HOW to PROVE it

SHAILESH SHIRALI

In this episode of "How To Prove It", we discuss some more applications of Ptolemy's theorem.

In the last episode of this series, we stated and proved Ptolemy's theorem. We also showcased a few elegant applications of the theorem. In this episode, we study more such applications. We also state and prove a related theorem which is associated with the names of the ancient Indian mathematicians Brahmagupta (7th century), Mahavira (9th century) and Paramesvara (15th century).

Ptolemy's theorem states the following (see Figure 1):

Theorem 1 (Ptolemy of Alexandria).
If ABCD is a cyclic quadrilateral, then we have the equality $AB \cdot CD + BC \cdot AD = AC \cdot BD$.

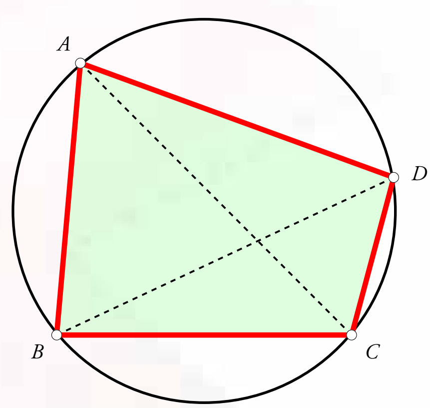


Figure 1. Ptolemy's theorem

Some applications of Ptolemy's theorem
We showcase below a few more applications of the theorem (we had discussed a few such in the previous article). The first is a proof of the most venerable theorem of all.

Keywords: Ptolemy, similar triangle, power of a point

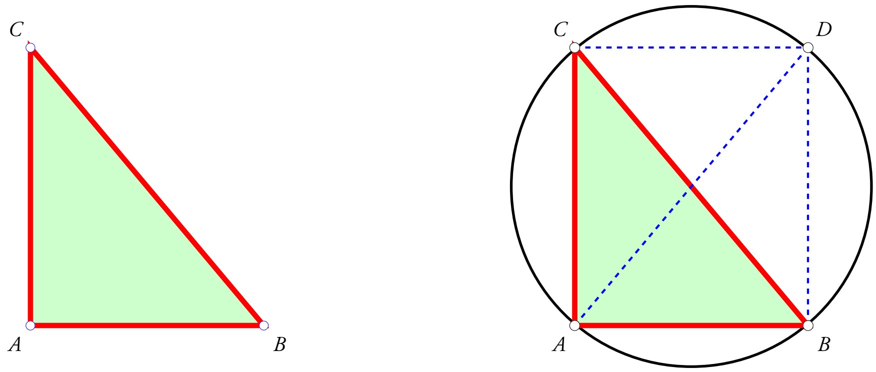


Figure 2. Application of Ptolemy's theorem--I: Proof of Pythagoras theorem

Theorem 2 (Pythagoras).
Let ABC be a right-angled triangle, with the right angle at vertex A. Then $b^2 + c^2 = a^2$.

Proof. Locate point D such that $ABDC$ is a rectangle (note the cyclic order of the vertices: $ABDC$, not $ABCD$; see Figure 2). Since $ABDC$ is cyclic (a rectangle is necessarily cyclic), Ptolemy's theorem applies to it.

Ptolemy's theorem tells us that $AB \cdot CD + AC \cdot BD = BC \cdot AD$. Since $AB = CD$, $AC = BD$ and $AD = BC$ (all these follow because $ABDC$ is a rectangle), we get: $AB^2 + AC^2 = BC^2$, i.e., $b^2 + c^2 = a^2$. □

Cosine rule. A slight alteration of the above yields a proof of the cosine rule.

Theorem 3 (Cosine rule).
Let ABC be an arbitrary triangle. Then we have the following relation: $a^2 = b^2 + c^2 - 2bc\cos A$.

Proof. Locate point D such that $ABDC$ is an isosceles trapezium, with $AB \parallel CD$, $AC = BD$ and $\angle CAB = \angle DBA$ (note again the cyclic order of the vertices: $ABDC$, not $ABCD$; see Figure 3). Since $ABDC$ is an isosceles trapezium, it is cyclic, so Ptolemy's theorem applies to it.

Since $AD = BC$ and $AC = BD$, we get: $BC^2 = AC^2 + AB \cdot CD$, or:

$$a^2 = b^2 + c \cdot CD.$$

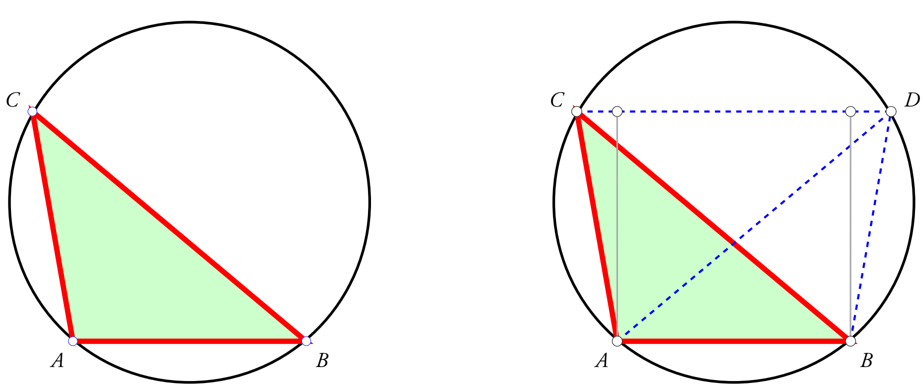
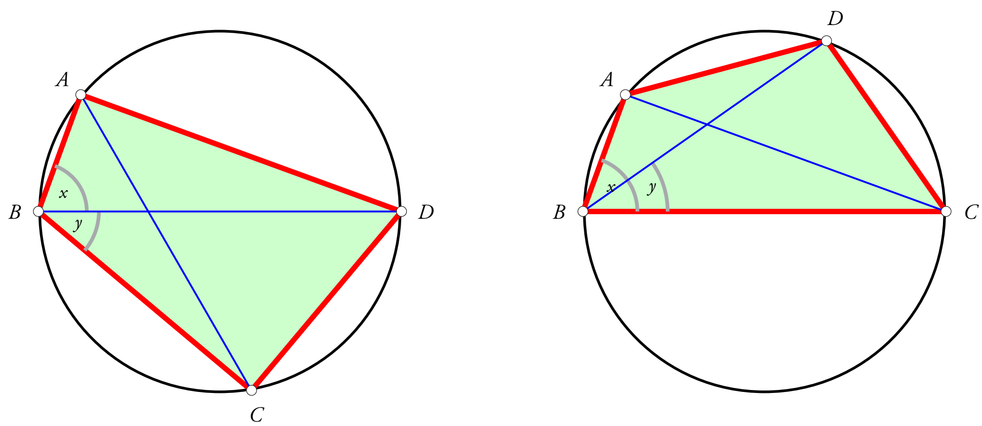


Figure 3. Application of Ptolemy's theorem--II: Proof of the cosine rule



$BD = 1$, $\angle ABD = x$, $\angle DBC = y$ $\qquad$ $BC = 1$, $\angle ABC = x$, $\angle DBC = y$

Figure 4. Derivation of the addition and difference rules for the sine function

By dropping perpendiculars from A and B to line CD (they have been shown using a very light colour to avoid creating a visual clutter), it follows readily that

$$CD - AB = 2\,AC \cos \angle ACD,$$

and since $\angle ACD$ and $\angle CAB$ are supplementary, it follows that $CD = c - 2b \cos A$. Therefore $a^2 = b^2 + c^2 - 2bc \cos A$, as required. □

Our third application features the proofs of two important trigonometric identities: the addition and difference rules for the sine and cosine functions. Historically, Ptolemy used his theorem to draw up tables of values of the trigonometric functions by traversing this very route. It is thus of interest to mathematicians as well as historians. Here is how it is done.

The addition and difference rules for the sine function.

The single result from geometry-trigonometry that we shall use repeatedly is this: in a circle with radius R, if a chord with length d subtends an angle x at the circumference, then we have: $d = 2R \sin x$.

Let acute angles x, y be given, $0 \le y \le x \le \pi/2$. Draw a circle with diameter $2R = 1$ unit; see Figures 4 (a) and (b).

In (a), BD is a diameter of the circle, and angles x, y are non-overlapping. Using the geometric-trigonometric result mentioned above, we get: $AB = \cos x$, $BC = \cos y$, $CD = \sin y$, $AD = \sin x$, $AC = \sin(x+y)$, $BD = 1$. Hence by Ptolemy's theorem:

$$\sin(x+y) = \sin x \, \cos y + \cos x \, \sin y,$$

and we have obtained the addition formula.

