tension and enjoy the triumph of discovery. Such experiences at a susceptible age may create a taste for mental work and leave their imprint on mind and character for a lifetime.

Thus, a teacher of mathematics has a great opportunity. If he fills his allotted time with drilling his students in routine operations he kills their interest, hampers their intellectual development, and misuses his opportunity. But if he challenges the curiosity of his students by setting them problems proportionate to their knowledge, and helps them to solve their problems with stimulating questions, he may give them a taste for, and some means of, independent thinking.

A good teacher should understand and impress on his students the view that no problem whatever is completely exhausted. There remains always something to do; with sufficient study and penetration, we could improve any solution, and, in any case, we can always improve our understanding of the solution.

VI. Ten commandments for teachers

On what authority are these commandments founded? Dear fellow teacher, do not accept any authority except your own well-digested experience and your own well-considered judgement. Try to see clearly what the advice means in your particular situation, try the advice in your classes, and judge after a fair trial.

1. *Be interested in your subject.* There is just one infallible teaching method: if the teacher is bored by his subject, his whole class will be infallibly bored by it.
2. *Know your subject.* If a subject has no interest for you, do not teach it, because you will not be able to teach it acceptably. Interest is an indispensable necessary condition; but, in itself, it is not a sufficient condition. No amount of interest, or teaching methods, or whatever else will enable you to explain clearly a point to your students that you do not understand clearly yourself.

   Between points #1 and #2, I put interest first because with genuine interest you have a good chance to acquire the necessary knowledge, whereas some knowledge coupled with lack of interest can easily make you an exceptionally bad teacher.
3. *Know about the ways of learning: the best way to learn anything is to discover it by yourself.*
4. *Try to read the faces of your students, try to see their expectations and difficulties, put yourself in their place.*
5. *Give them not only information, but "know-how", attitudes of mind, the habit of methodical work.*
6. *Let them learn guessing.*
7. *Let them learn proving.*
8. *Look out for such features of the problem at hand as may be useful in solving the problems to come — try to disclose the general pattern that lies behind the present concrete situation.*
9. *Do not give away your whole secret at once, let the students guess before you tell it; let them find out by themselves as much as is feasible.* Voltaire expressed it more wittily: The art of being a bore consists in telling everything.
10. *Suggest it, do not force it down their throats.* In other words: Let your students ask the questions; or ask such questions as they may ask for themselves. Let your students give the answers; or give such answers as they may give by themselves. At any rate avoid asking questions that nobody has asked, not even yourself.

Comment from the editors

There is great scope for developing these ideas. For example, take points #5 and #6 in the above list: Give them not only information, but know-how, attitudes of mind, the habit of methodical work; let them learn guessing. What are good mathematical habits? What are good attitudes of mind when it comes to teaching-learning mathematics? Why is 'guessing' important? What is mathematical know-how? We invite responses from the readers on these and related issues.

Recommended books by George Pólya on math teaching and math education

1. How to Solve It: A New Aspect of Mathematical Method
2. Mathematics and Plausible Reasoning: Volume I, Induction and Analogy in Mathematics
3. Mathematics and Plausible Reasoning: Volume II, Patterns of Plausible Inference
4. Mathematical Discovery: On Understanding, Learning and Teaching Problem Solving

Picture Puzzle 01

This is a photo taken at the Isha Home School, Coimbatore. Set in tranquil surroundings near Coimbatore, Tamil Nadu, the residential Home School is located at the foothills of the Velliangiri Mountains.

Let's zoom in on one particular curve noticed in the picture:

If you could transfer the lower curve to graph paper, could you find a quadratic function that modeled it?

For a soft copy of this photo visit **www.teachersofindia.org** and search for Snakes at Isha.

Answers may be submitted to **atria.editor@apu.edu.in**
Do remember to send in your working.

A Plethora

One Problem, Six Solutions

Connecting Trigonometry, Coordinate Geometry, Vectors and Complex Numbers

Most mathematics teachers have a soft corner for math problems which, in a single setting, offer a platform to showcase a variety of different concepts and techniques. Such problems are very useful for revision purposes, but they offer much more: they demonstrate the deep and essential interconnectedness of ideas in mathematics, and their consistency.

C⊗MαC

In this article we study a simple and easily stated problem (see Figure 1) which can be solved in a multiplicity of ways — half a dozen at last count. After presenting the solutions we find a bonus: an unsuspected connection with Pythagorean triples!

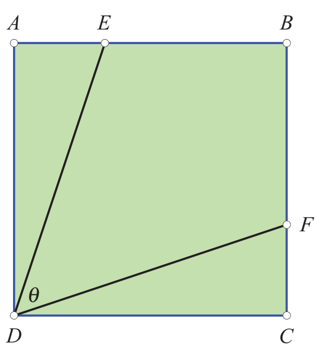


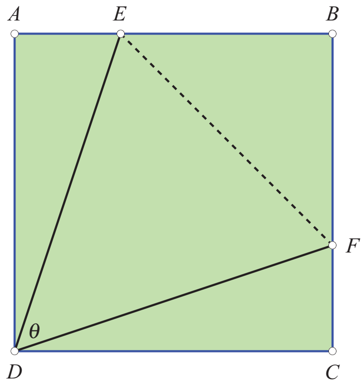
FIGURE 1. Statement of the problem

Problem.
$ABCD$ is a square; E and F are points of trisection of the sides AB and CB respectively, with E closer to A than to B, and F closer to C than to B (so $AE/AB = 1/3$ and $CF/CB = 1/3$). Segments DE and DF are drawn as shown.

Show that $\sin \angle EDF = 4/5$.

I. First solution, using the cosine rule

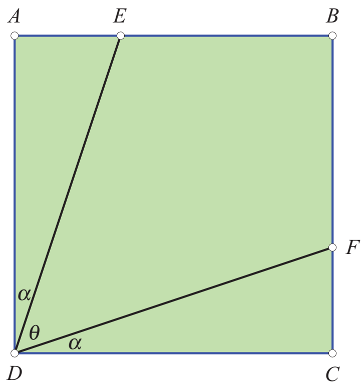
We take the side of the square to be 3 units; then $AE = CF = 1$ unit, and $BE = BF = 2$ units. Let $\angle EDF$ be denoted by θ. Join EF.



- Using the Pythagorean theorem we get $DE^2 = DF^2 = 10$, and $EF^2 = 8$.
- In $\triangle EDF$ we have, by the cosine rule: $EF^2 = DE^2 + DF^2 - 2\,DE \cdot DF \cdot \cos\theta$.
- So $\cos\theta = (10 + 10 - 8)/(2 \times 10) = 3/5$.
- Since θ is acute, $\sin\theta$ is positive. Hence: $\sin\theta = \sqrt{1 - 3^2/5^2} = 4/5$.

II. Second solution, using the trig addition formulas

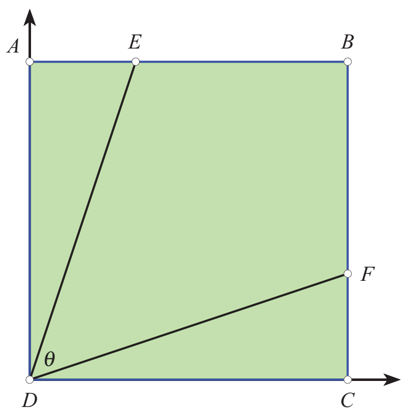
As earlier, we take the side of the square to be 3 units.



- Let $\angle ADE = \alpha$; then $\angle FDC = \alpha$ too.
- Since $AE = 1$ and $DE = \sqrt{10}$ we have $\sin\alpha = 1/\sqrt{10}$ and $\cos\alpha = 3/\sqrt{10}$.
- Since $\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$, we get $\cos 2\alpha = 9/10 - 1/10 = 4/5$.
- Since $\{2\alpha, \theta\}$ are complementary angles, the sine of either one equals the cosine of the other one.
- Hence $\sin\theta = 4/5$.

III. Third solution, using slopes

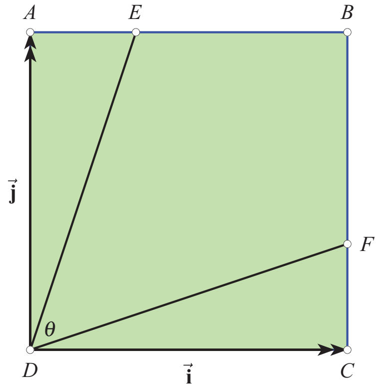
Let D be treated as the origin, ray $\overrightarrow{DC}$ as the x-axis, and $\overrightarrow{DA}$ as the y-axis.



- The slope of line DF is $1/3$.
- The slope of line DE is $3/1$.
- By the 'angle between two lines' formula, $\tan\theta = (3/1 - 1/3)/(1 + 3/1 \times 1/3)$, i.e., $\tan\theta = 4/3$.
- Hence $\sin\theta = 4/\sqrt{4^2 + 3^2} = 4/5$.

IV. Fourth solution, using the vector dot product

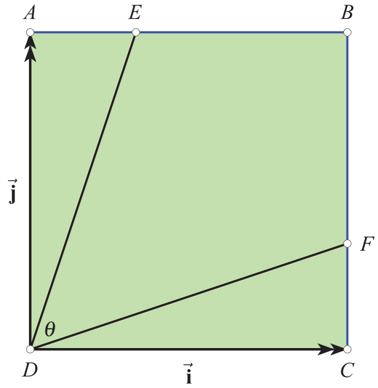
Let D be treated as the origin, ray $\overrightarrow{DC}$ as the unit vector $\vec{\mathbf{i}}$ along the x-axis, and $\overrightarrow{DA}$ as the unit vector $\vec{\mathbf{j}}$ along the y-axis. Recall that if $\vec{u}$ and $\vec{v}$ are two vectors, and the angle between them is ϕ, then $\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}| \cos\phi$.



- We have: $\overrightarrow{CF} = \vec{\mathbf{j}}/3$ and $\overrightarrow{AE} = \vec{\mathbf{i}}/3$.
- Hence $\overrightarrow{DF} = \vec{\mathbf{i}} + \vec{\mathbf{j}}/3$ and $\overrightarrow{DE} = \vec{\mathbf{i}}/3 + \vec{\mathbf{j}}$.
- Hence $\overrightarrow{DF} \cdot \overrightarrow{DE} = 1/3 + 1/3 = 2/3$.
- Also, $|\overrightarrow{DE}| = |\overrightarrow{DF}| = \sqrt{1 + 1/9} = \sqrt{10}/3$.
- Hence $\sqrt{10}/3 \cdot \sqrt{10}/3 \cdot \cos\theta = 2/3$, giving $\cos\theta = 2/3 \cdot 9/10 = 3/5$.
- Hence $\sin\theta = 4/5$.

V. Fifth solution, using the vector cross product

The same approach as in the fourth solution, but this time we use the cross product rather than the dot product. Let $\vec{\mathbf{k}}$ be the unit vector along the z-direction. Recall that if $\vec{u}$ and $\vec{v}$ are two vectors, and the angle between them is ϕ, then $|\vec{u} \times \vec{v}| = |\vec{u}||\vec{v}| \sin\phi$.

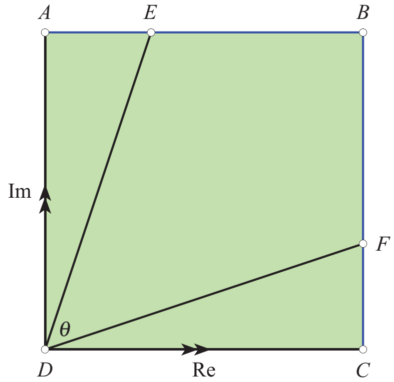


- We have: $\overrightarrow{DF} = \vec{\mathbf{i}} + \vec{\mathbf{j}}/3$ and $\overrightarrow{DE} = \vec{\mathbf{i}}/3 + \vec{\mathbf{j}}$.
- Hence $\overrightarrow{DF} \times \overrightarrow{DE} = (1 - 1/9)\ \vec{\mathbf{k}} = 8/9\ \vec{\mathbf{k}}$. (Remember that $\vec{\mathbf{i}} \times \vec{\mathbf{j}} = \vec{\mathbf{k}}$, and $\vec{\mathbf{j}} \times \vec{\mathbf{i}} = -\vec{\mathbf{k}}$.)
- Hence $|\overrightarrow{DF} \times \overrightarrow{DE}| = 8/9$.
- Also, $|\overrightarrow{DE}| = |\overrightarrow{DF}| = \sqrt{1 + 1/9} = \sqrt{10}/3$.
- Hence $\sqrt{10}/3 \cdot \sqrt{10}/3 \cdot \sin\theta = 8/9$, giving $\sin\theta = 8/10 = 4/5$.

VI. Sixth solution, using complex numbers

Our last solution uses the fact that multiplication by the imaginary unit $i = \sqrt{-1}$ achieves a rotation through 90° about the origin, in the counter-clockwise ('anti-clockwise') direction.

Let D be treated as the origin, line DC as the real axis, and line DA as the imaginary axis. Take the side of the square to be 3 units. Then the complex number representing F is $3 + i$, and the complex number representing E is $1 + 3i$.



- Let $z = \cos\theta + i \sin\theta$. Then $|z| = 1$, and multiplication by z achieves a rotation through θ about the origin 0, in the counter-clockwise direction.
- Hence $z \cdot (3+i) = 1+3i$. This equation in z may be solved by multiplying both sides by $3 - i$.
- Therefore $z = (1 + 3i)(3 - i)/(3^2 - i^2) = (6 + 8i)/10$.
- Hence $\sin\theta = 8/10 = 4/5$.

Remark. So there we have it: one problem with six solutions. Is there a 'best' among these solutions? We feel not. On the contrary: they complement each other very beautifully. (And there may be more such elegant solutions waiting to be found by you . . . .)

A PPT connection

Before closing we draw the reader's attention to a surprising but pleasing connection between this problem and the determination of Primitive Pythagorean Triples.

Observe the answer we got for the problem posed above: $\sin\theta = 4/5$. Hence θ is one of the acute angles of a right triangle with sides 3, 4, 5. Don't these numbers look familiar? Yes, of course: (3, 4, 5) is a PPT. Is this a happy coincidence?

Let's explore further . . . . Let us vary the ratio in which E and F divide segments AB and BC, while maintaining the equality $AE/EB = CF/FB$, and compute $\sin\angle EDF$ and $\cos\angle EDF$ each time. We summarized the findings below.

- If $AE/AB = CF/CB = 1/4$, we get $\sin\angle EDF = 15/17$ and $\cos\angle EDF = 8/17$. These values point to the PPT (8, 15, 17).
- If $AE/AB = CF/CB = 1/5$, we get $\sin\angle EDF = 12/13$ and $\cos\angle EDF = 5/13$. These values point to the PPT (5, 12, 13).
- If $AE/AB = CF/CB = 1/6$, we get $\sin\angle EDF = 35/37$ and $\cos\angle EDF = 12/37$. These values point to the PPT (12, 35, 37).
- If $AE/AB = CF/CB = 2/7$, we get $\sin\angle EDF = 45/53$ and $\cos\angle EDF = 28/53$. These values point to the PPT (28, 45, 53).

A PPT on every occasion! The connection is clearly something to be explored further. But we leave this task to the reader. (Note that we seem to have found a new way of generating PPTs!)

Acknowledgement

We first learnt of this multiplicity of ways from a long time colleague and friend, Shri S R Santhanam (Secretary, Talents Competition, AMTI).

The COMMUNITY MATHEMATICS CENTRE (CoMaC) is an outreach sector of the Rishi Valley Education Centre (AP). It holds workshops in the teaching of mathematics and undertakes preparation of teaching materials for State Governments, schools and NGOs. CoMaC may be contacted at comm.math.centre@gmail.com.

in the classroom

Triangles with sides in a Progression

A short write up which can spur the motivated teacher to design investigative tasks that connect geometry and sequences.

A Ramachandran

In *AtRiA* June 2012 we saw an analysis of right triangles with integer sides in arithmetic progression. In this context it is of interest to examine triangles with sides in some definite progression. In general, the least value for the constant increment/factor would give rise to an equilateral triangle; the largest value would lead to a degenerate triangle, with two sides adding up to the third side. An intermediate value would yield a right triangle. We consider separately three well known types of progression.

Sides in arithmetic progression

Take the sides to be $1-d, 1, 1+d$ where $d \geq 0$ is the constant difference. Then:

- The least possible value is $d = 0$, which yields an equilateral triangle.
- The case $d = 1/4$ (obtained by solving the equation $(1-d)^2 + 1 = (1+d)^2$) yields a right triangle with sides $3/4, 1, 5/4$ (this is similar to the triangle with sides 3, 4, 5).

- Since we must have $1-d+1 \geq 1+d$ there is a maximum possible value of d, namely $d = 1/2$, which yields a degenerate triangle with sides $1/2, 1, 3/2$.

The three 'critical' numbers 0, 1/4, 1/2 are themselves in A.P.

