Method in Mathness

CATALOGUING 1-UNIFORM TILINGS

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In my earlier two articles on Tessellations – *Covering the Plane with Repeated Patterns* Parts I and II – which appeared in *At Right Angles* (March 2014 and July 2014), I tried to provide a glimpse into the topic of recognizing regular polygons that can tessellate. In an accompanying article, *Enumeration of Semi-regular Tessellations* (AtRiA, March 2014), the authors arithmetically enumerated the combinations of regular polygons whose interior angles could fit together to make a complete angle (i.e., 360°); 17 such combinations were listed. However, not all of these tessellate. The present article may be viewed as an extension of the earlier ones; I have tried to identify and shortlist the combinations that extend to create semi-regular tiling patterns. (*Note from the editor: See the glossary at the end of the article for explanations of terms with which you may not be familiar. Some potentially unfamiliar terms have been highlighted for you.*)

Code	Number of faces						Code	Number of faces					
	n_1	n_2	n_3	n_4	n_5	n_6		n_1	n_2	n_3	n_4	n_5	n_6
A	3	7	42				**K**	6	6	6			
B	3	8	24				**L**	3	3	4	12		
C	3	9	18				**M**	3	3	6	6		
D	3	10	15				**N**	3	4	4	6		
E	3	12	12				**P**	4	4	4			
F	4	5	20				**Q**	3	3	3	4	4	
G	4	6	12				**R**	3	3	3	3	6	
H	4	8	8				**S**	3	3	3	3	3	3
J	5	5	10										

Table 1: The 17 combinations

Keywords: *Tessellation, tiling, enumeration, regular polygon, edge-to-edge tessellation, regular tessellation, semi-regular tessellation, demi-regular tessellation, Archimedean tessellation*

Enumeration of tiling patterns has been sporadic. The credit for categorizing tessellations goes to Johannes Kepler. In Book II of *Harmonices Mundi* (1619), Kepler enumerated tessellations which have the property that the way in which the polygons are arranged around each vertex is the same for all vertices; tessellations with this property are called **1-*uniform vertex-homogeneous tilings***. Subsequently, Krotenheerdt, Chavey and Galebach succeeded in cataloguing tessellations in which there is more than one vertex-homogeneity in the pattern. In other words, they systematised tessellations that have *k* kinds of vertices in the pattern. Tessellations with this property are called ***k-uniform*** **(*k* ≥ 2)** ***vertex tilings***. (See references [1], [2] and [3].)

An ***edge-to-edge tessellating pattern*** is one in which two polygons which touch each other do so along complete edges. An immediate consequence of this requirement is that the edge lengths of all polygons in the pattern are the same. Figure 1a shows an example of such a pattern, and Figure 1b shows an example of a tessellating pattern which is *not* edge-to-edge. In this article, we consider only tessellating patterns which are edge-to-edge.

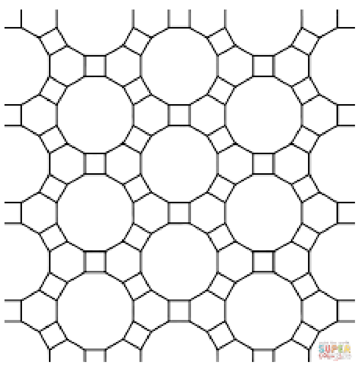

Example of edge-to-edge tessellation

Figure 1a

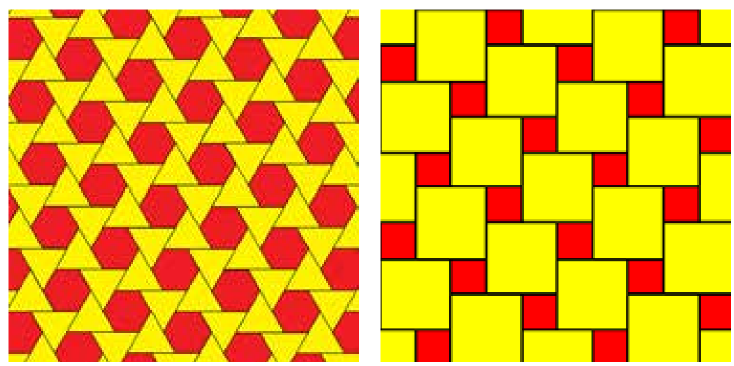
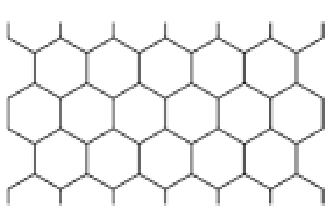
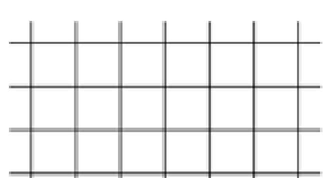

