FAGNANO'S PROBLEM
A Geometric Solution

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When we see a problem on maximization or minimization, we generally think of calculus or linear programming. But in reality, problems are never bound to a specific tool and we are always free to try something different, like high school geometry and even a bit of physics!

Here is one such minimization problem: *Inspired by the American ministry of defense headquarters, the Pentagon, the Department of Mathematics has built* ***The Triangle*** *– a triangular building with three wings named A, B, C after Aryabhata, Brahmagupta and Chandrasekhar, respectively. The gates of the wings opening into the central courtyard are connected to one another with a triangular walkway. To ensure close interaction among the mathematicians, it is decided to minimize the length of this walkway, by optimally positioning the gates of the wings.*

Can you find the length (perimeter) of the shortest walkway?

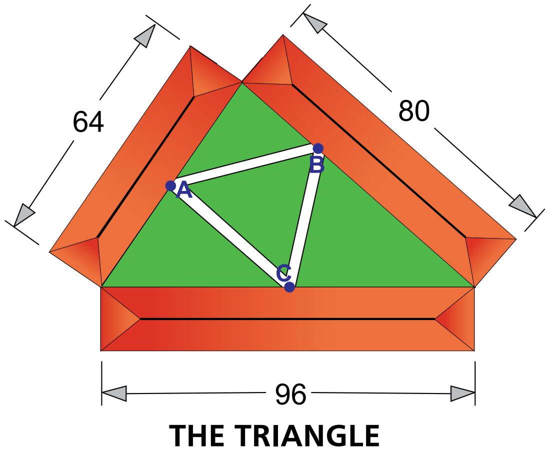

Figure 1

Keywords: optimization, triangle, ellipse, confocal, similar triangle, cosine rule, reflection, Fermat's Principle

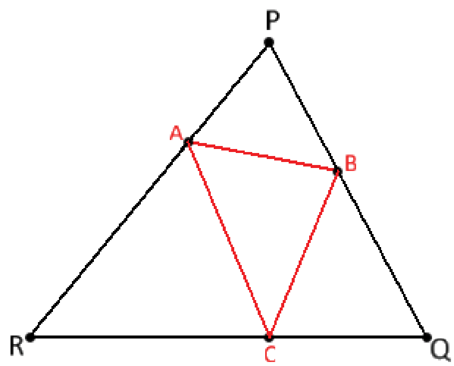

Figure 2

Historical reference: The abstract form of this problem was first solved by Giovanni Fagnano in 1775, stated as: *Given an acute-angled triangle (PQR), find the inscribed triangle (ABC) of minimum perimeter.*

[Comment: The restriction that triangle PQR should be acute-angled may not seem clear. What happens if instead the triangle is right-angled or obtuse-angled? It turns out that if PQR is either right-angled or obtuse-angled, the optimization problem is rather trivial. The reason for this is explained in an addendum to this article.]

Fagnano used calculus to solve it, but let's try some geometry and perhaps a dash of physics!

To start with, imagine that we have partially solved the problem and already found the best positions of gates A and B and now we want to minimize the length A-C-B.

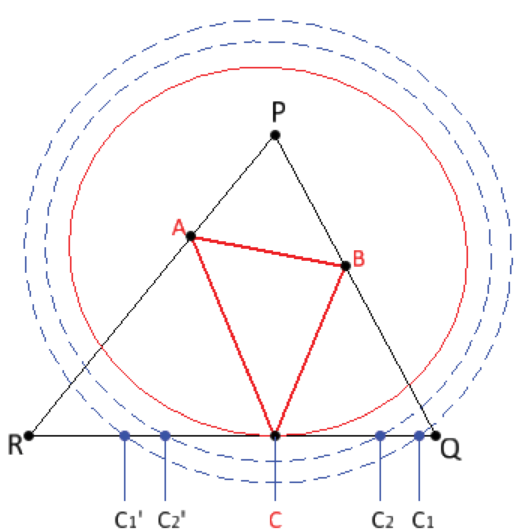

Figure 3

Consider any path A-C-B as a string with its ends A and B fixed with pins. Does that remind you of something? The string and pins method of plotting an ellipse, of course!

So for a given length of A-C-B, point C lies on an ellipse with A and B as its foci, shown by dotted blue lines in the figure.