Sides in geometric progression

Take the sides to be $1/r, 1, r$ where $r \geq 1$ is the constant ratio. Then:

- The least possible value is $r = 1$, which yields an equilateral triangle.
- The condition for the triangle to be right-angled is $1/r^2 + 1 = r^2$. This is a quadratic equation in r^2, and it yields, on applying the quadratic formula:

$$r^2 = \frac{\sqrt{5}+1}{2}, \quad r = \sqrt{\phi} \approx 1.272,$$

  where $\phi \approx 1.618$ is the *golden ratio*.
- It may not be obvious that there is a maximum possible value of r. But we realize it when we see that the inequality $1/r + 1 > r$ must fail when r is sufficiently large (indeed, it fails when $r = 2$). What is the 'critical' value beyond which it fails? To find it we solve the equation $1/r + 1 = r$. We obtain $r = \phi$, the golden ratio.

Curiously, the three critical numbers $1, \sqrt{\phi}, \phi$ are themselves in G.P.

The above mentioned right triangle (sides $1/\sqrt{\phi}$, 1, $\sqrt{\phi}$) represents the only 'shape' that a right triangle with sides in G.P. can have. One of its angles is the only acute angle whose cos and tan values are the same. The angle in question is approximately $38^\circ 10'$.

Sides in harmonic progression

Three non-zero numbers are in harmonic progression (H.P.) if their reciprocals are in arithmetic progression. So for the sides of the triangle we may use the values

$$\frac{1}{1+d}, \quad 1, \quad \frac{1}{1-d},$$

where $0 \leq d < 1$. We note the following.

- The least possible value is $d = 0$, which yields an equilateral triangle.
- The condition for the triangle to be right-angled is $1/(1+d)^2 + 1 = 1/(1-d)^2$, which leads to a fourth degree ('quartic') equation:

$$\left(d^2-1\right)^2 = 4d,$$
$$\therefore\ d^4 - 2d^2 - 4d + 1 = 0.$$

  This unfortunately does not yield to factorization. Solving the equation numerically, we get $d \approx 0.225$.
- The greatest value of d is found by solving the equation

$$\frac{1}{1+d} + 1 = \frac{1}{1-d},$$

  which yields $d = \sqrt{2} - 1 \approx 0.414$. For this d, the triangle is degenerate.

Exploring the geometric properties of these triangles would be of interest.

A RAMACHANDRAN has had a long standing interest in the teaching of mathematics and science. He studied physical science and mathematics at the undergraduate level, and shifted to life science at the postgraduate level. He has been teaching science, mathematics and geography to middle school students at Rishi Valley School for two decades. His other interests include the English language and Indian music. He may be contacted at ramachandran@rishivalley.org.

An application of graphs

Centigrade–Fahrenheit Conversion

Understanding Algorithms

Which would you rather do? Parrot a formula for temperature conversion or heat up the class room with the excitement of understanding and using new concepts such as 'invariant points' in the application of a linear function? Read the article if you choose the latter option . . . .

C⊗MαC

In this note, which is based on an e-mail posted to a mailing list by noted math educator Prof Jerry Becker, we describe a striking way of converting from the Centigrade (Celsius) scale to the Fahrenheit scale and vice versa (see Figure 1).

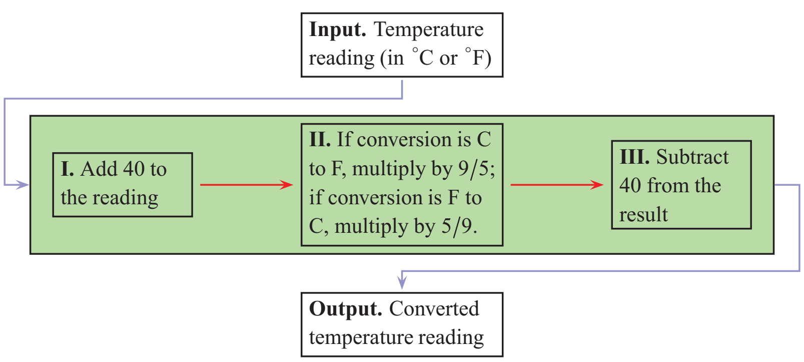


FIGURE 1. A 'symmetric' C to F and F to C converter: both ways we add 40 at the start, and subtract 40 at the end

Example 1 (C to F) Suppose the reading is 40°C. Step I: Add 40; we get $40 + 40 = 80$. Step II: Multiply by 9/5; we get: $80 \times 9/5 = 144$. Step III: Subtract 40; we get: $144 - 40 = 104$. Hence 40°C is the same as 104°F.

Example 2 (F to C) Suppose the reading is 50°F. Step I: Add 40; we get $50 + 40 = 90$. Step II: Multiply by 5/9; we get: $90 \times 5/9 = 50$. Step III: Subtract 40; we get: $50 - 40 = 10$. Hence 50°F is the same as 10°C.

Explanation

The algorithm works because of a basic way in which all linear non-constant functions (i.e., functions of the form $f(x) = ax + b$ where a, b are constants with $a \neq 0$) behave. The graph of such a function is a straight line with slope a. Call the line ℓ; then ℓ is not parallel to the x-axis.

Suppose $a \neq 1$. Then ℓ is not parallel to the line $y = x$ and hence intersects it at some point P. Since P lies on the line $y = x$, its coordinates have the form (c, c) for some c (Figure 2).

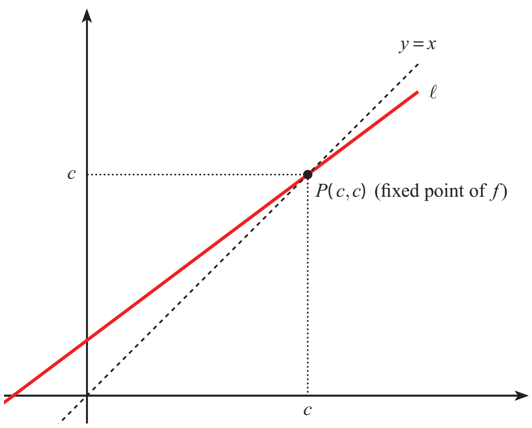


FIGURE 2

By construction, $f(c) = c$. *So f maps c to itself.* For this reason, c is called a *fixed point* or *invariant point* of f. We now cast f in a different form, using the fixed point.

From $f(x) = ax + b$ we get the following.

$$f(x) - c = ax + b - c,$$

$$\therefore\ f(x) - c = ax + b - (ac + b), \quad \text{because } c = f(c) = ac + b,$$

$$\therefore\ f(x) - c = a(x - c),$$

$$\therefore\ f(x) = a(x - c) + c.$$

So we have found an alternate expression for f, in terms of its invariant point.

Such an expression may always be found for the linear form $f(x) = ax + b$, provided $a \neq 1$.

Note carefully the 'shape' of the expression $a(x - c) + c$: we first *subtract* the value c, *multiply* by the factor a, then *add* back the value c.

Here is a numerical example. Suppose $f(x) = 2x - 3$. The fixed point for this function is $c = 3$, obtained by solving the equation $f(x) = x$. Therefore we can write the expression for f as: $f(x) = 2(x - 3) + 3$.

What makes this finding significant as well as useful is that the inverse function has a very similar form. For:

$$f(x) = a(x - c) + c,$$

$$\therefore\ a(x - c) = f(x) - c,$$

$$\therefore\ x = \frac{f(x) - c}{a} + c \quad \text{(remember that } a \neq 0\text{)},$$

$$\therefore\ f^{(-1)}(x) = \frac{x - c}{a} + c.$$

Note the form of *this* expression for the inverse function: we *subtract* c, *divide* by a, then *add* back c.

Observe that the prescription for $f^{(-1)}$ has the same form as the one for f; in both cases we subtract c at the start, and add back c at a later point; the only difference is that 'multiply' has been replaced by 'divide'.

Back to temperature scale conversion

Consider the formula used for C to F conversion:

$$F = \frac{9C}{5} + 32.$$

The associated function here is

$$f(x) = \frac{9x}{5} + 32.$$

The fixed point of f is found by solving the equation $f(x) = x$. A quick computation shows that the fixed point is $c = -40$; thus, -40 is the 'common point' of the two scales: −40°C is the same as −40°F (this is well known). Hence the expression for f may be written as:

$$f(x) = \frac{9(x+40)}{5} - 40.$$

This explains the 'C to F' conversion rule: *Add* 40, *multiply by* 9/5, *then subtract* 40. And the inverse function is:

$$f^{(-1)}(x) = \frac{5(x+40)}{9} - 40.$$

This explains the 'F to C' conversion rule: *Add* 40, *multiply by* 5/9, *then subtract* 40.

References

This article is based on an e-mail posted by Prof Jerry Becker to a mailing list and a document by François Pluvinage attached to that mail, in which Pluvinage proves a general result: *Every dilation of the number line is a translation or has an invariant point*. Many thanks to Prof K Subramaniam (HBCSE) for bringing the mail to our attention.

Riddles?

01

Four people – Racer, Jogger, Walker and Meditator – need to cross a bridge under the following conditions:

1. They are all initially on the same side of the bridge.
2. It is dark, the bridge is unlit, and they have just one working torch between them.
3. The bridge is narrow and weak, and at most two people can cross at the same time.
4. They cannot cross without the torch.
5. The torch cannot be thrown across; it must be carried across by them.
6. Racer can cross the bridge in 1 minute, Jogger in 2 minutes, Walker in 5 minutes and Meditator in 10 minutes.
7. A pair walking together must walk at the slower person's pace.

What is the shortest time in which the entire group of four can transfer to the other side of the bridge?

02

A card has precisely four statements printed on it, as follows:

- On this card exactly one statement is false.
- On this card exactly two statements are false.
- On this card exactly three statements are false.
- On this card exactly four statements are false.

Assuming that each statement is either true or false, how many false statements are there on the card?

These riddles have been adapted from similar riddles given in Christian Constanda's book, Dude Can You Count? (Springer, 2010)

For answers see page no. 66

tech space

Exploring problems in geometry

In a dynamic geometry environment

Sneha Titus &
Jonaki Ghosh

In the articles on Pólya in this issue, a key ingredient listed in his recipe for successful problem solving is *persistence*. Would a 21st century student exercise greater persistence in problem solving if, instead of paper and pencil, the medium used was technology? A dynamic geometry software (DGS) allows the user to construct an object and explore its properties by dragging its component parts. Under different dragging modes, certain properties of the object remain invariant while others may vary. In particular, the logical dependencies are preserved. Observations regarding the object can be made through measurement in the algebra view. For example, by dragging the vertices of a triangle the user can 'see' in the algebra panel that the sum of its interior angles remains 180° irrespective of the lengths of the sides or shape of the triangle. Thus, exploring a figure in a DGS can play a vital role in enabling the student to explore geometrical ideas and concepts. Our view is that such software is a valuable addition to the problem solvers' toolkit.

The geometrical investigation described in this article is a student's attempt to explore a problem in geometry, in a classroom situation where access to technology was provided. The problem was from Pólya's list, and students were given access to *GeoGebra* (open source dynamic geometry software) to carry out their investigations. *GeoGebra* is available as a free download on the Internet at **www.geogebra.org/**. An excellent guide on the use of *GeoGebra* is available here. For the benefit of first-time users of *GeoGebra*, the investigation has been described as a series of steps used by the student. Later, some 'Teacher's Notes' are included. The key point of the article is to emphasise the pedagogical opportunities presented by a dynamic geometry environment, particularly to suggest that such explorations can enable learners to observe patterns and form conjectures. After some initial remarks by the teacher to facilitate the investigation, the student (whose solution will be described) was able to arrive at the solution of the problem.

The problem posed to the student was: **Given two intersecting straight lines, construct a circle of a given radius r that touches the two lines.** Note that lines are given, as is the radius of the circle to be drawn.

The following steps were used by the student in Geogebra

Step 1: Open a new file in Geogebra. A snapshot of the screen is shown in Fig. 1

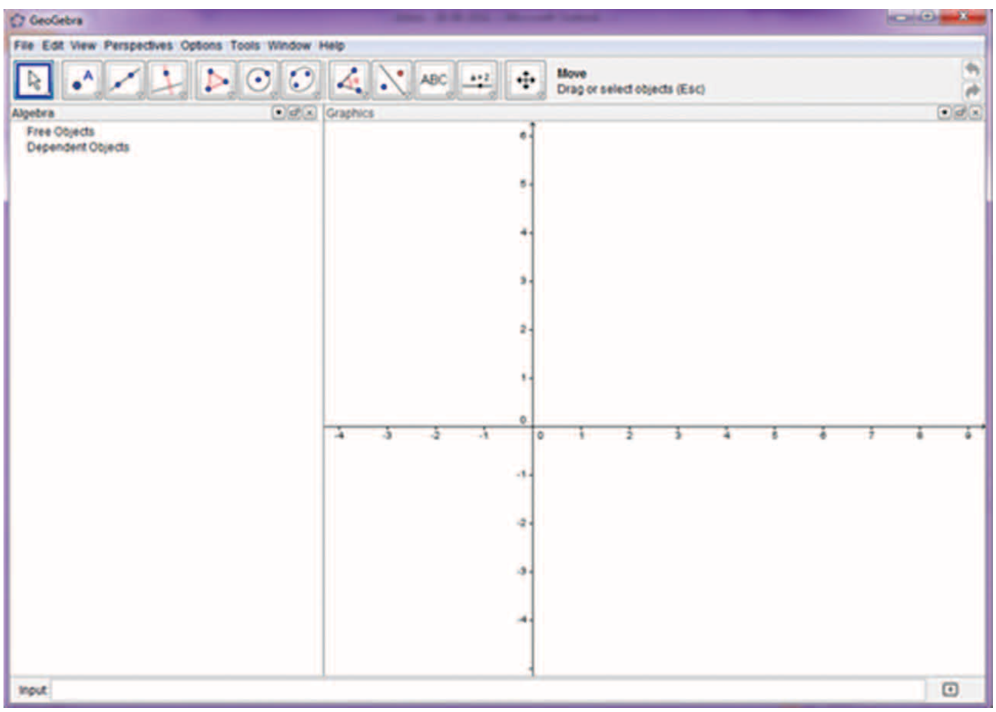


Figure 1

Step 2. Maximise the window. At the top is the toolbar showing several icons. Clicking at the bottom of an icon produces a drop down menu from which you can select a desired tool.

Step 3. For this activity, we do not need the graph grid. If it is visible, go to **View** ⇨ **Grid**. This is a toggle switch; clicking on Grid will make the grid disappear.

Step 4. Go to the Tool bar and select the **line tool** (line through 2 points) and click twice anywhere on the blank screen to produce a line. Moving the cursor will change the slope of the line. Repeating the process will create a second line. Make sure that it intersects the first line (see Fig. 2). Name their point of intersection 'O'.

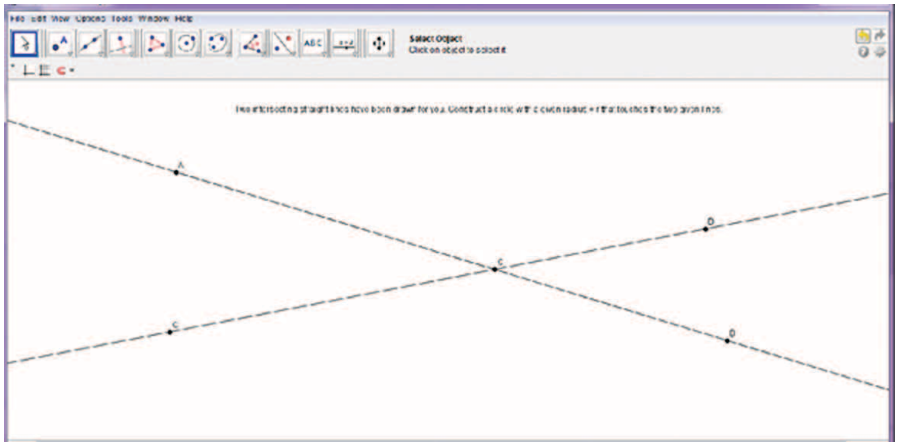


Figure 2

Here is a snapshot of the next stage of the construction as done by the student (Fig. 3):

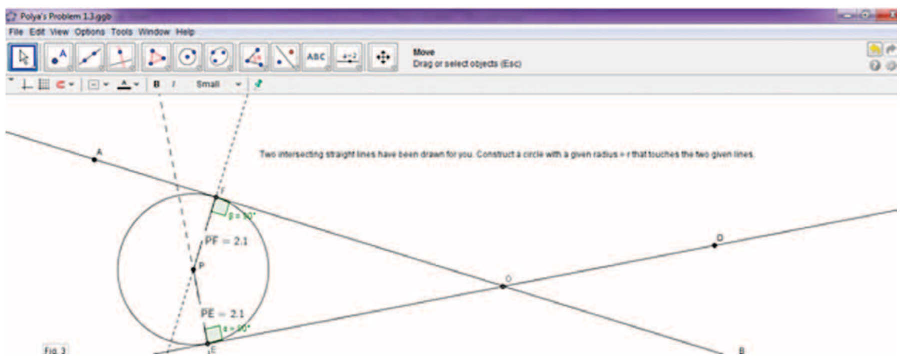


Figure 3

The student explained her reasoning as follows:

- The centre of the circle must lie between the two lines. (See Teacher's Comments, below.)
- Straight lines drawn from the centre to the two lines will be perpendicular to them and of equal length as they are the radii of the required circle and the two lines are, by definition, tangents to the circle.