Examples of non-edge-to-edge tessellation of regular polygons

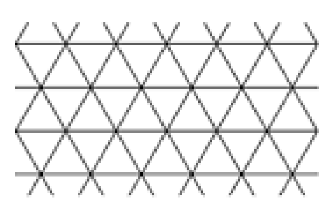
Figure 1b

Edge-to-edge tessellating patterns which also maintain vertex-homogeneity have been classified as ***regular, semi-regular*** and ***demi-regular.*** In regular tessellations and semi-regular tessellations, there is a single uniform vertex configuration all through the pattern (such tilings are also called 1-***uniform Archimedean tilings***), whereas in a demi-regular tessellation there are two or more vertex configurations. In other words, tessellations in which a combination of two or more polygons repeat to cover a plane are termed as semi-regular, and patterns in which two or more vertex configurations co-exist are termed as demi-regular. *(Note: Some mathematicians define a demi-regular tessellation as a combination of semi-regular tessellations.)* Though there does not exist much consensus on the number of demi-regular tessellations, we are confident that there are only three regular tessellations and only eight semi-regular tessellations. In this article, we study the eight configurations that give rise to semi-regular tessellations. The ideas presented in this article can be extended for identifying, constructing and cataloguing demi-regular tessellations.

The first category of 1-uniform vertex tessellation is that of regular tessellations, in which the component polygons all have the same number of edges. Combinations K, P and S (Table 1) make regular tessellations (see Figure 2).


6.6.6
Figure 2a


4.4.4.4
Figure 2b


3.3.3.3.3.3
Figure 2c

To catalogue semi-regular tessellations, let us work with the remaining 14 combinations in Table 1. Through the enumeration process we had found all combinations of polygons that meet to form 360° at a vertex. It so happens that this process conveys little about the actual space fitting of the polygons. For example, the lexicon 3.3.3.4.4 only tells us that three equilateral triangles and two squares join together to complete the vertex. It does not, however, convey any information about the spatial arrangement of these triangles and squares. Would the three triangles come together as 3.3.3.4.4, or would they alternate as 3.4.3.3.4? Will both the arrangements tessellate? Will these make 1-uniform vertex tessellations?

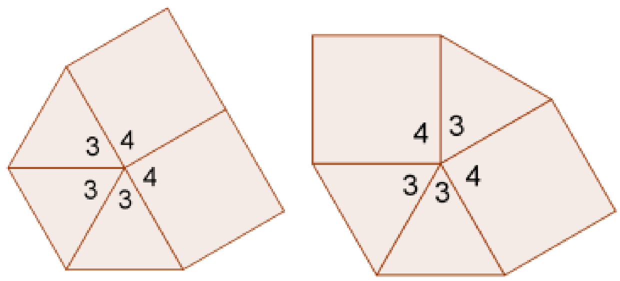


Figure 3

Three Regular Polygons at a Vertex

Let us start with a combination of three polygons of which one is a regular triangle: 3.A.B. In this lexicon, 3 refers to a regular triangle, A and B are regular polygons with A sides and B sides, respectively. The edge lengths of all the polygons are the same. To maintain a uniform vertex-homogeneity, the triangle must have the same spatial configuration of polygons at all its three vertices; i.e., they must be 'surrounded' by polygons in the same manner.

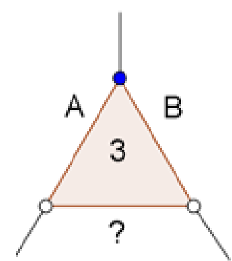


Figure 4

In the above representation, polygons A and B meet to complete the blue vertex. However, the white vertices will get completed only when the polygon at '?' is either A or B. The only way to do this is by setting A = B; so the pattern becomes 3.A.A. This eliminates combinations 3.7.42, 3.8.24, 3.9.18 and 3.10.15. Thus there exists only one combination, E (3.12.12), that extends to make 1-uniform vertex tessellation.

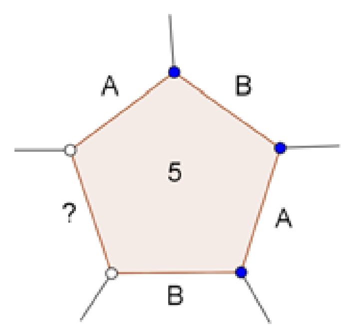


Figure 5

The same argument holds for all odd sided polygons incident to two other regular polygons. This eliminates F (4.5.20, read it as 5.20.4) and J (5.5.10).