Such an ellipse, in general, will intersect line RQ in two points (say C_1 & C_1')

Now, if we shorten the 'string', the ellipse shrinks inwards, and we get a smaller 'confocal' ellipse which brings the two points of intersection (C_2 & C'_2) closer. If we continue to shorten the string, ultimately the two points of intersection will merge into a single point (C), which will make the ellipse tangential to line RQ at (C). And then we can use an interesting property of a tangent to an ellipse.

Property: The tangent to an ellipse is equally inclined to the lines (F_1T & F_2T) joining the point of tangency (T) to the two foci (F_1 & F_2).

Think of the two foci as two gates already found and (T) as the gate of the third wing represented by the tangent.

Conclusion: The sides of the optimized inscribed triangle ABC will be equally inclined to the sides of the outer triangle PQR.

Interestingly, this result can be reached via physics too, using what is called as Fermat's Principle.

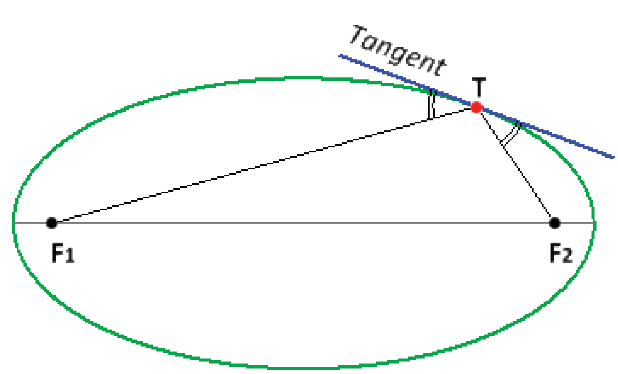

Figure 4

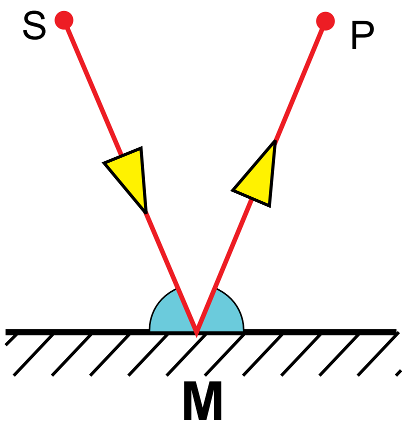

Figure 5

Fermat's Principle: Light always takes the quickest path.

We can apply it to a ray of light coming from a source (S), getting reflected from a plane mirror at (M) and reaching point (P) as shown. From Fermat's principle, S-M-P must be the shortest path from (S) to (P) via the mirror.

We know from basic optics, that the incident ray (SM) and the reflected ray (MP) are equally inclined to the normal and hence also to the mirror – a conclusion reached earlier using geometry!

So we conclude that **all** sides of the optimized inscribed triangle (ABC) must be equally inclined to the sides of the outer triangle (PQR).

In ΔARC, ΔPAB and ΔBCQ (Figures 6 and 7):

$\angle R + \gamma + \alpha = 180^\circ$, $\angle P + \alpha + \beta = 180^\circ$, $\angle Q + \beta + \gamma = 180^\circ$.

Adding the three equations, we get

$(\angle R + \angle P + \angle Q) + 2(\alpha + \beta + \gamma) = 3 \times 180.$

This yields $\alpha + \beta + \gamma = 180^\circ$, and $\angle R = \beta$ $\angle P = \gamma$ $\angle Q = \alpha$.

Thus ΔARC, ΔPAB and ΔBCQ are similar to ΔPQR

Let,

$x, y, z =$ the scale factors of ΔARC, ΔPAB and ΔBCQ relative to ΔPQR respectively.

$p, q, r =$ sides RQ, PR and PQ respectively; then,

$p = xq + zr, \quad q = xp + yr, \quad r = yq + zp.$

Solving simultaneously for x, we get

$x = \dfrac{q^2 + p^2 - r^2}{2pq}$ which is equal to $\cos R$.

Thus,

$RC = RP \cos R.$

This can happen if ΔRPC is a right angled triangle. In other words, (C) is the foot of the altitude from (P).