The construction steps are detailed below.

Step 5: Select the **Point icon** and click in the space between the lines to create the point **P**. (Right clicking on the point created will allow you to rename the point as you wish.)

Step 6: Select the **Perpendicular Line** icon and click first on P and then on line AB. Repeat with P and line CD. This will create perpendiculars from P to lines AB and CD.

Step 7: Using the drop down menu from the **Point** icon, choose the **Intersect two objects** icon and click on the two pairs of perpendicular lines in succession. This will create points E and F. The lengths of PE and PF can be measured using the **distance tool** available under the measurement icon.

Step 8: Drag P until distances PE and PF are equal (you can see the distances in the Algebra pane at the left).

Step 9: Using the drop down menu from the **Circle** icon, choose **Circle with centre and radius** and click first on **P** and then enter the radius as PE.

So far, the teacher had not intervened and had allowed students to work in pairs. She now began to interact with students to find out how they had attempted to solve the problem. She urged the student to explore the effect of dragging point D (used to draw the original two lines), and to check if the two lines remain tangent to the circle. (Actually the lines are fixed. But see Teacher's Comments, below.)

The resulting sketch is shown below (Fig. 4)

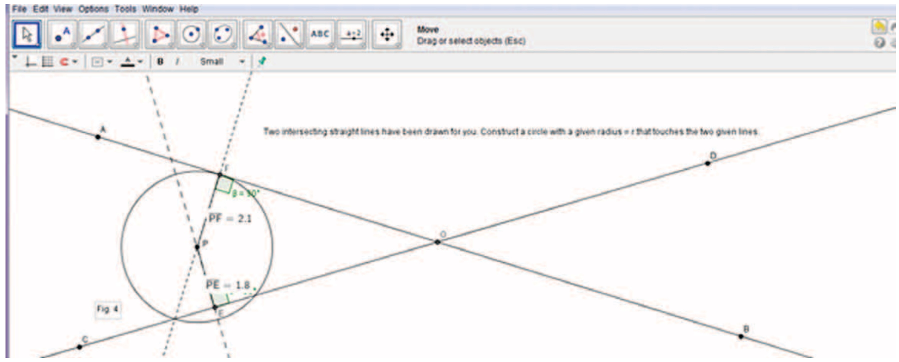


Figure 4

Clearly, the circle no longer met the conditions specified in the problem. Since the student seemed baffled by the problem now, the teacher suggested constructing the line PO and measuring the two angles POF and POE as shown in Fig. 5

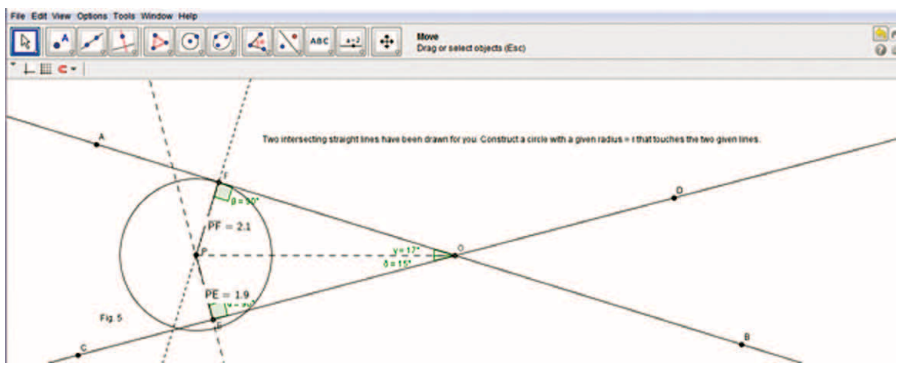


Figure 5

Now as the student dragged the point D back to its original position, it was clearly seen that when the two lines were indeed tangents to the circle, the two angles were equal.

From this point on, the student proceeded rapidly. She reasoned that for the lines AB and CD to be tangent to the circle, triangles POF and POE had to be congruent (Right angles, Hypotenuse PO and Sides PF and PE had to be equal). Consequently, angles POF and POE had to be equal and PO therefore had to be the angular bisector of angle FOE.

The student then re-did the entire construction selecting P not as a random point between the lines but as a point on the angular bisector (option available on Geogebra) of angle FOE. Constructing one perpendicular from P to either AB or CD and getting either point E or point F was sufficient to construct the circle with centre P and radius PE (or PF). In this way she was able to construct the circle, but she was unable to control its radius. (See the Teacher's Notes below for a way of getting the circle to be of the required radius r.)

Teacher's Notes

Since the above report led from an actual investigation, it is not a generalized solution to the problem. Note that the circle can also be constructed in the supplementary angle AOD; the student's assumption that the centre P must be in the angle AOC is not valid.

In fact, in the actual problem, the angle between the lines is fixed. But dragging the point D in the above sketch was a vital step in leading the student to the conjecture that the centre of the required circle must lie on the bisector of angle

EOF. Further, her reasoning and going on to proving the congruency of the triangles POF and POE to validate her conjecture was a crucial step towards the solution of the problem. Once the student understands that the centre must lie on the bisector of the angle, then fixing the angle becomes a particular configuration of the general solution.

We had noted that the student was not able to control the radius of the circle, but this can be done in the following manner. We know that the lengths PE and PF are equal to r, and that P lies on the angle bisector. Therefore, P may be located by drawing a line LM parallel to AB at a distance r from it; P is then the point where LM meets the angle bisector. (In fact two such lines can be drawn, one on each side of AB, and this allows us to construct both the circles that meet the given specifications.)

This example suggests that a dynamic geometry environment, when used in a pedagogically appropriate way, can play a vital role in facilitating the cognitive transition from verification and conjecturing to formal abstract concepts and proof.

GeoGebra is an interactive geometry, algebra, and calculus application, intended for teachers and students. GeoGebra is written in Java and thus available for multiple platforms.

Its creator, Markus Hohenwarter, started the project in 2001 together with the help of open-source developers and translators all over the world. Currently, the lead developer of GeoGebra is Michael Borcherds, a secondary school maths teacher.

Most parts of the GeoGebra program are licensed under GPL and CC-BY-SA[3], making them free software. One of the sites from which it can be downloaded is http://www.geogebra.org/cms/. An excellent manual for new users of Geogebra is available for download at http://www.geogebra.org/book/intro-en.zip

Using Geogebra, geometry becomes a dynamic activity. Constructions can be made with points, vectors, segments, lines, polygons, conic sections, inequalities, implicit polynomials and functions. All of them can be changed dynamically afterwards. Elements can be entered and modified directly on screen, or through the Input Bar.

Precisely because of its capabilities, Geogebra is often used as a demonstration tool to illustrate theorems and results. But Geogebra is also a tool for visualizations and mathematical investigations based on which conjectures can be made. A worksheet which provides suitable scaffolding in the form of guided reasoning can enable the students to move towards a convincing proof.

SNEHA TITUS. a teacher of mathematics for the last twenty years has resigned from her full time teaching job in order to pursue her career goal of inculcating in students of all ages, a love of learning the logic and relevance of Mathematics. She works in the University Resource Centre of the Azim Premji Foundation. Sneha mentors mathematics teachers from rural and city schools and conducts workshops using the medium of small teaching modules incorporating current technology, relevant resources from the media as well as games, puzzles and stories which will equip and motivate both teachers and students. She may be contacted on sneha.titus@azimpremjifoundation.org

JONAKI GHOSH is an Assistant Professor in Dept. of Elementary Education, Lady Sri Ram College, University of Delhi where she teaches courses related to math education. She obtained her Ph.D in Applied Mathematics from Jamia Milia Islamia University, New Delhi and her masters degree from IIT, Kanpur. She taught mathematics at DPS, R K Puram for 13 years, where she was instrumental in setting up the Math Lab & Technology Centre. She has started a Foundation through which she conducts professional development programmes for math teachers. Her primary area of research is in the use of technology in math instruction. She is a member of the Indo Swedish Working Group on Mathematics Education. She regularly participates in national and international conferences. She has published articles in proceedings and journals and has authored books for school students. She may be contacted at jonakibghosh@gmail.com

Fun Problems

C⊗MαC

Digital Problems for the Digital Age

Consider all three digit numbers with the property that *the first digit equals the sum of the second and third digits*. Examples of such numbers are 413, 615 and 404. We call this property ♡. Let X be the sum of all three digit numbers that have property ♡.

Next, consider all four digit numbers with the property that *the sum of the first two digits equals the sum of the last two digits*. Examples of such numbers are 4123, 6372 and 4013. We call this property ♣. Let Y be the sum of all four digit numbers with property ♣.

Problem:
Show that both X and Y are divisible by 11.

Note that the problem does not ask for the actual values of X and Y; *it only asks you to show that they are multiples of* 11. Could there be a way of proving this without actually computing X and Y? We shall show that there is such a way. First, some notation.

Notation 1: $\overline{AB}$ denotes the two digit number with tens digit A and units digit B; $\overline{ABC}$ denotes the three digit number with hundreds digit A, tens digit B and units digit C; $\overline{ABCD}$ denotes the four digit number with thousands digit A, hundreds digit B, tens digit C and units digit D; and so on. We use the bar notation to avoid confusion, for example, between the two digit number $\overline{AB}$ and the product AB which means $A \times B$.

Notation 2: Given a number with two or more digits, by its '**TU portion**' we mean *the number formed by its last two digits*. ('TU' stands for 'tens-units'.) For example, the TU portion of 132 is 32, and the TU portion of 1234 is 34.

Notation 3: Given a number with three or more digits, by its '**H portion**' we mean its hundreds digit.

Showing that X is divisible by 11. A three digit number $\overline{ABC}$ has property ♡ if $A = B + C$. Observe that if $\overline{ABC}$ has property ♡, so does $\overline{ACB}$. If $B = C$ then these two numbers are the same. In this case $\overline{ACB}$ has the form $\overline{ABB}$.

Now observe that $\overline{BB} = 11B$ is a multiple of 11; so too is $\overline{BC} + \overline{CB} = 11(B + C)$. Hence:

- The sum of the TU portions of $\overline{ABC}$ and $\overline{ACB}$ is a multiple of 11.
- The TU portion of $\overline{ABB}$ is a multiple of 11.

It follows that *for each fixed value of A, the sum of the TU portions of the numbers $\overline{ABC}$ having property ♡ is a multiple of* 11.

Now we shall show that the sum of the H portions of the numbers having property ♡ is a multiple of 11. To show this we adopt a different strategy.

With $A = 1$ there are *two* numbers with property ♡ (101 and 110). With $A = 2$ there are *three* such numbers (202, 211 and 220). With $A = 3$ there are *four* such numbers, with $A = 4$ there are *five* such numbers, . . . , and with $A = 9$ there are *ten* such numbers. It follows that the sum of the H portions of the three digit numbers having property ♡ is

$$(1 \times 2) + (2 \times 3) + (3 \times 4) + (4 \times 5) + (5 \times 6) + (6 \times 7) + (7 \times 8) + (8 \times 9) + (9 \times 10) = 330,$$

which is a multiple of 11.

Since the sum of the H portions of all the numbers with property ♡ is a multiple of 11, and so is the sum of the TU portions, it follows that X must be a multiple of 11.

Showing that Y is divisible by 11. We shall use the same strategy. If $\overline{ABCD}$ is a number with property ♣ then $A + B = C + D$; hence $\overline{ABDC}$ too has the property. Since $\overline{CC} = 11C$ and $\overline{CD} + \overline{DC} = 11(C + D)$ are multiples of 11, it follows that for each fixed (A, B) pair, the sum of the TU portions of the numbers $\overline{ABCD}$ with property ♣ is a multiple of 11.

Now we focus on the front two digits.

Suppose that $\overline{ABCD}$ has property ♣, and B is non-zero. Then $\overline{BACD}$ too is a four digit number with property ♣. The sum of the numbers associated with the front two digits is $\overline{AB} + \overline{BA} = 11(A + B)$, which is a multiple of 11.

What if $B = 0$? Then the number has the form $\overline{A0CD}$, with $A = C + D$. This number can be matched with the three digit number $\overline{ACD}$ which has property ♡. We have already shown (in the above section) that the sum of the A-values of all such numbers $\overline{ACD}$ is a multiple of 11. This proof implies that the sum of the A-values of all numbers $\overline{A0CD}$ with property ♣ is a multiple of 11.

Thus Y is a sum of various multiples of 11, and hence is a multiple of 11.

It is worth reflecting on the solution strategies used. *We did not at any stage attempt to compute the actual sum of all the numbers. Instead we grouped them in a way that would make the divisibility property perfectly visible*.

Problems for Solution

Problem II-1-F.1

Solve the following cryptarithm:

$$\overline{EAT} + \overline{THAT} = \overline{APPLE}.$$

Problem II-1-F.2

Solve the following cryptarithm:

$$\overline{EARTH} + \overline{MOON} = \overline{SYSTEM}.$$

Problem II-1-F.3

Given that $\overline{IV} \times \overline{VI} = \overline{SIX}$, and $\overline{SIX}$ is not a multiple of 10, find the value of $\overline{IV} + \overline{VI} + \overline{SIX}$.

Problem II-1-F.4

Explain why the following numbers are all perfect squares:

$$1, \quad 121, \quad 12321, \quad 1234321, \quad 123454321, \quad 12345654321, \quad \ldots .$$

Problem II-1-F.5

Explain why the following numbers are all perfect squares:

$$1089, \quad 110889, \quad 11108889, \quad 1111088889, \quad 111110888889, \quad \ldots .$$

Solutions of Problems from Issue-I-2

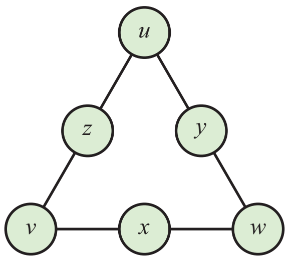
Problem I-2-F.1

Show that in a magic triangle, the difference between the number at a vertex and the number at the middle of the opposite side is the same for all three vertices.

We must prove that $u - x = v - y = w - z$. We know that $u + z + v = v + x + w = w + y + u$.

From the equalities we get: $u + z - x - w = 0$, hence $u - x = w - z$. In the same way we get $w - z = v - y$.

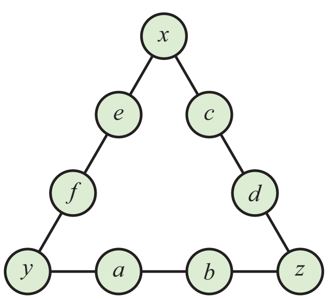
Hence proved.



Problem I-2-F.2

Explore the analogous problem in which the digits 1, 2, 3, 4, 5, 6, 7, 8, 9 *are placed along the sides of a triangle, one at each vertex and two on the interiors of each side, so that the sum of the numbers on each side is the same.*

Let the configuration be as shown in the figure, with the numbers x, y, z at the corners of the triangle, and the numbers a, b, c, d, e, f on the interiors of the sides. Then, by requirement, the sums $x + y + e + f$, $y + z + a + b$ and $z + x + c + d$ are all equal to some constant s, say. Let $C = x + y + z$ be the sum of the corner numbers, and let $M = a + b + c + d + e + f$ be the sum of the 'middle' numbers.

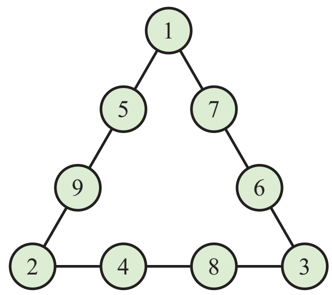


By addition we get $3s = 2C + M$. Also $C + M = 45$; hence $C = 3s - 45$, and we see that C is a multiple of 3; so is M. Next, the least possible value of C is $1 + 2 + 3$, and the largest possible value is $7 + 8 + 9$. So $6 \leq C \leq 24$.

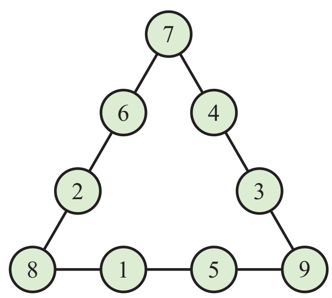
Hence $51 \leq 3s \leq 69$, leading to $17 \leq s \leq 23$.

Each s-value between 17 and 23 (and hence each C-value between 6 and 24 which is a multiple of 3) can be 'realized' by a suitable magic triangle. Two such possibilities are shown below.

$s = 17, C = 6$



$s = 23, C = 24$



Problem I-2-F.3

Show that the cryptarithm $\overline{AT} + \overline{RIGHT} = \overline{ANGLE}$ *has no solutions.*

Since the hundreds digits of $\overline{RIGHT}$ and $\overline{ANGLE}$ are the same, we infer that the addition of $\overline{AT}$ to $\overline{RIGHT}$ has only affected the tens and units digits, with no 'carry' to the hundreds digit. Hence the leading two digits must stay unaffected; we must have $\overline{AN} = \overline{RI}$. This violates a basic rule concerning cryptarithms: that different letters cannot represent the same digit. Therefore the problem has no solution.