With a four-sided regular polygon, incidental to two other regular polygons, the configuration would be 4.A.B. Figures 6a and 6b explain that there can be only two ways in which 4.A.B would extend while maintaining single vertex-homogeneity in the plane – either A is equal to B or A and B are placed alternately. This qualifies H (4.8.8) and G (4.6.12).

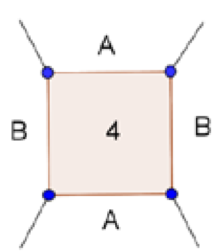


Figure 6a

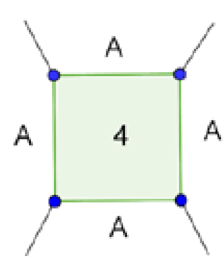
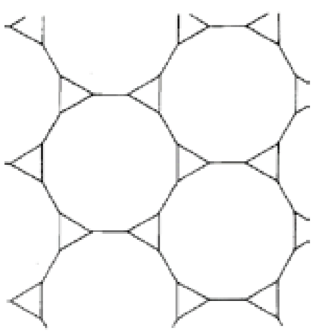
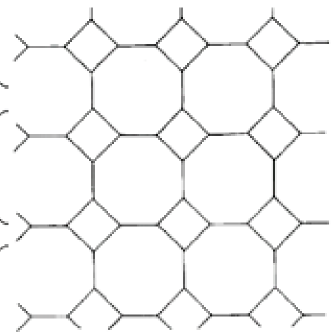


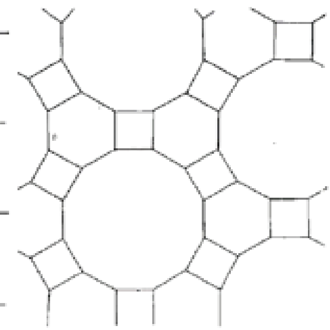
Figure 6b



3.12.12
Figure 7a



4.8.8
Figure 7b



4.6.12
Figure 7c

Summarizing, there can be only three 1-uniform vertex Archimedean tessellations of three regular polygons meeting at a vertex, and these are 3.12.12, 4.8.8 and 4.6.12 (see Figure 7).

Four Polygons

Let the tessellation have two triangles and two other polygons: 3.3.A.B. These polygons can be arranged as 3.3.A.B or 3.A.3.B. Let us begin by placing polygons A and B at the top blue vertex (Figure 8a). The polygons read in cyclic order would be 3.3.A.B. To have the same configuration at the red vertex, the polygons must be arranged in counter-clockwise order. At the white vertex, however, there will be three triangles making the sum total of angles 180º, leaving no possibility of fitting any other polygon. Thus, 3.3.A.B will not make a semi-regular tessellation. This shows the impossibility of both 3.3.4.12 (L) and 3.3.6.6 (M).

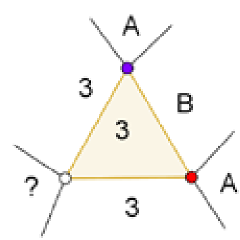


Figure 8a

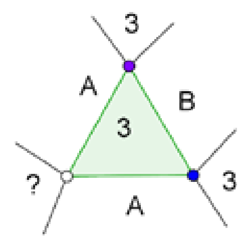
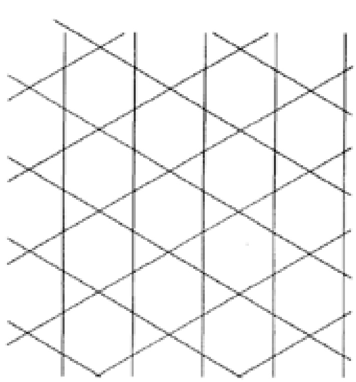


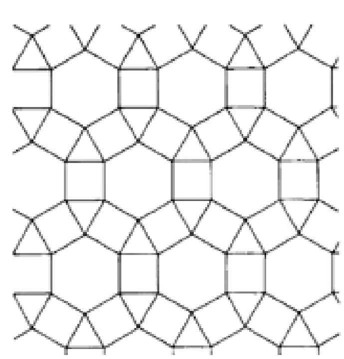
Figure 8b

The arrangement 3.A.3.B is depicted in Figure 8b; to maintain the same configuration at the white vertex, polygons A and B would have to be the same. Thus, two triangles, a square and a dodecagon (L) will not tessellate, but the combination M when rearranged as 3.6.3.6 will lead to a 1-uniform vertex tiling Figure 9a.

Similar reasoning leads to the conclusion that for Code N (3.4.4.6), there cannot be a semi-regular tessellation. However, if the polygons are rearranged as 3.4.6.4, then as shown in Figure 9b we are able to form a semi-regular tessellation.