By symmetry, points (A) and (B) too would lie at the feet of the altitudes from (Q) and (R) respectively.

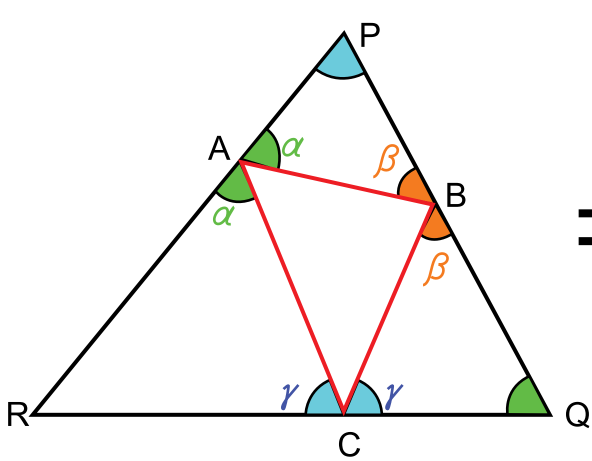

Figure 6

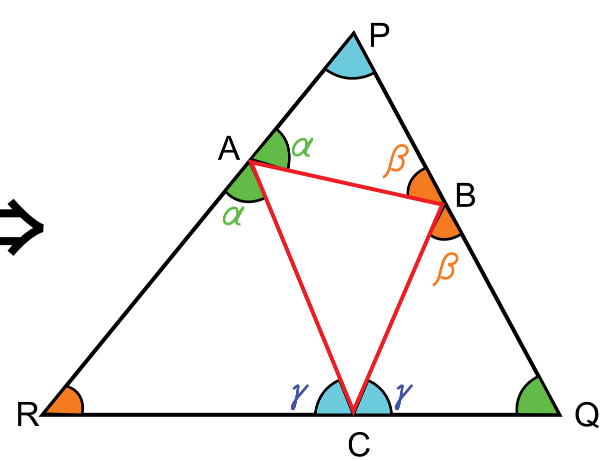

Figure 7

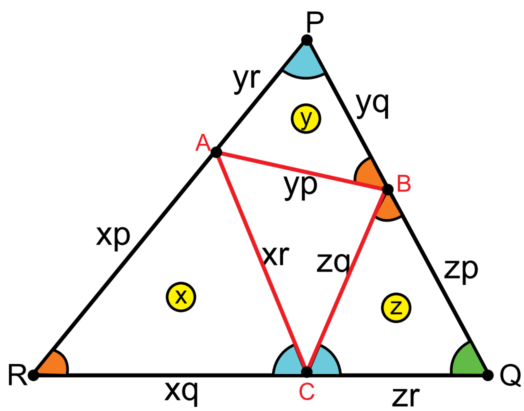

Figure 8

Thus the desired triangle is the **Orthic Triangle** – a triangle with the feet of the three altitudes as its vertices.

Now for the perimeter itself:

Perimeter $= p \cos P + q \cos Q + r \cos R$.

Substituting the given lengths of wings:
$p = 96, q = 64, r = 80$,

$$p \cdot \cos P = 96 \times \frac{64^2 + 80^2 - 96^2}{2(64)(80)} = 12,$$

$$q \cdot \cos Q = 64 \times \frac{80^2 + 96^2 - 64^2}{2(80)(96)} = 48,$$

$$r \cdot \cos R = 80 \times \frac{64^2 + 96^2 - 80^2}{2(64)(96)} = 45.$$

So the minimal perimeter is $45 + 48 + 12 =$ **105** units.

Additional note from the author: I have created two videos of the solution and placed them online. Here are their short URLs:

http://tinyurl.com/FagnanoAnalytical

http://tinyurl.com/FagnanoPhysical

Please view them in full screen. Both videos have closed captions for their entire length.

UJJWAL RANE is a Mechanical Engineer by training and has master's degrees from IIT Madras (Machine Dynamics) and Arizona State University, USA (Computer Aided Geometric Design). He has a particular interest in finding visual representations and solutions of mathematical and physical phenomena and in using this approach in classroom teaching of Engineering, Physics and Math, and also online via his channel on YouTube (https://www.youtube.com/user/UjjwalRane). He can be reached on ujjukaka@gmail.com.