Problem I-2-F.4

Solve the following cryptarithm:
$\overline{CATS} \times 8 = \overline{DOGS}$.

Since $8 \times S$ has units digit S, it follows that $S = 0$. Since $\overline{CATS} \times 8$ is a four-digit number, $\overline{CATS} < 1250$. Hence $C = 1$ and $A = 2$ (since 0 and 1 have been 'used up'), and $T = 3$ or 4. Only the first possibility yields an answer (if $T = 4$ we get $G = 2 = A$). So the answer is: $\mathbf{1230 \times 8 = 9840}$.

Problem I-2-F.5

Solve this cryptarithm:
$\overline{ABCDEF} \times 5 = \overline{FABCDE}$.

We must have $E = 0$ or 5. We must also have $A = 1$ (since A is the leading digit of a six-digit number for which multiplication by 5 yields another six-digit number); and $F \geq 5$. If $E = 0$ then F is even, else it is odd. We now arrive at the answer by simultaneously proceeding from 'each end' of the number to the 'opposite end'. The argument is easier to present 'live' on a blackboard than in print, so you (the reader) will have to set up a multiplication display and follow the reasoning there.

$$\begin{array}{cccccc} A & B & C & D & E & F \\ & & & & \times & 5 \\ \hline F & A & B & C & D & E \\ \hline \end{array}$$

$$\begin{array}{cccccc} \_ & \_ & \_ & \_ & \_ & \_ \\ & & & & \times & 5 \\ \hline \_ & \_ & \_ & \_ & \_ & \_ \\ \hline \end{array}$$

If $E = 0$ then $F = 6$ or 8. If $F = 6$ then $D = 3$, leading to $C = 5$, $B = 6$ and $A = 2$ which cannot be; we already know that $A = 1$. So the option $F = 6$ does not work. If $F = 8$ then $D = 4$, hence $C = 0$; but this means that $C = E$. So this fails too.

Therefore, $E \neq 0$. Hence $E = 5$, and $F = 7$ or 9.

If $F = 9$ then $D = 9$ (from $25 + 4 = 29$); hence $D = F$. So this too does not work. The only possibility now left is $F = 7$. This leads to $D = 8$ (from $25 + 3 = 28$), $C = 2$ (from $40 + 2 = 42$), $B = 4$ (from $10 + 4 = 14$). Everything has now worked out, and we have the answer: $\mathbf{142857 \times 5 = 714285}$.

Remark.

It is not a coincidence that the answer corresponds exactly to the repeating part of the decimal expansion of $1/7 = 0.142857\,142857\,142857\,\ldots$. But we will elaborate on the connection later.

Problems for the Middle School

problem corner

Problem Editor : R. ATHMARAMAN

Problems for Solution

The problems in this selection are all woven around the theme of GCD ('greatest common divisor', also called 'highest common factor') and LCM ('least common multiple').

Problem II-1-M.1

Two-digit numbers a and b are chosen ($a > b$). Their GCD and LCM are two-digit numbers, and a/b is not an integer. What could be the value of a/b?

Problem II-1-M.2

The sum of a list of 123 positive integers is 2013. Given that the LCM of those integers is 31, find all possible values of the product of those 123 integers.

Problem II-1-M.3

Let a and b be two positive integers, with $a \leq b$, and let their GCD and LCM be c and d, respectively. Given that $a + b = c + d$, show that: (i) a is a divisor of b; (ii) $a^3 + b^3 = c^3 + d^3$.

Problem II-1-M.4

Let a and b be two positive integers, with $a \leq b$, and let their GCD and LCM be c and d, respectively. Given that $ab = c + d$, find all possible values of a and b.

Problem II-1-M.5

Let a and b be two positive integers, with $a \leq b$, and let their GCD be c. Given that $abc = 2012$, find all possible values of a and b.

Problem II-1-M.6

Let a and b be two positive integers, with $a \leq b$, and let their GCD and LCM be c and d, respectively. Given that $d - c = 2013$, find all possible values of a and b.

Solutions of Problems in Issue-I-2

Solution to problem I-M-S.1 *Using the digits* $0, 1, 2, 3, 4, 5, 6, 7, 8, 9$ *once each, can you make a set of numbers which when added and subtracted in some order yields* 100*?*

If the problem had said only 'added' (with subtraction not allowed) the answer is that this is not possible! For, the sum $0 + 1 + 2 + \cdots + 8 + 9 = 45$ is a multiple of 9, hence *any* set of numbers made using these digits and added together will yield a multiple of 9. For example, the sum $125 + 37 + 46 + 80 + 9$ equals 297, which is a multiple of 9. So an answer of 100 would be impossible to achieve.

However with subtraction permitted, the task *is* possible. Let A represent the part which is 'added' and B the part which is subtracted. Then we want $A - B = 100$. For reasons already explained, $A + B \equiv 0 \pmod 9$; also, $A - B \equiv 1 \pmod 9$. These two relations yield $A \equiv 5 \pmod 9$ and $B \equiv 4 \pmod 9$. Our task now is to partition the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 into two subsets, with sums $5 \pmod 9$ and $4 \pmod 9$ respectively, and try to create numbers using the two sets of digits whose sums differ by 100. One possible approach is to initially leave out the digits 0, 3, 6, 9 and work only with the digits 1, 2, 4, 5, 7, 8. After some play we find that the following partition works: {1, 2, 4, 7} and {5, 8}; observe that $1 + 2 + 4 + 7 = 14 \equiv 5 \pmod 9$ and $5 + 8 = 13 \equiv 4 \pmod 9$. A convenient possibility is $72 + 14 - 85 = 1$. Now if we can somehow create 99 using the remaining digits, our task is done. This is possible: $90 + 6 + 3 = 99$. So we have our answer: $\mathbf{90 + 6 + 3 + 72 + 14 - 85 = 100}$.

Solution to problem I-M-S.2 *To find a formula for the n-th term of the sequence of natural numbers from which the multiples of* 3 *have been deleted:* $1, 2, 4, 5, 7, 8, \ldots$.

We make use of the *floor function*, defined as follows: $[x]$ = the largest integer not exceeding x. Example: $[2.3] = 2$, $[10.7] = 10$, $[-1.7] = -2$. Let $f(n)$ denote the n-th term of the sequence $1, 2, 4, 5, 7, 8, \ldots$. Then the sequence $f(n) - n$ has the following terms: $0, 0, 1, 1, 2, 2, 3, 3, \ldots$. The n-th term for this is easy to work out: it is simply $[(n-1)/2]$. Hence $\mathbf{f(n) = n + [(n-1)/2]}$.

Solution to problem I-M-S.3 *To find a formula for the n-th term of the sequence of natural numbers from which the squares have been deleted:* $2, 3, 5, 6, 7, 8, 10, 11, 12, \ldots$.

We again use the floor function. Let $g(n)$ denote the n-th term of the sequence $2, 3, 5, 6, 7, 8, 10, 11, 12, \ldots$. Then the sequence $g(n) - n$ has the following terms: $1, 1, 2, 2, 2, 2, 3, 3, 3, 3, 3, 3, 4, 4, 4, 4, \ldots$. Note the pattern: two 1s, four 2s, six 3s, eight 4s, …. The last 1 comes at position 2; the last 2 comes at position 6; the last 3 comes at position 12; …. It is easy to see that the last k must come at position $k(k+1)$. Hence $g(n) - n = k$ precisely when $(k-1)k < n \le k(k+1)$. Solving these inequalities for k we find that

$$\mathbf{g(n) = n + \left[\frac{[\sqrt{4n}] + 1}{2}\right]}.$$

It turns out that this can be expressed in a much more pleasing form:

$$\mathbf{g(n) = n + \left[\sqrt{n + \sqrt{n}}\right]}.$$

The proof of this surprising equality is left to the reader.

Solution to problem I-M-S.4 *Amar, Akbar and Antony are three friends. The average age of any two of them is the age of the third person. Show that the total of the three friends' ages is divisible by* 3. By focusing on the age of the oldest among the three persons, or the youngest among them (assuming there is an oldest), we easily deduce that their ages are identical. Hence the sum of the ages is a multiple of 3.

Solution to problem I-M-S.5 *A set of consecutive natural numbers starting with* 1 *is written on a sheet of paper. One of the numbers is erased. The average of the remaining numbers is* $5\frac{2}{9}$. *What is the number erased?* Let the largest number be n, so the sum of the numbers is $n(n+1)/2$; let the number erased be x, where $1 \le x \le n$. Then we have the following equation which we must solve for n and x:

$$\frac{\frac{1}{2}n(n+1) - x}{n-1} = \frac{47}{9}.$$

Cross-multiplying and simplifying (we leave the details to you) we get:

$$9n^2 - 85n + 94 = 18x.$$

From this we see that $9 \mid 85n - 94$, hence $9 \mid 4n - 4 = 4(n-1)$, hence $9 \mid n - 1$. (Recall that $a \mid b$ means: 'a is a divisor of b'.) Therefore $n \in \{1, 10, 19, 28, 37, 46, \ldots\}$.

Next, since $1 \le x \le n$, it follows that

$$\frac{\frac{1}{2}n(n+1) - n}{n-1} \le \frac{47}{9} \le \frac{\frac{1}{2}n(n+1) - 1}{n-1}.$$

We solve these two inequalities for n. The one on the left gives:

$$\frac{9n(n-1)}{2} \le 47(n-1), \quad \therefore\ 9n \le 94,$$
$$\therefore\ n \le 10,$$

since n is a whole number. The one on the right gives:

$$\frac{9(n-1)(n+2)}{2} \ge 47(n-1), \quad \therefore\ 9(n+2) \ge 94,$$
$$\therefore\ n \ge 9.$$

Hence $n \in \{9, 10\}$. Invoking the earlier condition we get $n = 10$, and the number removed is $x = (900 - 850 + 94)/18 = 144/18 = \mathbf{8}$.

Solution to problem I-M-S.6 *The average of a certain number of consecutive odd numbers is A. If the next odd number after the largest one is included in the list, then the average goes up to B. What is the value of $B - A$?*

The sum of k consecutive odd numbers starting with $2n + 1$ is $(k + n)^2 - n^2 = k^2 + 2nk$, hence the average of these numbers is $k + 2n$. The average of $k + 1$ consecutive odd numbers starting with $2n + 1$ is clearly $k + 1 + 2n$. The difference between these two is 1. Hence $\mathbf{B - A = 1}$.

Solution to problem I-M-S.7 101 *marbles numbered from* 1 *to* 101 *are divided between two baskets A and B. The marble numbered* 40 *is in basket A. This marble is removed from basket A and put in basket B. The average of the marble numbers in A increases by* 1/4*; the average of the marble numbers in B also increases by* 1/4*. Find the number of marbles originally present in basket A. (1999 Dutch Math Olympiad.)*

Let baskets A and B have n marbles and $101 - n$ marbles at the start, and let the averages of baskets A and B be x and y, respectively. Then the totals of the numbers in the two baskets are, respectively, nx and $(101 - n)y$. Since the total across the two baskets is $1 + 2 + 3 + \cdots + 101 = 101 \times 102/2 = 5151$, we have:

$$nx + (101 - n)y = 5151. \qquad (1)$$

After the transfer of marble #40 from A to B, the individual basket totals are $nx - 40$ and $(101 - n)y + 40$, and the new averages are, respectively:

$$\frac{nx - 40}{n - 1}, \quad \frac{(101 - n)y + 40}{102 - n}.$$

We are told that the new averages exceed the old ones by 1/4. Hence:

$$\frac{nx - 40}{n - 1} - x = \frac{1}{4},$$
$$\frac{(101 - n)y + 40}{102 - n} - y = \frac{1}{4}.$$

Hence:

$$nx - 40 - (n - 1)x = \frac{n - 1}{4},$$
$$(101 - n)y + 40 - (102 - n)y = \frac{102 - n}{4}.$$

These yield, on simplification:

$$x - 40 = \frac{n - 1}{4}, \quad 40 - y = \frac{102 - n}{4}. \qquad (2)$$

We must solve (1) and (2). Substituting from (2) into (1) we get:

$$n\left(\frac{n - 1}{4} + 40\right) + (101 - n)\left(40 - \frac{102 - n}{4}\right)$$
$$= 5151.$$

This yields:

$$\frac{101(n + 29)}{2} = 5151,$$
$$\therefore\ n + 29 = 51 \times 2 = 102,$$

giving $n = \mathbf{73}$.

problem corner

Problems for the Senior School

Problem editors: PRITHWIJIT DE & SHAILESH SHIRALI

We start this column with a problem posed by a reader from Romania. It looks daunting but turns out on closer examination to be a simple consequence of a well known fact.

A Cryptarithmic Inequality

Problem posed by Stanciu Neculai (Department of Mathematics, 'George Emil Palade' Secondary School, Buzau, Romania; E-mail: <stanciuneculai@yahoo.com>) Let A, B, C, D, E denote arbitrary digits. Prove the inequality

$$\overline{ACDEA} \times \overline{BCDEB} \le \overline{ACDEB} \times \overline{BCDEA}. \tag{1}$$

Example. Let $(A, B, C, D, E) = (1, 2, 3, 4, 5)$. The stated relation then reads

$$13451 \times 23452 \le 13452 \times 23451,$$

and this statement is true: the quantity on the left side equals 315452852, while the quantity on the right equals 315462852.

Solution. Note that the sum of the two numbers on the left of (1) equals the sum of the two numbers on the right:

$$\overline{ACDEA} + \overline{BCDEB} = \overline{ACDEB} + \overline{BCDEA}. \tag{2}$$

To see why, note that the middle three digits are the same in the four numbers (namely: C, D, E), and they occur in the same order too; and the first and last digits have simply swapped places ($\overline{A\ldots A}$ and $\overline{B\ldots B}$ on the left side, $\overline{A\ldots B}$ and $\overline{B\ldots A}$ on the right side).

Now when you have two pairs of positive numbers with equal sum, which pair has a greater product? We can state the same question geometrically: If we have two rectangles with equal perimeter, which of the two has greater area? To guide our number sense we may consider various pairs of numbers with sum 20, e.g., $(19, 1), (18, 2), (17, 3), (16, 4), \ldots$. The products associated with these pairs are 19, 36, 51, 64, ... The trend is easy to spot: *The closer the two numbers, the larger the product*. Stated geometrically: *The rectangle which is closer in appearance to a square has the greater area.*

We may prove this statement rigorously as follows. Let p, q be two numbers whose sum is a constant. We wish to examine the behaviour of the product pq. We now draw upon the following simple identity:

$$4pq + (p-q)^2 = (p+q)^2. \tag{3}$$

Since $p+q$ is constant, the sum of $4pq$ and $(p-q)^2$ is constant; so as one of them increases, the other decreases by an equal amount. Hence: *The larger the difference between p and q, the smaller the product pq; the smaller the difference, the larger the product.*

So we ask: *Of the two pairs* $\{\overline{ACDEA}, \overline{BCDEB}\}$ *and* $\{\overline{ACDEB}, \overline{BCDEA}\}$, *which pair is closer together?*

Of course it is the second pair (we assume that $A \neq B$; if $A = B$ then the two pairs are identical); for the difference between the numbers in the first pair is $10001|A - B|$ while the difference between the numbers in the second pair is $9999|A - B|$. Inequality (1) follows.

Comment. We see that the problem is merely a special case of a very well known fact: that when the sum of two numbers is kept constant, their product is larger when they are closer to each other. So we may have any number of such inequalities:

$$\overline{AA} \times \overline{BB} \le \overline{AB} \times \overline{BA},$$

$$\overline{ACA} \times \overline{BCB} \le \overline{ACB} \times \overline{BCA},$$

$$\overline{ACDA} \times \overline{BDCB} \le \overline{ACDB} \times \overline{BCDA}, \quad \ldots.$$

Problems for Solution

Problem II-1-S.1

Drawn through the point A of a common chord AB of two circles is a straight line intersecting the first circle at the point C, and the second circle at the point D. The tangent to the first circle at the point C and the tangent to the second circle at the point D intersect at the point M. Prove that the points M, C, B, and D are concyclic.

Problem II-1-S.2

In triangle ABC, point E is the midpoint of the side AB, and point D is the foot of the altitude CD. Prove that $\angle A = 2\angle B$ if and only if $AC = 2ED$.

Problem II-1-S.3

Solve the simultaneous equations: $ab + c + d = 3, bc + d + a = 5, cd + a + b = 2, da + b + c = 6$, where a, b, c, d are real numbers.

Problem II-1-S.4

Let x, y, and a be positive numbers such that $x^2 + y^2 = a$. Determine the minimum possible value of $x^6 + y^6$ in terms of a.

Problem II-1-S.5

Let p, q and y be positive integers such that $y^2 - qy + p - 1 = 0$. Prove that $p^2 - q^2$ is not a prime number.