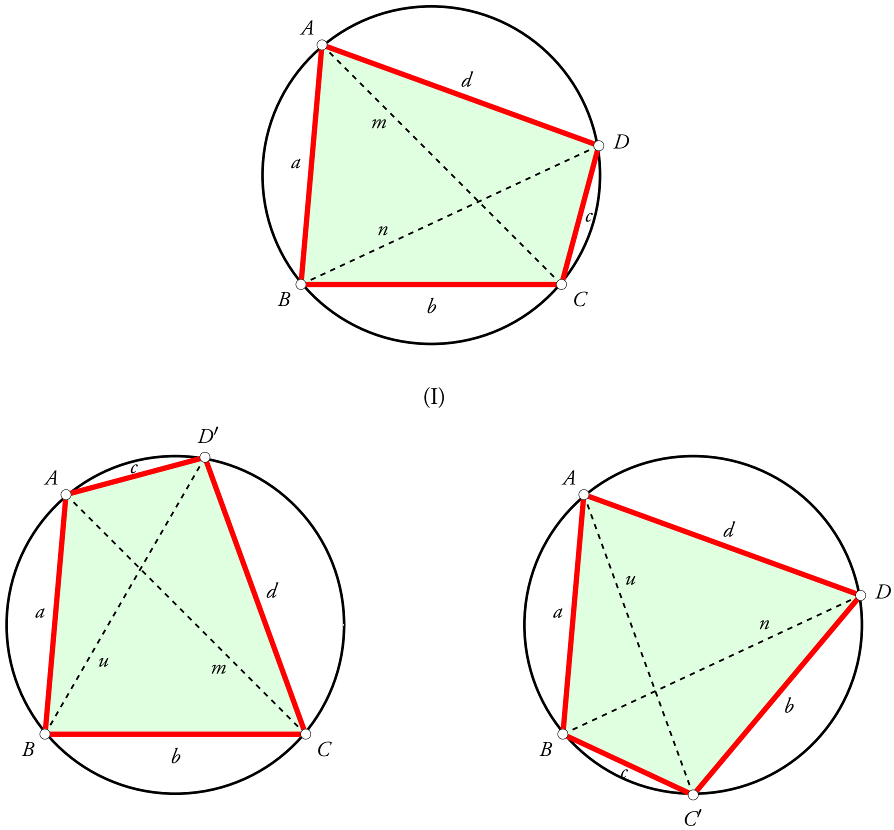
In (b), BC is a diameter of the circle, and angles x, y are overlapping. This time we get: $AB = \cos x$, $BC = 1$, $CD = \sin y$, $AD = \sin(x-y)$, $AC = \sin x$, $BD = \cos y$. Hence by Ptolemy's theorem:

$$\sin x \, \cos y = \cos x \, \sin y + \sin(x-y),$$

and hence:

$$\sin(x-y) = \sin x \, \cos y - \cos x \, \sin y.$$

We have obtained the difference formula.



(I) → (II): Swap c and d (I) → (III): Swap b and c

Figure 5. Formulas for the lengths of the diagonals

The Brahmagupta-Mahavira Identities

The formulas presented here give the lengths m, n of the diagonals of the cyclic quadrilateral, expressed in terms of the sides a, b, c, d, the sides being in the cyclic order a, b, c, d; see Figure 5 (I). The proofs of the formulas are amazingly simple; they are the work of Paramesvara (15th century), though the formulas themselves were discovered by Brahmagupta (7th century) and again by Mahavira (9th century).

We perform two independent operations on (I), giving Figures (II) and (III). To get (II), we swap sides c and d (this is equivalent to reflecting D in the perpendicular bisector of AC). The sides of the resulting quadrilateral (which is still cyclic) are in the order a, b, d, c, and the diagonals are m, u (diagonal m stays unchanged).

To get (III), we swap sides b and c (this is equivalent to reflecting C in the perpendicular bisector of BD). The sides of the resulting quadrilateral (which is still cyclic) are in the order a, c, b, d, and the diagonals are n, u (diagonal n stays unchanged). The crucial point here is that diagonal AC' has the same length as diagonal BD'. Hence the choice of the symbol u to denote their common length.

Using Ptolemy's theorem, we obtain the following three relations:

$$mn = ac + bd,$$

$$mu = ad + bc,$$

$$nu = ab + cd.$$

The last two relations yield:

$$\frac{m}{n} = \frac{ad + bc}{ab + cd}.$$

Combining this with the first relation, we get the Brahmagupta-Mahavira identities:

$$m^2 = \frac{(ac + bd)(ad + bc)}{ab + cd}, \qquad n^2 = \frac{(ac + bd)(ab + cd)}{ad + bc}.$$

We have obtained formulas which yield the lengths of the diagonals. The relation

$$\frac{m}{n} = \frac{ad + bc}{ab + cd}$$

is sometimes called the *ratio form of Ptolemy's theorem*.

References

1. A. Bogomolny, "Ptolemy's Theorem" from *Interactive Mathematics Miscellany and Puzzles*, http://www.cut-the-knot.org/proofs/ptolemy.shtml, Accessed 03 September 2016
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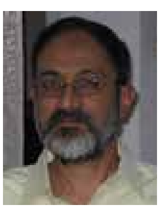

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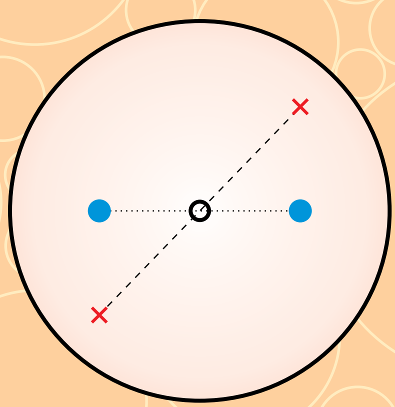
SOLUTION TO FILLING A CIRCLE

Problem on page ....13........

Rahul: 'The winning strategy is based on the perfect symmetry of a circle. I can ensure my win if I play first and if I do, then I place the first coin exactly at the centre of the table. Subsequently, I simply imitate your actions by mirroring them in the centre of the table. If you have space for a coin, on the table, then so will I. So you will be the first to be unable to find space for a coin.'



The winning strategy. Clearly, Rahul is a winner!