Solutions of Problems in Issue-I-2

Solution to problem I-2-S.1

To find the sum of the first 100 *terms of the series, given that it begins with* 2012 *and is in AP as well as GP.*

Let r be the common ratio of the geometric progression. Since the numbers are in AP and GP, the numbers $1, r, r^2$ are both in AP and GP, hence $1 + r^2 = 2r$; this yields $r = 1$, implying that the sequence is a constant sequence. Thus the sum of the first 100 terms is **201200**.

Solution to problem I-2-S.2

To find the sum of all four digit numbers such that the sum of the first two digits equals the sum of the last two digits, and to compute the number of such numbers.

We first show that there are 615 such numbers. Let A refer to the block of the first two digits, and B to the block of the last two digits. Let s be the sum of the digits in A (and therefore in B as well); then $1 \le s \le 18$. We shall count separately the numbers corresponding to each value of s.

If $s = 1$ then the digits in A and B must be 1, 0. For A the only possibility is $(1, 0)$ and for B the possibilities are $(1, 0)$ and $(0, 1)$; so there are 1×2 possibilities. If $s = 2$, the possibilities for A are $(2, 0)$ and $(1, 1)$; the possibilities for B are $(2, 0)$, $(1, 1)$ and $(0, 2)$; hence there are 2×3

possibilities. If $s = 3$ we get 3×4 possibilities the same way. This pattern continues till $s = 9$, with 9×10 possibilities. For $s = 10$ the zero digit becomes unavailable, and we get 9^2 possibilities; for $s = 11$ there are 8^2 possibilities; and so on down to $s = 18$, with just 1^2 possibility. Hence the total number of possibilities is

$$(1 \times 2 + 2 \times 3 + \cdots + 9 \times 10) + (1^2 + 2^2 + \cdots + 9^2) = 330 + 285 = \mathbf{615}.$$

Now we compute the sum of all such numbers; we show that the sum is 3314850. But we give the solution in outline form and leave the task of filling some details to the reader.

As a first step, we find *the sum of all three digit numbers* $\overline{ABC}$ *whose first digit equals the sum of the last two digits, i.e.,* $A = B + C$ *or* $C = A - B$. The number equals $100A + 10B + A - B = 101A + 9B$; here $1 \leq A \leq 9$ and $B \leq A$. With A fixed, there are $A + 1$ such numbers, and their sum is $101A(A+1) + 9(0+1+2+\cdots+A) = 211\binom{A+1}{2}$. Hence the sum of all such numbers is $211\left(\binom{2}{2}+\binom{3}{2}+\binom{4}{2}+\cdots+\binom{10}{2}\right) = 211\cdot\binom{11}{3} = 34815$.

Next, we find *the sum of all two digit numbers with a given digit sum* s. We shall leave it to you to show that if $1 \leq s \leq 9$ the sum equals $11(1+2+3+\cdots+s) = 11\binom{s+1}{2}$, while if $10 \leq s \leq 18$ the sum equals $11\left(45 - (1+2+\cdots+(s-10)\right) = 11\left(45 - \binom{s-9}{2}\right)$.

Now we are ready to compute the sum of all four digit numbers for which the sum of the first two digits and the sum of the last two digits equal a given number s, where $1 \leq s \leq 18$, but with 0 permitted as the leading digit. Using the result derived in the preceding paragraph we find that the sum equals $\frac{1111}{2}s(s+1)^2$ for $1 \leq s \leq 9$, and $\frac{1111}{2}s(19-s)^2$ for $10 \leq s \leq 18$. Hence the sum of all such numbers is

$$\frac{1111}{2}\left(\sum_{s=0}^{s=9} s(s+1)^2 + \sum_{10}^{18}(19-s)s^2\right) = \frac{1111}{2}(2640 + 3390) = 3349665.$$

This is not the final answer, because in the collection of four digit numbers we have included numbers whose leading digit is 0. To get the required answer we must subtract the sum of all three digit numbers for which the first digit equals the sum of the last two digits. Hence the desired answer is $3349665 - 34815 = \mathbf{3314850}$.

Solution to problem I-2-S.3

To show that no term of the sequence 11, 111, 1111, 11111, 111111, ... *is the square of an integer.*

Every integer in the sequence is odd and of the form $100k + 11$ for some non-negative integer k. We know that the square of an odd integer is one more than a multiple of four. But all integers in the given sequence are three more than a multiple of four. Therefore none of them is the square of an integer.

Solution to problem I-2-S.4

The radius r *and the height* h *of a right-circular cone with closed base are both an integer number of centimetres, and the volume of the cone in cubic centimetres is equal to the total surface area of the cone in square centimetres; find the values of* r *and* h.

The given condition leads to the equation

$$\frac{1}{3}\pi r^2 h = \pi r^2 + \pi r\sqrt{r^2 + h^2}.$$

Simplifying we obtain $r^2 = 9h/(h-6)$. Since $r^2 > 0$ we get $h > 6$.

We also write the previous relation as $r = \sqrt{9 + 54/(h-6)}$. Since r is an integer, $h - 6$ must divide 54 and the expression under the square root sign must be a perfect square. Thus $h - 6 \in \{1, 2, 3, 6, 9, 18, 27, 54\}$. On checking these values we find that r is an integer only when $h - 6 = 2$. Hence $\mathbf{h = 8}$ and $\mathbf{r = 6}$.

Solution to problem I-2-S.5

Given a $\triangle ABC$ *and a point* O *within it, lines* AO, BO *and* CO *are drawn intersecting the sides* BC, CA *and* AB *at points* P, Q *and* R*, respectively; prove that* $AR/RB + AQ/QC = AO/OP$.

Denote by $[PQR]$ the area of the triangle PQR (see Figure 1). Observe that

$$\frac{AQ}{QC} = \frac{[ABQ]}{[CBQ]} = \frac{[AOQ]}{[COQ]} = \frac{[ABQ] - [AOQ]}{[CBQ] - [COQ]} = \frac{[AOB]}{[BOC]}.$$

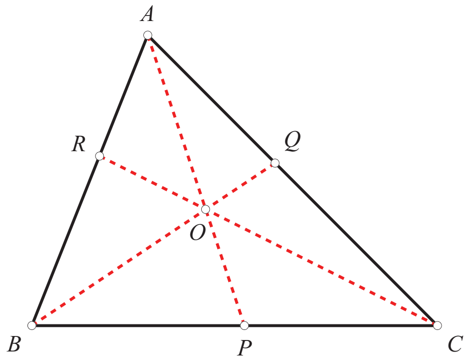


FIGURE 1.

Similarly we get $AR/RB = [AOC]/[BOC]$. Hence

$$\frac{AQ}{QC} + \frac{AR}{RB} = \frac{[AOB] + [AOC]}{[BOC]}.$$

Now,

$$\frac{AO}{OP} = \frac{[AOB]}{[POB]} = \frac{[AOC]}{[POC]}$$
$$= \frac{[AOB] + [AOC]}{[POB] + [POC]} = \frac{[AOB] + [AOC]}{[BOC]}.$$

Therefore, $AQ/QC + AR/RB = AO/OP$.

Solution to problem I-2-S.6

To show that every triangular number > 1 is the sum of a square number and two triangular numbers.

We consider separately the cases where n is even and odd. If n is even, there exists a natural number k such that $n = 2k$. Then:

$$\frac{n(n+1)}{2} = 2k^2 + k = (k^2 + k) + k^2$$
$$= \frac{k(k+1)}{2} + \frac{k(k+1)}{2} + k^2.$$

If $n > 1$ is odd there exists a natural number k such that $n = 2k + 1$. So we can write the n^{th} triangular number as

$$(2k+1)(k+1) = k(k+1) + (k+1)^2$$
$$= \frac{k(k+1)}{2} + \frac{k(k+1)}{2} + (k+1)^2.$$

Remark.

We have established a stronger result: each triangular number exceeding 1 can be expressed as the sum of a square number and twice a triangular number.

Picture Puzzle 02

This photo is from the Field Institute Office of the Azim Premji Foundation, Puducherry.

If an equilateral triangle of side ‘s’ was created by the 3 stones and you were lighting a fire in this space and cooking some food, what would be the smallest radius of a cylindrical cooking vessel placed on the stones? (A smaller vessel would drop into the gap.)

review

Pólya to the rescue

When you don't know the solution to a problem

Tips from a master

Is problem solving ability simply a gift, innate and inborn? A gift that some people have and others do not? Or is there something that can be learnt about problem solving? It is widely regarded as a gift, but the mathematician George Pólya thought differently. In this article we learn about the ways in which he looked at this important educational issue, and the approaches he recommended.

K. Subramaniam

Like most students, I too went through school learning solutions to problems and doing my best to remember them in an exam. We prepare for exams by 'practising' the solutions to problems – trying to remember a formula, a method, a trick, the steps, and so on. But, as my friend and fellow math educator Dr. Hridaykant Dewan puts it, *learning to solve* problems is different from *learning solutions* to problems. Learning to solve problems is learning how to tackle problems to which you don't know the solution already. You haven't just forgotten the solution; rather, it is a problem that you haven't solved before.

For most students, this may seem too difficult a task, even impossible. How do you solve the problem if you don't know the solution, if you have never been taught the solution? Many people try a mathematics problem for about 5 or 10 minutes at the most. If they cannot solve it in this short time they decide that they are incapable of solving the problem. But that's not really true. Good problems may take a long time to solve – an hour or more, sometimes even days. Mathematicians often think about a problem for a very long time trying to find a solution. So many people, who give up after a short time and think that they cannot solve a maths problem, are simply making a wrong judgement about their own capability.

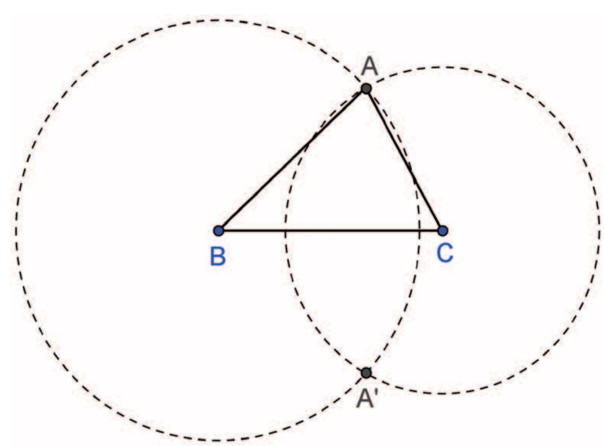
Why do some people – even some young students – keep at a problem for so long? Is it competition that drives them, refusing to be beaten by a problem, or wanting to prove a point? It may be that, partly. But often what keeps them going is the sheer pleasure that comes at the end of solving a problem and the joy at having discovered and learnt something worthwhile, purely by one's own effort. Often, when one struggles with a problem, not only does one learn the solution to the problem, but one makes other discoveries around the problem, and gets a glimpse into what doing mathematics is really like.

The master who wanted to bring the riches of problem solving to students, even those who have been turned away from mathematics, was the mathematician George Pólya. His most famous book is *How to solve it*, published in 1945 [1]. A number of problem solving books have been written over the years, but all of them trace their lineage back to this classic. In his book, Pólya gives a detailed account of how one could become a problem solver.

There are several elements which together make a person a good problem solver. As I said above, belief in one's own capability and the willingness to stick with a problem are important. Everybody knows, of course, that one must know some maths to be able to solve math problems. But just knowing the maths is not enough – many people know the maths, but they cannot apply it to find a solution. In his book, Pólya presented a stock of thumb rules useful when facing a new or unfamiliar situation, which he called *heuristics*. Heuristics are like approximate techniques: they suggest ways in which a problem can be solved. They don't guarantee a solution, but they are useful in pointing the way towards a possible solution. Generally all good problem solvers use heuristics consciously.

Pólya continued to write on problem solving, publishing other volumes. Twenty years later, Pólya published his last writings on the art of problem solving in two volumes titled *Mathematical Discovery* [2]. These volumes contain a more systematic presentation of heuristics than his earlier writings and an excellent collection of carefully chosen problems. They are a wonderful introduction to the world of problem solving for a high school student. Alan Schoenfeld, the Berkeley mathematician and maths educator, who used the book in his problem solving courses, has this to say: "... *Mathematical Discovery* is a classic.... It represents the capstone of Pólya's career..." [3].

Pólya begins the first volume of *Mathematical Discovery* with geometric construction problems. You are given some data about a geometric figure and you need to find a way of constructing the figure using straight edge and compasses. (You don't actually need to construct the figure, only find the procedure.) Pólya begins with one of the simplest of such problems: Given the three sides of a triangle ABC, construct the triangle. Nearly every high school student knows how to do this. Draw a line segment equal to the side BC. With B as centre and radius equal to AB, draw an arc, or better, draw a circle. Similarly, draw a circle with C as centre and radius equal to AC. The intersection of the two circles gives the point A. (A' is an alternative point; you get a congruent triangle if you use A'.)



Pólya shows how much there is to learn in this simple problem. The solution is an application of a heuristic he calls "the pattern of two loci", a heuristic that can be used for many other, more difficult, problems. "Locus" (plural "loci") means a path, a line on which a point with certain properties can lie. The steps in finding the pattern of the two loci are as follows:

1. Use some of the data in the problem to construct an initial geometric object. (This is the line segment BC in the problem above.)
2. Reduce the problem to finding a single point. (After drawing BC, once we find point A the problem is solved, all that remains is to join A to B and C, a mechanical task. But how do we find A?)
3. Of the data in the problem, use some and ignore the rest to obtain a locus for the missing point. (The circle with centre B and radius AB is the locus of all points that are at a distance AB from B. The point A lies somewhere on this circle. Note that we have ignored the information about the length AC.)
4. Use the remaining data to obtain another locus. (The circle with centre C and radius equal to AC is the locus of all points at a distance of AC from C. A is somewhere on this locus.)
5. The intersection of the two loci (in step 3 and step 4) is the required point.

If you think about it, this is an interesting way of looking at the problem. And the pattern generalizes to other problems. Before we look at more construction problems, here are some problems on finding the locus of a point given certain properties. Think of how you can construct the locus in each case with a straight edge and compass.

(Answers are given on page 66 of this issue)

1. A point moves so that it is always at a fixed distance *d* from a given point P. What is its locus?
2. A point moves so that it is at a fixed distance *d* from a given straight line *l*. What is its locus?
3. A moving point remains equally distant from two given points P and Q; what is its locus?
4. A moving point remains at equal distance from two given parallel straight lines *m* and *n*; what is its locus?
5. A moving point remains at equal distance from two given intersecting straight lines *l* and *m*; what is its locus?
6. Two vertices, A and B, of the triangle ABC are marked for you. Angle C, opposite to the side AB, is also given. The triangle is not determined, since the point C can vary. What is the locus of the point C?

A small sample of problems from Pólya's book is given below. These are from the exercise just after he discusses the pattern of the two loci. You can try them out. Take your time. Maybe one or more of the problems will need many hours to solve. Maybe you will need to return to them after a break, think about them the next day. But don't give up easily.

1. Construct a ΔABC given the length two sides BC = *a* and AC = *b* and the length of the median (*mA*) drawn from the vertex A to the side BC.
2. Construct a ΔABC given the length of BC, the length of the altitude (*hA*) from A to BC and the length of the median from A to BC (*mA*).
3. Construct a ΔABC given the length of BC, the length of the altitude (*hA*) from A to BC and angle A.
4. Two intersecting straight lines have been drawn on paper for you. Construct a circle with a given radius = *r* that touches the two given lines.

   (*See pages 44-48 for a discussion of this problem.)*
5. A straight line *l* is drawn on paper for you. A point P outside *l* is marked on the paper for you. Construct a circle with a given radius = *r*, such that the point P lies on its circumference and the line *l* is tangent to it. (Under what condition is it impossible to construct this circle?)

These are only a small selection of the many interesting problems that Pólya presents that can be solved using the pattern of the two loci. Notice that knowing the heuristic doesn't guarantee that you will find the solution. In fact, in each problem you will discover something interesting about the conditions of the problem. Perhaps you can come up with some new problems yourself.

Pólya describes other heuristics in the book and goes on to discuss more problems in geometry, algebra and combinatorics. In all, the book is a wonderful treat of problems both for a teacher and for a student, to be savoured slowly. Mulling over a problem even after one has found a solution is often a learning exercise: Are there any other solutions? What made it difficult (or easy) to find the solution? What happens if I vary some of the conditions in the problem? Are there similar problems that I can think of?

One of the problems related to the pattern of two loci has been a favourite in the problem solving sessions that I used to conduct some years ago for high school students. Again, it was Schoenfeld's writings which called my attention to this problem. This problem is surprisingly hard when students start on it. Often, we have spent the whole session discussing possible approaches to solving the problem, without solving the problem. Once the solution is found, it doesn't look so hard at all. So it is worth thinking about why it seemed hard in the first place. (Maybe thinking about this can also help in finding the solution.) Here is the problem:

- Construct a triangle ABC given the length of side BC, the measure of angle A, and the radius of the incircle of the triangle *r*.

An important rule in our problem solving sessions is that the teacher doesn't tell the solution even if he or she knows it. The teacher helps the students think aloud, notes down their ideas on the board for others to see and explore, and encourages students to think of alternative approaches. The session might end without the solution being found. But the students go back and think about it and often find the solution to the problem given above themselves.

Pólya devoted his life to describing many heuristics and making collections of problems that can be tackled using one or more of these heuristics. But there is no list of heuristics, memorizing which, will make one a problem solver. Heuristics are essentially approximate thumb rules, they help one to think more creatively, but don't guarantee solutions. As one gains experience in solving problems, one builds up a personal repertoire of what has been most useful in problem solving. Pólya only opened the doors, but one has to make the journey oneself.

(Solutions to some of the problems above will be given in the next issue. If you want to discuss solutions with the author, please send an email to subra@hbcse.tifr.res.in)

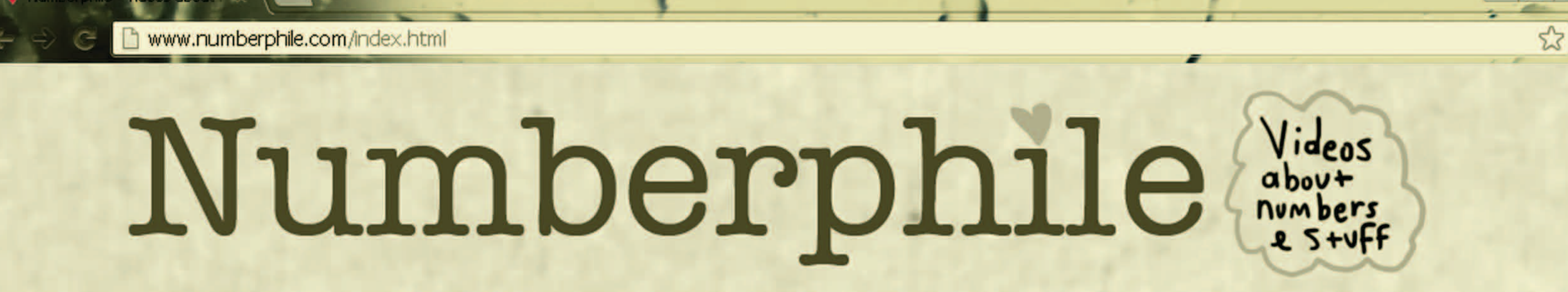
References

[1] Pólya, G. (1990). *How to solve it: A new aspect of mathematical method*. Penguin Books.

[2] Pólya, G. (1981). *Mathematical Discovery: on understanding, learning, and teaching problem solving*. (Combined paperback edition of both volumes), Wiley, New York.

[3] Schoenfeld, A. H. (1987). Pólya, problem solving, and education. *Mathematics magazine*, *60*(5), 283-291.

K. SUBRAMANIAM is associate professor of mathematics education at the Homi Bhabha Centre for Science Education, Mumbai. His areas of research are characterizing learning strands for topics in middle school mathematics, like fractions and algebra, and developing models for the professional development of mathematics teachers. He has an interest in cognitive science and philosophy, especially in relation to education and to maths learning. He has contributed to the development of the national curriculum framework in mathematics (NCF 2005), and to the development of mathematics textbooks at the primary level.



WEBSITE REVIEW by Rajkishore Patnaik

In Darren Aronofsky-directed 1998 surrealist psychological thriller Pi, the leading character Maximillian Cohen mutters to himself: “Restate my assumptions: One, Mathematics is the language of nature. Two, everything around us can be represented and understood through numbers. Three: If you graph the numbers of any system, patterns emerge. Therefore, there are patterns everywhere in nature. Evidence: The cycling of disease epidemics; the waxing and waning of caribou populations; sun spot cycles; the rise and fall of the Nile. So, what about the stock market? The universe of numbers that represents the global economy. Millions of hands at work, billions of minds.... A vast network, screaming with life... My hypothesis: Within the stock market, there is a pattern as well... Right in front of me... hiding behind the numbers. Always has been.”

To me, the website ‘Numberphile’ represents Cohen’s mutterings. Perhaps, when the celebrated mathematics professor and BBC presenter Marcus Du Sautoy introduces himself as a ‘pattern finder’ he was enunciating Numberphile’s mission statement.

Don’t be unimpressed by this rather dull homepage of the website **www.numberphile.com**:

Started late in November 2011, it is a clickable page for one of the most popular YouTube channels that has over 320,900 subscribers around the world. This website, of an Australian video journalist Brady Haran, made numbers ever so seductive. He says his aim is to make videos for every number from “1 to ∞” (hey Brady! ∞ is not a number!) Peppered with the passion of leading practitioners of mathematical art & craft including Professors Phil Moriarty, James Grime, Roger Bowley, Alex Bellos (of *Alex’s Adventures in Numberland* fame) among others, Numberphile is a whiff of fresh air that helps you enjoy patterns, see connections and find absorbing trivia for coffee break discussions.

Not restricted to just the interestingness of a number, the videos delve deeper into its evolution (http://bit.ly/VChvn6), etymology (http://bit.ly/Km81p3) and existence (http://bit.ly/K5GngC)

Sample this: the videos span across themes like googol to dyscalculia to Rubik cubes to number card tricks to Usain Bolt to Enigma code to Nepal’s flag to Batman equation to Olympic rings to Kaprekar; and, before I forget, Ramanujan’s taxicab numbers.

My favorite of course is *Sounds of Pi* video (http://bit.ly/zSkuuK) that was hosted to celebrate World π Day (there are 4 dedications for π from this channel). It talks about my favorite science populariser Richard Feynman. What does Feynman have to do with the near mythical π? Watch it for yourself. The duration of the video is 6.28 min. The numeric coincidence is *not* intentional.

As I am writing this, Brady has a Nelson’s Number of Numberphile videos on the YouTube. Hey, how come he hasn’t talked about this number as it also has a bit of cricketing trivia associated with it?

RAJKISHORE is a science communicator and math enthusiast who taught mathematics and coordinated the sciences and global perspectives at the Blue Mountains School, Ooty. At present he works at the Educational Technology and Design team at Azim Premji University where, in addition to his professional duties, he follows his interest in digital curation.

An Impossible Construction?

It is well known that there is no general procedure for trisecting an angle using only a compass and unmarked ruler (naturally, we must stick to the rules governing geometric constructions). In particular, a 60° angle cannot be so trisected. This implies that a 20° angle cannot be constructed using such means. However, here is a construction which appears to do the impossible! Throughout, the notation 'Circle(P,Q)' means: "circle with centre P, passing through Q" (for a given pair of points P, Q). We start with any two points O and A (see Figure 1) and follow the steps given below.

1. Draw Circle(O,A) and Circle(A,O). Let B be one of their points of intersection.
2. Draw Circle (A,B) and Circle(B,A). Let C be their point of intersection other than O.
3. Let D be the point other than O where Circle(A,B) meets ray OA.
4. Draw Circle(C,D) and Circle(D,C). Let E be their point of intersection other than A. Join OE.
5. Measure $\measuredangle EOD$. It appears to be a 20° angle!

So: has the impossible been achieved?

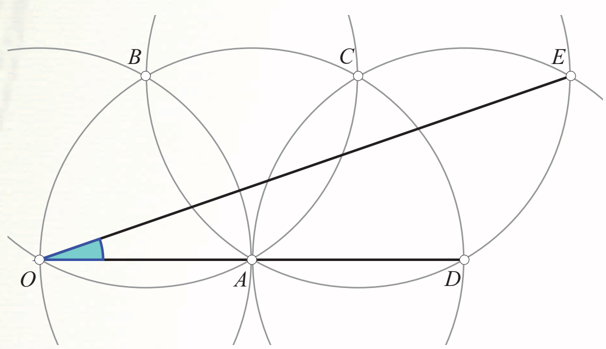


Figure 1. Supposed construction of a 20° angle

Construction sent to us by Shri Ashok Revankar of Dharwar. The work is that of his student Subra Jyoti, of KV Dharwar.

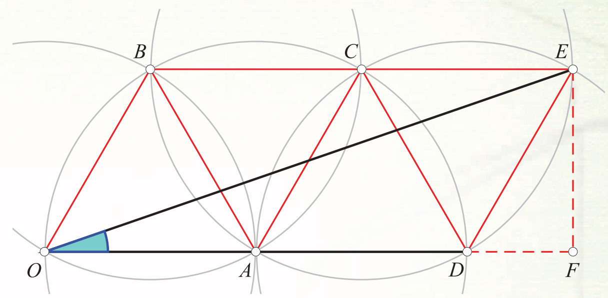


Figure 2.

No, the impossible has not been achieved! We shall show that $\measuredangle EOD$ is close to 20° but is not equal to it.

Draw the segments joining the centres of the circles; we get a lattice of equilateral triangles (Figure 2). If we take OA = 2 units, then B, C, E all lie at a perpendicular distance of $\sqrt{3}$ units from line OD. (Use Pythagoras's theorem to see why.) Let F be the foot of the perpendicular from E to line OD; then $EF = \sqrt{3}$ and DF = 1, hence OF = 2+2+1 = 5, and:

$$\tan \measuredangle EOD = \frac{\sqrt{3}}{5}.$$

Using a scientific calculator we get: $\measuredangle EOD = \tan^{-1} \sqrt{3}/5 \approx 19.1066°$. So $\measuredangle EOD$ is quite close to 20°. The difference is small enough that the eye will most likely not notice it.

With some experimentation you will be able to find more such constructions, which come close to 'doing the impossible'. (There are many such impossibilities in plane geometry, and we shall be examining more such examples in the following issues.) In such cases, doing an error analysis of the kind we have done can be most instructive.

The COMMUNITY MATHEMATICS CENTRE (CoMaC) is an outreach sector of the Rishi Valley Education Centre (AP). It holds workshops in the teaching of mathematics and undertakes preparation of teaching materials for State Governments, schools and NGOs. CoMaC may be contacted at comm.math.centre@gmail.com.

Answers to some problems posed in

"When you don't know the solution to a problem".

1. A circle with centre P and radius equal to *d*.
2. A line parallel to *l* at a distance of *d*. (Think of how you would construct this with straight edge and compass.)
3. The perpendicular bisector of the line segment PQ.
4. A line parallel to and exactly mid-way between *m* and *n*.
5. Angles are formed where the given lines intersect. The bisectors of these angles are the required loci. How many such loci are there?
6. The locus is an arc lying on the circumcircle of the triangle and hence passing through A and B. To draw the arc, find the circumcentre O by drawing the perpendicular bisector of AB and using the information about angle C. Hint: What is angle OAB in terms of angle C?

SOLUTIONS TO Riddles ?

01

The least possible time is 17 minutes.

Explanation: The pairings should be made such that the two slowest people cross together, and the torch is always returned by one of the quick crossers. So Walker and Meditator must cross together. But they cannot be the first ones to cross, as the torch would then be stuck on the wrong side of the bridge. This means that the first to cross must be Racer and Jogger. This now yields the solution:

1. Racer and Jogger cross over, together: 2 minutes
2. Racer returns alone, bringing the torch: 1 minute
3. Walker and Meditator cross over, together: 10 minutes
4. Jogger returns alone, bringing the torch: 2 minutes
5. Racer and Jogger cross over, together: 2 minutes
6. Total: 17 minutes

Note that in steps 2 and 4 we can swap 'Racer' and 'Jogger'. The total time taken is still 17 minutes.

02

There are 3 false statements on the card.

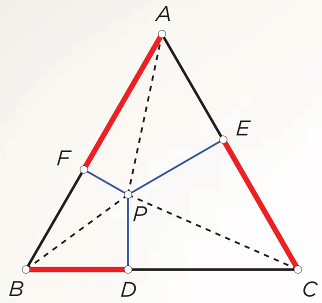
Explanation: Since the four statements contradict each other, at most one of them can be true. So the number of true statements is either 0 or 1.

If the number of true statements is 0, then all the statements are false, but that makes statement #4 true! So the number of true statements cannot be 0. Hence it must be 1.

So there must be 1 true statement and 3 false statements on the card. The sole true statement is #3.

Cousin to Viviani's Theorem = Clough's Conjecture = Clough's Theorem

In Issue-I-2 of *At Right Angles* we had presented an account of Viviani's theorem, proved it using vector algebra, and found that the proof gave rise to a corollary in an unexpected and yet very natural way. We called it a 'cousin' to Viviani's theorem:



Let ΔABC be equilateral with side a, and let P be a point in its interior. Let perpendiculars PD, PE, PF be dropped to sides BC, CA, AB respectively. Then $BD+CE+AF = 3a/2$ for all positions of P.

It turns out that this result has been known for a decade, and has a curious history behind it. In the literature it is known as *Clough's Conjecture*. We came to know this through a letter received from Professor Michael de Villiers of the Department of Math Education, University of KwaZulu-Natal, South Africa. He refers us to a paper of his, "An example of the explanatory and discovery function of proof". It was presented at ICME 12 and has now been published in the online journal 'Pythagoras' at: http://www.pythagoras.org.za/index.php/pythagoras/article/view/193.

Readers are urged to download this very readable paper and learn how the result was discovered *empirically* by Duncan Clough, a Cape Town grade 11 student, during a dynamic geometry session in which the students were exploring Viviani's theorem and attempting to prove it; he reported it to his teacher Marcus Bizony, who wrote to de Villiers; and that's how it got the name "Clough's Conjecture" (but it is now a theorem, proved by de Villiers himself). In the paper, the author notes that the incident provides an illustration of the fact that the search for proof sometimes uncovers new results. This is the central thesis of the paper, and it is a matter worth dwelling on as it has important pedagogic implications. He also provides a few proofs of the theorem, shows that it follows from the main Viviani theorem, and deduces some extensions, e.g., to a rhombus and to an equi-angular pentagon.

Many thanks to Prof de Villiers for this communication.

— *The Editors*

A poem on the imaginary number i

THE MATHEMATICAL "i"

by Punya Mishra

The negative numbers were full of dismay
We have no roots, they were heard to say
What, they went on, would be the fruit
Of trying to compute our square root?
Matters seem to be getting out of hand
Since the negatives have taken a stand,
On the fact that positives have two roots, while they have none
They plead, would it have killed anybody to give us just one?
The square roots of 4 are + and – 2! As for – 4? How unfair,
He has none! None at all. Do the math gods even care?
We suspect a plan sinister, our value to undermine
Just because we are on the left of the number line
Among the more irrational negatives, one even heard cries
It is time, they said, it is time, to radicalize!!
Hearing this non-stop (somewhat justified) negativity
Mathematicians approached the problem with some levity
And suggested a solution, kinda cute and fun
Let's rename, they said, the square root of minus 1!
In essence let's re-define the problem away, on the sly
By just calling this number (whatever it may be) i.
i times itself would be one with a negative sign
Every negative could now say, a square root is mine!
This simple move would provide the number -36
With two roots, + and –, i times 6!

All in all, an awesome fix.

The positives grumbled, what could be dumber

Than this silly imaginary number

But it was too late, much too late you see

To bottle this strange mathematical genie.

i was now a part of the symbolic gentry

Finding use, of all places, in trigonometry.

And with time i began its muscles to flex

Extending the plane, making it complex!

In fact, hanging out with the likes of e and Pi

i got bolder, no longer hesitant and shy.

And combined to form equations, bold and profound,

Patterns that, even today, do not cease to astound.

Consider for a moment the equation

*e to the power Pi * i plus 1*

It was Euler who first saw, how these variables react

To come up with a beautiful mathematical fact,

To total up to, (surprise!) the number zero.

Could we have done it without our little imaginary hero?

Even today Euler's insight keeps math-lovers in thrall

One equation to rule them all.

So if you want to perceive the value of this little guy

I guess you have to just develop your mathematical i.

It also reminds us just how often we forget to see

The significance, to human life, of the imaginary.

Michigan State University
punya@msu.edu, http://punyamishra.com

ANNOUNCEMENTS

TIME 2013 + ATCM 2013, a joint session of 18th Asian Technology Conference in Mathematics and 6th Technology & Innovations in Math Education.

Dates: 07-11 December, 2013

Location: Department of Mathematics, Indian Institute of Technology, Powai, Mumbai 400076, India

URL: http://atcm.mathandtech.org/ and http://www.math.iitb.ac.in/TIME2013

Organizer: Inder Kumar Rana / email Address: ikr@math.iitb.ac.in

Description: The ATCM conferences are international conferences addressing technology-based issues in all Mathematical Sciences. The 17th ATCM December 16-20, 2012 was held at SSR University, Bangkok, Thailand. About 400 participants from over 30 countries around the world participated in the conference. The TIME conferences are national (Indian) conferences held every two years. TIME conferences serve a dual role: as a forum in which math educators and teachers come together to discuss and to probe major issues associated with the integration of technology in mathematics teaching and learning, and as a place where they share their perspectives, personal experiences, and innovative teaching practices.

Mathematics of Planet Earth (MPE2013)

International Centre for Theoretical Sciences of the Tata Institute of Fundamental Research (ICTS-TIFR) is a partner institution in Mathematics of Planet Earth (MPE2013), http://www.mpe2013.org – a year-long worldwide program designed to showcase ways in which the mathematical sciences can be useful in tackling our planet's problems. MPE-2013 activities in India began with the Srinivasa Ramanujan Lectures by Andrew Majda of Courant Institute and an associated week-long discussion meeting on "Mathematical Perspectives on Clouds, Climate, and Tropical Meteorology" during 22-26 January 2013 at ICTS, Bangalore: http://www.icts.res.in/lecture/details/1631/. An upcoming MPE2013 ICTS program is on "Advanced dynamical core modelling for atmospheric and oceanic circulations" during 18-23 February 2013 at National Atmospheric Research Laboratory (NARL), Gadanki, Andhra Pradesh, India. Two more such events are under planning.

ICTS and TIFR Centre for Applicable Mathematics (CAM) have partnered to announce "Explore & Exhibit: an Intercollegiate All-India Exhibition Competition" for MPE2013 exhibition modules, along the lines of the global competition. We hope that students, individuals, faculty, and researchers from different institutes within the country will be excited to learn about and to express in an engaging manner some of the mathematics underlying the various processes on the Planet Earth, and submit an entry to this competition. The best ideas will be shortlisted by an eminent panel of judges and selected to be awarded and showcased in an exhibition of physical and virtual modules. Such an exhibition will be hosted by ICTS towards the end of the year. ICTS and CAM are also conducting many related activities such as workshops for college students, camps for school students, public lectures to spread the awareness amongst the general public. For more information please write to mpe2013@icts.res.in.

The Closing Bracket . . .

In the context of the furious debates going on about education in many parts of the world, it is useful to recall the gentle words of a great educator - Richard Skemp (1919–1995), who studied towards a mathematics degree, became a math teacher, and then, convinced that he needed to understand how children learn, returned to college to study psychology. A deep conviction of Skemp's was that young children have the capacity to learn with engagement and understanding, and in consonance with that belief he produced a complete curriculum framework for primary school, called "Structured Activities in Intelligent Learning"; these 'SAIL' books may be freely downloaded from http://www.grahamtall.co.uk/skemp/sail/index.html.

The theme that children are capable of intelligent learning recurs repeatedly in Skemp's writings, and it reflects in a piece he wrote which has now become a classic, "Relational and instrumental understanding". Here he distinguishes between two varieties of understanding (by 'relational understanding' he refers to an understanding where one grasps the subject matter in terms of its network of relationships, connectedness and pathways; 'instrumental understanding' refers to mastery of skills and procedures), asks why teachers the world over seem to prefer teaching for instrumental understanding, plays the Devil's advocate and sets out some of its attractive features, then demonstrates convincingly the lasting value of relational understanding. The article is available at http://www.grahamtall.co.uk/skemp/pdfs/instrumental-relational.pdf.

We recall the words of Noam Chomsky: "An essential part of education is fostering the impulse to challenge authority and think critically." There has never been a time when the need for critical thinking is greater than at present, when fundamentalist forces threaten our very existence, when the strident need for identity and the acceptance of authority have begun to dominate individual lives, and consumerism is ripping apart the Earth. What role can a mathematics teacher play with regard to this great need?

It is sobering to reflect on the state of education in India. (Refer to ASER 2012, available at http://www.asercentre.org/.) Given that we are struggling with matters of basic literacy, basic numeracy and basic amenities available to students and teachers, it may seem surreal to talk of what education can be in a deeper sense.

But in fact the demand becomes all the more vital. Is it not incumbent on us – those who possess the facilities and the wherewithal to do so – to not restrict our teaching to mere instrumental understanding, to not restrict schooling to a mere acquisition of skills meant to sharpen one's competitive instincts to rise up the social ladder, but to allow education its full and deepest expression? As J Krishnamurti (1895–1986) put it in a talk to students in Rishi Valley School, "Education is not only learning from books, memorizing some facts, but also learning how to look, how to listen to what the books are saying, whether they are saying something true or false. . . . Education is not just to pass examinations, take a degree and a job, get married and settle down, but also to be able to listen to the birds, to see the sky, to see the extraordinary beauty of a tree, and the shape of the hills, and to feel with them, to be really, directly in touch with them. As you grow older, that sense of listening, seeing, unfortunately disappears because you have worries, you want more money, a better car, more children or less children. You become jealous, ambitious, greedy, envious; so you lose the sense of the beauty of the earth. You know what is happening in the world. You must be studying current events. There are wars, revolts, nation divided against nation. In this country too there is division, separation, poverty, squalor and complete callousness. Man does not care what happens to another so long as he is perfectly safe. And you are being educated to fit into all this. . . . Is this right, is this what education is meant for, that you should willingly or unwillingly fit into this mad structure called society?" (From the text *Krishnamurti on Education*.) What is our response to this, as mathematics teachers?

- Shailesh Shirali

Specific Guidelines for Authors

Prospective authors are asked to observe the following guidelines.

1. Use a readable and inviting style of writing which attempts to capture the reader's attention at the start. The first paragraph of the article should convey clearly what the article is about. For example, the opening paragraph could be a surprising conclusion, a challenge, figure with an interesting question or a relevant anecdote. Importantly, it should carry an invitation to continue reading.
2. Title the article with an appropriate and catchy phrase that captures the spirit and substance of the article.
3. Avoid a 'theorem-proof' format. Instead, integrate proofs into the article in an informal way.
4. Refrain from displaying long calculations. Strike a balance between providing too many details and making sudden jumps which depend on hidden calculations.
5. Avoid specialized jargon and notation — terms that will be familiar only to specialists. If technical terms are needed, please define them.
6. Where possible, provide a diagram or a photograph that captures the essence of a mathematical idea. Never omit a diagram if it can help clarify a concept.
7. Provide a compact list of references, with short recommendations.
8. Make available a few exercises, and some questions to ponder either in the beginning or at the end of the article.
9. Cite sources and references in their order of occurrence, at the end of the article. Avoid footnotes. If footnotes are needed, number and place them separately.
10. Explain all abbreviations and acronyms the first time they occur in an article. Make a glossary of all such terms and place it at the end of the article.
11. Number all diagrams, photos and figures included in the article. Attach them separately with the e-mail, with clear directions. (Please note, the minimum resolution for photos or scanned images should be 300dpi).
12. Refer to diagrams, photos, and figures by their numbers and avoid using references like 'here' or 'there' or 'above' or 'below'.
13. Include a high resolution photograph (author photo) and a brief bio (not more than 50 words) that gives readers an idea of your experience and areas of expertise.
14. Adhere to British spellings – organise, not organize; colour not color, neighbour not neighbor, etc.
15. Submit articles in MS Word format or in LaTeX.

Call for Articles

At Right Angles welcomes articles from math teachers, educators, practitioners, parents and students. If you have always been on the lookout for a platform to express your mathematical thoughts, then don't hesitate to get in touch with us.

Suggested Topics and Themes

Articles involving all aspects of mathematics are welcome. An article could feature: a new look at some topic; an interesting problem; an interesting piece of mathematics; a connection between topics or across subjects; a historical perspective, giving the background of a topic or some individuals; problem solving in general; teaching strategies; an interesting classroom experience; a project done by a student; an aspect of classroom pedagogy; a discussion on why students find certain topics difficult; a discussion on misconceptions in mathematics; a discussion on why mathematics among all subjects provokes so much fear; an applet written to illustrate a theme in mathematics; an application of mathematics in science, medicine or engineering; an algorithm based on a mathematical idea; etc.

Also welcome are short pieces featuring: reviews of books or math software or a YouTube clip about some theme in mathematics; proofs without words; mathematical paradoxes; 'false proofs'; poetry, cartoons or photographs with a mathematical theme; anecdotes about a mathematician; 'math from the movies'.

Articles may be sent to :
AtRiA.editor@apu.edu.in

Please refer to specific editorial policies and guidelines below.

Policy for Accepting Articles

'At Right Angles' is an in-depth, serious magazine on mathematics and mathematics education. Hence articles must attempt to move beyond common myths, perceptions and fallacies about mathematics.

The magazine has zero tolerance for plagiarism. By submitting an article for publishing, the author is assumed to declare it to be original and not under any legal restriction for publication (e.g. previous copyright ownership). Wherever appropriate, relevant references and sources will be clearly indicated in the article.

'At Right Angles' brings out translations of the magazine in other Indian languages and uses the articles published on The Teachers' Portal of Azim Premji University to further disseminate information. Hence, Azim Premji University holds the right to translate and disseminate all articles published in the magazine.

If the submitted article has already been published, the author is requested to seek permission from the previous publisher for re-publication in the magazine and mention the same in the form of an 'Author's Note' at the end of the article. It is also expected that the author forwards a copy of the permission letter, for our records. Similarly, if the author is sending his/her article to be re-published, (s) he is expected to ensure that due credit is then given to 'At Right Angles'.

While 'At Right Angles' welcomes a wide variety of articles, articles found relevant but not suitable for publication in the magazine may - with the author's permission - be used in other avenues of publication within the University network.

Azim Premji University

Publications that discuss key themes & issues in school education

For Teachers | Teacher Educators | Researchers | Education Administrators & everyone passionate about education.

Learning Curve

A theme-based publication focussing on topics of current relevance to the education sector

Earlier issues of the Learning Curve can be downloaded from **http://www.azimpremjiuniversity.edu.in/content/publications**

To find out more or order a printed copy of the earlier issues or the upcoming one, e-mail your postal address and the issue number to **learningcurve@azimpremjifoundation.org**

At Right Angles

A resource for school mathematics

Earlier issues of At Right Angles can be downloaded from **http://www.azimpremjiuniversity.edu.in/content/publications**

To find out more or order a printed copy of the earlier issues or the upcoming one, e-mail your postal address and the issue number to **AtRightAngles@apu.edu.in**

Language and Language Teaching

A publication focusing on issues and practices relevant to language teaching

Earlier issues of Language and Language Teaching can be downloaded from **http://www.azimpremjiuniversity.edu.in/content/publications**

To subscribe or find out more write to **jourllt@gmail.com**

Teacher Plus

A monthly magazine packed with ideas and addressing the concerns of the practicing teacher

Teacher Plus can be accessed online at **http://www.teacherplus.org/**.

To subscribe or find out more write to editorial@teacherplus.org

Exchanging Experiments and Experiences in Education

At Right Angles
A Resource for School Mathematics

The entire issue can be freely downloaded from:
http://azimpremjiuniversity.edu.in/content/publications

For a print copy, kindly send a mail giving your complete postal address and institutional affiliation to the following e-mail ID: **AtRightAngles@apu.edu.in**

Azim Premji University

A publication of Azim Premji University
together with Community Mathematics Centre,
Rishi Valley

PLACE VALUE

PADMAPRIYA SHIRALI

A PAPER KIT APPROACH

At Right Angles
A Resource for School Mathematics

INTRODUCTION

Teaching of the place value system happens in the context of teaching numbers and is very closely related to counting, grouping objects to aid counting, usage of number decomposition, learning the patterns in number names, learning the written representations of numbers, learning the patterns in the relationships between consecutive places, and developing a proper number sense. Children develop facility with numbers and a sound understanding of the number system only if sufficient care is taken in building all the above mentioned areas.

PRE-REQUISITES BEFORE TEACHING PLACE VALUE SYSTEM

- Recognizing and identifying in terms of objects, the numbers 1 to 9
- Reciting, reading and writing of numerals, number names 1 to 9
- Functional understanding of 0
- Ordering numbers 1 to 9
- Basic addition facts
- Addition facts of 0
- Complementary addition facts of 9 and 10

ACTIVITY ONE

Objective
Introduction of 10 and the relationship between ten and a unit

Materials required:
- Loose sticks or straws, rubber bands
- Loose colour papers, clips
- Dot sheets
- Place value card

Importance

Even though this is the first activity in the teaching of place value and is a fairly simple activity for the child it lays the foundation of the place value system. It needs to be done repeatedly in various situations as will be explained later to help children understand the relationship between a ten and a unit.

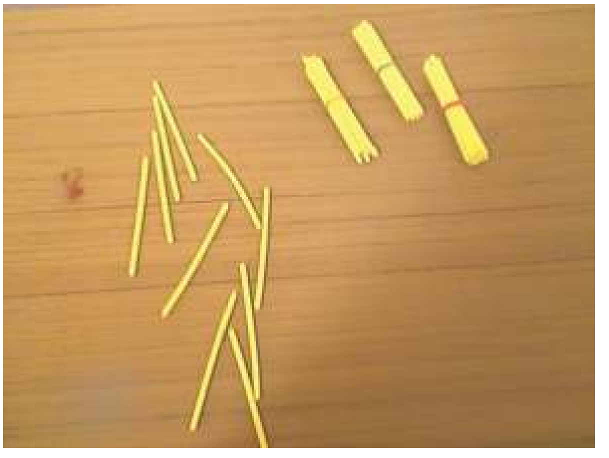
Initially the teacher should count out the sticks (slowly, saying aloud 1, 2, 3, etc.) till he reaches 10 and show them that he is making a bundle of 10 sticks. He should clearly differentiate between the word **sticks** and **bundle** as the sticks are 10 but the bundle is 1.

Let each child count ten sticks carefully and make a bundle of 10 sticks with a rubber band.

The teacher can pick up 7 sticks and ask: "How many more sticks do I need to make a bundle of 10 sticks?" Since we expect children to know complementary facts of 10 by now, they should be able to answer this.

In a similar way the teacher can pick up 12 sticks and ask: "I need to make a bundle of 10 sticks. What do I do?" The children will suggest that he remove 2 sticks and bundle the rest.

Children can be given some seeds and asked to make a group of ten. It is important however to use and emphasize the right language: "This is a group of 10 seeds."



Tens and units sticks

They can be given coloured square paper sheets which they can count and clip. "This is a bundle of 10 papers."

They can also be given dot paper and asked to line 10 dots or circle 10 dots. "This is a group of 10 dots."

They can now be shown how to write ten using a place value card with headers. The use of place value cards (see photograph) facilitates placing of materials and the corresponding number cards in the right places. From the beginning children must see clearly the relationship between the activity or the manipulative and the procedural rules of recording and writing.

ACTIVITY TWO

Objective
Learning to count in tens: 1 ten, 2 tens, and so on, up to 9 tens; and their number names ten, twenty, etc.

Materials required:
- Loose sticks or straws, rubber bands
- Loose colour papers, clips
- Dot sheets
- Place value card
- Flash cards for number names, numerals, objects
- Beads and string

We now repeat activity I by working with more sticks and making several bundles of 10 sticks.

Point out that the bundle that they are making has 10 sticks.

As mentioned earlier one needs to emphasize the language aspect by saying: "Here is 1 bundle of sticks. How many sticks?" Ten. "Here are 2 bundles of sticks. How many sticks?" Twenty.

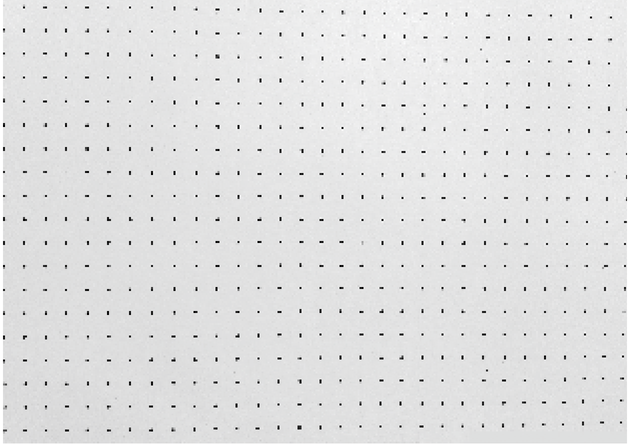
Now the teacher can ask various children to make different numbers of bundles and teach number names for those. They can record them using the place value cards.

The teacher can pick up some bundles and ask "How many sticks?" They first answer by counting the number of bundles and then verify their answer by opening up the bundle and counting the sticks.

Children can also do some exercises with dot paper. They should also be given worksheets which require them to write the numbers for given pictures and draw pictures for given numbers. They can build bead strings with different tens.

Finally children can be given flash cards consisting of pictures of bundles and corresponding number names for matching.

Bead string for tens



Dot sheet

ACTIVITY THREE

Objective
Counting, recording and writing numbers
- From 11 to20
- From 20 to 99

Materials required:
- Loose colour papers, clips
- Ten square strips, loose square slips
- Dot sheets
- Place value card
- Flash cards for number names, numerals, objects
- Number line strip(0 to 99); permanent number line can be drawn below the blackboard
- Number cards

We can now repeat Activity II by working with several bundles of 10 sticks and loose sticks.

Let the children count objects not exceeding 100 (objects kept loose). Show how grouping them into tens makes the task easier.

Let them count objects not exceeding 100 (objects kept in tens and ones).

Let them count both discrete objects (seeds, beads) and continuous objects (line of tiles, strings of beads or flowers, paper rolls with regular markings) not exceeding 100.

Show them some tens and some ones.

Ask them to show fewer sticks than what you have put in front of them.

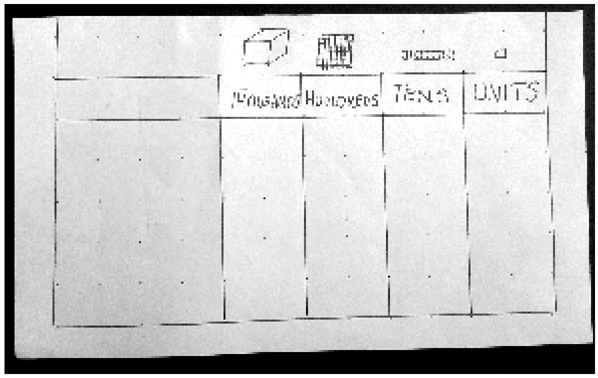
Give them a number and ask them to pick out the required number of tens and ones

Give them a sheet of paper with some dots and let children circle the tens and ones when you call out a number.

Give them various activities which make them record and write different numbers.

Common errors: When asked to write thirty one, a child writes 13. He has not understood that 13 is 1 ten and 3 ones, whereas 31 is 3 tens and 1 one.

One major difficulty with teaching the writing of numbers from 11 to 19 is caused by the mismatch between the way the number is written and the name by which it is called; e.g., 14 is 'fourteen': the word four comes first, which does not happen for numbers



Place value card

from twenty onwards. 61 is 'sixty one'; the number name matches with the way it is written. This problem exists to varying degrees in other languages as well.

Hence while teaching children to record numbers from 11 to 20 it is necessary to emphasize their decomposition: ten and one make eleven, ten and two make twelve, etc., so they associate the tens place digit and units place digit with the correct number.

Practice: The teacher can ask the children to turn to the correct page of a book, given the page number.

Children should also be given worksheets which require them to write the numbers for given pictures and draw pictures for given numbers. The semi-concrete representation is necessary till the children reach the take-off stage.

GAME

Game 1: Double Nine

Objective: Developing number sense

Materials required:

- Ten square strips and loose square slips.
- Dice

Children can be divided into groups of 5. One child becomes a banker and has a stock of loose square slips and strips. Each child throws the dice in turns and collects that many ones (square slips) from the banker. As the children continue to play, they collect more ones. Each time they have a collection of ten ones they exchange it for a strip with the banker. They continue till one of them reaches 99.

ACTIVITY FOUR

Objective

Developing sequential nature of numbers

- From 11 to 20
- From 20 to 99

Materials required:

- Number line

Number line

Many number line exercises can be created which will help in visualizing the sequential nature of numbers.

Teachers should consciously help children to achieve understanding of the succession of numbers by using different manipulatives.

Both forward counting and backward counting should be practised.

GAME

Game 2: Flags

Objective: Sequencing numbers

Materials required:

- Long string
- Number cards

Variation 1: Tie the string across the room. Take some number cards at random and let each child pick up one number card. By turn each can clip it on the string ensuring that they are in increasing order.

Variation 2: The teacher can put up a card on the string and ask questions like: "Who has the nearest card to this?", "Who has the card furthest away from this?", "Who has the nearest ten to this card?", "Who has 5 more than this?", "Who has 10 less than this?", "Who has the card where the tens and units are interchanged?" These questions will stimulate discussion amongst students leading to comparison of numbers, adding, subtracting and paying close attention to the place values.

ACTIVITY FIVE

Objective
Developing number sense, approximation and estimation skills

Materials required:
- Newspapers
- A few textbooks
- A collection of seeds

The teaching of numbers should be accompanied by activities which develop a number sense – i.e., a sense of the size of the number, its relationship with other numbers, properties of the number, proximity to multiples of ten, etc.

GAME

Game 3: Guess the number

Objective: Developing number sense

Let each child take a fistful of seeds and pour them out on his table. Let the child guess the number of these seeds and write it down. Now ask the child to count them by making it into groups of ten.

Ask the children to open a particular page in a textbook. Ask them to guess the number of words on the page (ensure that it is less than 100), or in a given paragraph. Let the child record his guess and then count the words to check how close his guess was.

Ask the children to bring a newspaper. Ask them to circle 50 words (by guessing and not counting). Let them later count the words and check how close their guess was to the actual number.

Number patterns: Plenty of number pattern exercises can be done to build number sense leading to an understanding of number behavior.

ACTIVITY SIX

Objective
Fixing place value through headers: tens (t), units (u) and arrow cards

Materials required:
- Ten square strips and loose square slips
- Place value card
- Arrow cards

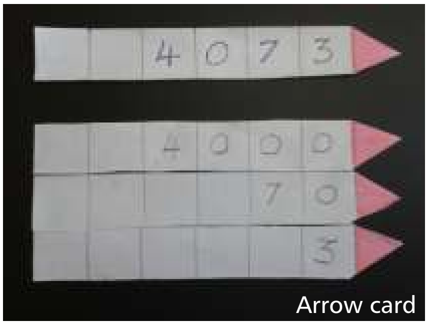
Common errors: Integrating part and whole:

When asked to write twenty three, a child writes 203.

What has led to this error? If the child were now asked to read it, how does he/she read it?

This is a situation of not being able to integrate parts with the whole – the child is treating the tens separately and the three units separately. What form of teaching will prevent these errors?

Arrow cards help in remedying this kind of a situation and making the hidden values explicit for children.



Arrow card

Let the children show the given number on the place value card with strips and slips.

Let them build the number using arrow cards as shown, one below the other and later by placing one over the other to integrate the parts with the whole.

Let the children write the number for the given picture and build the number.

ACTIVITY SEVEN

Objective
Reinforcing place value through the usage of an abacus

Materials required:
- Abacus, beads
- Strips of ten squares and square slips
- Place value cards

Abacus is a useful device in demonstrating place values. But the teacher must keep in mind that it does involve abstraction as one bead in the tens place represents a ten and a bead in the hundred's place represents a hundred.

Introduction to the abacus needs to be done slowly and carefully by actually showing how numbers from 1 to 9 are represented, and that when we need to represent a ten we move to the tens place as the units place can be used for only nine beads. (It may be best to use a model of an abacus which can only accommodate nine beads.) By placing one bead after another progressively we show how numbers 11 to 99 are represented on an abacus. One has to make sure that children grasp the point that each time we have ten ones an extra bead gets added to the tens place. The transitions from 9 to 10, 10 to 11, 19 to 20, 20 to 21, 29 to 30 and 30 to 31 are important; the teacher needs to make the actions clear by giving a 'running commentary'. It is also important to go backwards from 99 to 1 by removing one bead at a time.

Practice: You can make groups of 3 children and give an abacus to the first child, the strips of ten squares and the square slips to the second, and the place value cards to the third. One child shows a number on the abacus, and the other two show the same with their materials. Another now shows a different number using strips and square slips; the other two have to show that number using their materials. And so on.

They should record the work in square ruled note books with appropriate drawings and recordings of numbers.

GAME

Game 4: Reach zero
Objective: Exchanging tens and units

Make a group of 4 children. One child can be the banker. Start with any number say 30, represented on the abacus. Children take turns throwing the die. After each throw, they take away that number from the abacus. If in the first round a child gets four the child will have to remove 1 bead from the tens place and exchange for 10 beads and place 6 back on the units rod. They continue to play till they reach zero.

ACTIVITY EIGHT

Objective
Introducing hundred

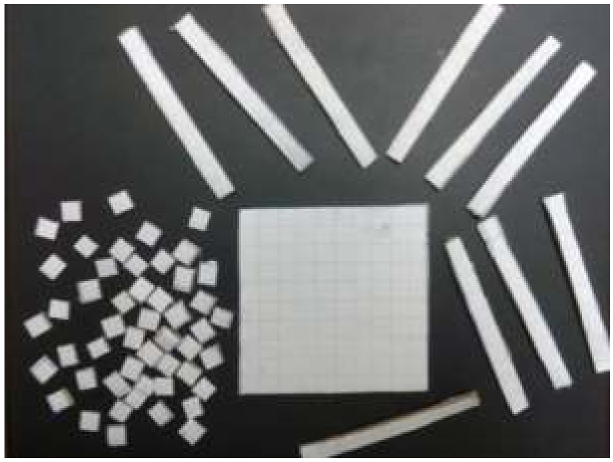
Materials required:
- Abacus, beads
- Hundred square sheets, Ten square strips and Square slips
- Place value cards
- 100 square board

Initially count out the ten square strips (slowly, saying aloud: 91, 92, 93, etc.) till you reach 99 and show them that when one more square slip is added there will be 10 ten square strips (the 10 loose units get exchanged for a ten strip). Now tell them that 10 such strips together make a hundred (they can be exchanged for a hundred square) and show them how it is written. (Let the children verify for themselves that the hundred square is made up of ten strips.) They must see that a hundred equals 10 tens and also 100 units.

You can demonstrate this on an abacus, counting from 99 onwards. It is important to approach the teaching of a new place value in a progressive way, so that children see its relationship to numbers they have met earlier, and the place values.

Now use the place value cards by progressively changing cards from 91 to 99 and point out how the units and tens places both have a 9, and how as you add one more unit to it, a new place (hundred gets created), and the units and tens places both become zero.

Many a time, teachers conduct activities with children without adequate commentary, explanations and questions; without pausing and drawing children's attention to the crucial aspects. The activity will not produce the desired benefits (in terms of improving understanding) if this is not done. Also, the activity needs to be repeated by the teacher and the students a sufficient number of times for the concept to be internalized. It is important to correlate the activity with the materials and the writing by asking questions such as: "How many units do I have here now?" Nine. "What happens when I bring in 1 more unit?" There are 10 units. "Now I exchange the ten units for a strip.



Hundreds, tens and units

So how many ones do I have now?" Zero. "So I write a zero in the ones place." … "How many tens did I have in the beginning?" Nine. "How many tens are there now?"10 tens. "I can exchange 10 Tens for a hundred square. How many tens are there now?" Zero. "So I write a zero in the tens place. How many hundreds do I have now?" One. "So I write a 1 in the hundreds place." And so on.

Explore: Let children write all the numbers from 1 to 100 in a 10 by 10 square. There are many patterns in a number square which the children can notice and share. For example, if they look at the numbers vertically (along the columns) they see 23, 33, 43, 53, etc., leading to understanding of addition by tens. If they look at the numbers along the diagonals they see how the units and tens places are changing. They notice what happens when they increase any number by nine. A modified version of snakes and ladders can be played using a 100 square board.

ACTIVITY NINE

Objective
Teaching 101 to 999

Materials required:
- Hundred square sheets, Ten square Strips and Square slips
- Place value cards
- Abacus, beads
- Arrow cards

The initial focus is on numbers 100 to 200.

Each child must have a place value kit (hundreds, tens and units material, arrow cards, place value card, abacus) which can be used for depicting any number between 100 and 200.

Handling the concrete material should be followed by a representation (semi-concrete) in the square ruled note book, accompanied by the written form of the number.

Common errors: When asked to write the number which comes after 129, a child writes 1210.

What could be the causes for this?

The child has not understood that when the units increase to ten, it alters the tens place and the units place.

The child has also not grasped that any place can hold only one digit.

The child may not be reading the number as a whole –not as 'one hundred and twenty-nine' but as 'one two nine'.

A child who has handled concrete materials for a sufficient length of time would have internalized the relevant concept, and this would have prevented and corrected these types of problems.

It is important to focus on these transition points in numbers: 119-120-121,129-130-131,139-140-141, etc. Children need to perceive the patterns present here. Many text books do have exercises which expect children to fill a 10 by 10 square grid with numbers from 101 to 200. This exercise is meaningful if teachers pose questions based on this, requiring them to observe and record different kinds of patterns and helping them to generalize from the observations.

Once children are thorough with numbers from 100 to 200, one can proceed to 200 to 999.

Common error: When asked, "How many tens are there in 342?" a child responds by saying '4 tens'.

This error comes from the child not having understood that each higher place is composed of the lower places.

The teacher will need to show that the hundreds are composed of tens and 3 hundreds are composed of 30 tens. So the number 342 contains 34 tens and 2 units. While discussing place value it is important to help the child to realize that tens are composed of units, hundreds are composed of tens and units and so on.

As practice one needs to pose exercises like: 254 = ___ tens + ____units, with the blanks to be filled.

Common error: While comparing numbers, a child writes '97 > 102'.

This is an error of incorrect application of procedures. The child is comparing the starting digits in both the numbers without reading the whole numbers with their place values.

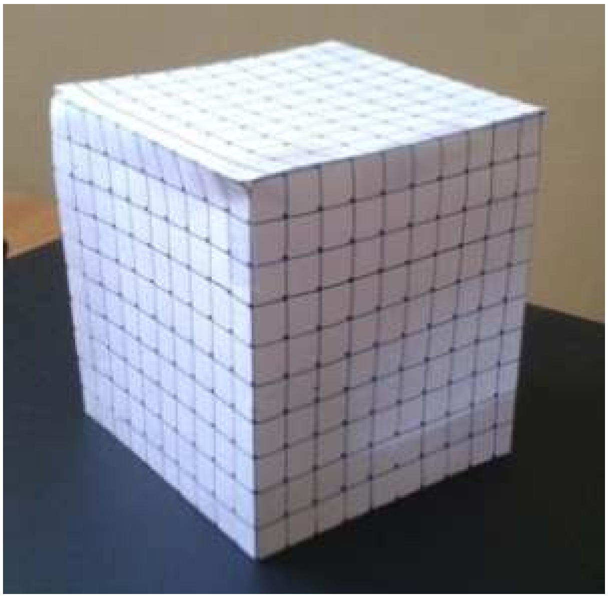
ACTIVITY TEN

Objective
Introducing thousand

Materials required:
- Wooden cube or a card board cube (as shown in the picture), hundred square sheets, ten square Strips and Square slips
- Place value cards
- Abacus, beads

Use materials or drawings to show 991, 992, 993, etc., till you reach 999, and ask the children what would happen when one more unit is added. Lead them to discover that one more unit will increase the tens by one more ten, one more ten will lead to an increase in hundreds by one more hundred and that we will then have 10 hundreds. Now you can tell them that 10 hundreds is called a thousand and is written as 1000. You can show the model for 1000. If you show them a cube (as shown in the picture) you will need to discuss with the children the layers of hundreds in them, get them to count these layers so that they are able to visualize the 1000. They need to see that a thousand equals each of the following: 10 hundreds, 100 tens, 1000 units. As the numbers grow larger and counting is not an option any longer, children need to notice relationships and patterns, generalize them and develop the capacity for abstraction.

You can now demonstrate this on an abacus, counting from 990 onwards.



Paper cube

ACTIVITY ELEVEN

Objective
Thousands and larger numbers

Materials required:
- Abacus, beads

Beyond thousand it is not necessary to use concrete materials as the child would have internalized the relationships between different places, and would be in a position to extend his understanding to other places.

However usage of abacus does help many children, particularly children who have difficulties with visualization and abstraction. It aids in gaining a functional understanding of zero. Many children make errors while writing large numbers with several zeroes in them. Abacus as a visual aid helps in strengthening the memory of the place value order.

Some activities and games given above can be used for teaching thousands and larger numbers.

It is important that the teacher helps the children to develop both factual understanding as well as procedural understanding. In the context of place value, children must understand that a thousand is equal to 10 hundreds, 100 tens, 1000 units. They must also understand that each place is created by taking 10 times the lower place. Once they are clear on the procedures, they will then be able to generalize them and apply them to higher place values (up to lakhs or millions at the appropriate age).

Right from the beginning we must help children to learn the place values in order from the right most number. For example: 32,504. We need to point to 4 (while saying units) and move step by step mentioning each place value so that the child notes the order. This will address the problem of wrong reading of numbers. The point that one has to stress is that the value of a place is determined by reference to the right most place.

"What number precedes 2,01,010?" A child writes 2, 00,009.

We find that the child has not completely understood the way numbers increase.

Teachers should consciously help children to achieve understanding of succession of numbers by discussing many such problems.

GAME

Game 5: Twenty questions

Tell the children that your number lies between 100 and 200. The children have to find the number by asking twenty questions. They can only ask questions of the kind which require an answer "yes" or "no". They may ask a question like "Is the number more than 130?" Teacher can draw a number line on the board, and after each yes/no answer, cross out the irrelevant part to help children in visualizing the range within which the number lies. It also helps children learn how to ask good questions, how to eliminate the unnecessary parts, and how to use diagrams in problem solving.

Padmapriya Shirali

Padmapriya Shirali is part of the Community Math Centre based in Sahyadri School (Pune) and Rishi Valley (AP), where she has worked since 1983, teaching a variety of subjects – mathematics, computer applications, geography, economics, environmental studies and Telugu. For the past few years she has been involved in teacher outreach work. At present she is working with the SCERT (AP) on curricular reform and primary level math textbooks. In the 1990s, she worked closely with the late Shri P K Srinivasan, famed mathematics educator from Chennai. She was part of the team that created the multigrade elementary learning programme of the Rishi Valley Rural Centre, known as 'School in a Box'. Padmapriya may be contacted at **padmapriya.shirali@gmail.com**