3.6.3.6
Figure 9a



3.4.6.4
Figure 9b

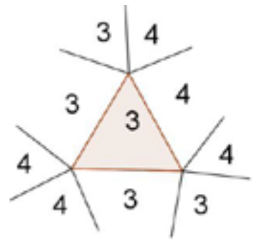


Figure 10a

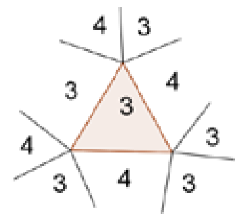
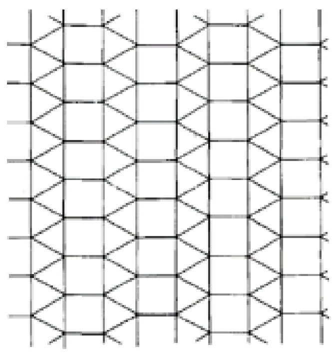
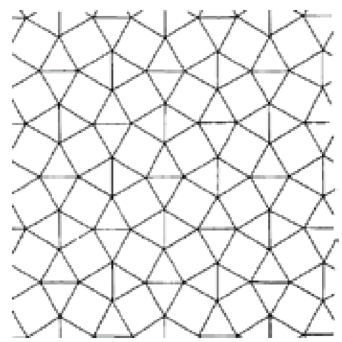


Figure 10b



3.3.3.4.4
Figure 11a

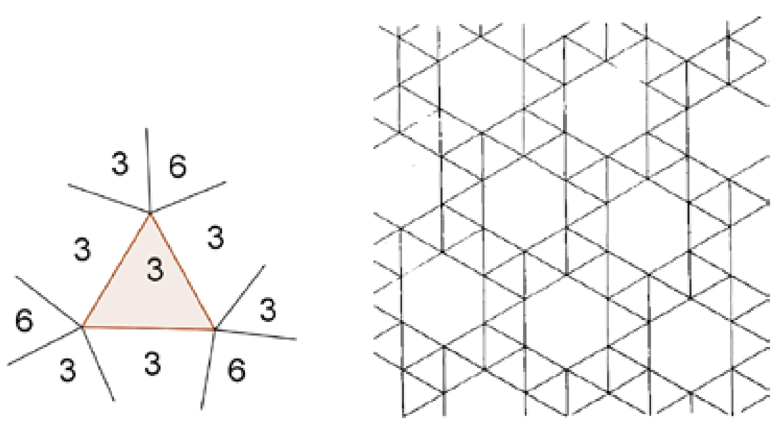


3.4.3.4.3
Figure 11b

Five Polygons
There are two configurations Q and R of five polygons. We will consider these separately.

The configuration Q makes two arrangements: 3.3.3.4.4 and 3.3.4.3.4. The arrangements in Figure 10a and Figure 10b depict schematically how these arrangements partition the plane, maintaining single homogeneity of vertices, to create semi-regular tiling patterns.(Figures 11a and 11b respectively)

And finally, Figure 12 depicts the tessellation of R (3.3.3.3.6):



3.3.3.3.6
Figure 12

Thus, only 8 of the 14 combinations of regular polygons partition the plane while conforming to 1-uniform vertex-homogeneity: 3.12.12, 4.8.8, 4.6.12, 3.6.3.6, 3.4.6.4, 3.3.3.4.4, 3.3.4.3.4 and 3.3.3.3.6. These then are our eight 1-uniform Archimedean or semi-regular tessellations.

Though there is a consensus on the possible number of regular and semi-regular tessellations, there is no precise way of concluding the same for demi-regular tessellations. To explore demi-regular tessellations you may choose to consider tile-homogeneity or vertex-homogeneity or edge-homogeneity as the criterion for listing. Krötenheerdt, 1969 (as stated in Grünbaum and Shephard, 1986) established 124 uniform-vertex tessellations on the basis of vertex-homogeneity. Chavey (1984) considered edge-homogeneity to list more than 165 demi-regular tessellations.

In this article we have taken an explorative approach to catalogue semi-regular tessellations. Middle grade teachers may let students hunt for tessellations combinatorially as well as geometrically and explore their many properties. This work can be extended to identify and catalogue demi-regular tessellations. Including such hands-on activities as part of middle grade mathematics teaching provides opportunities to experience the interplay of shapes, space, position and symmetry.

Further Reading

For more about demi-regular classification, readers can refer to the following:

1. Chavey, D.P. (1984). *Periodic Tilings and Tilings by Regular Polygons*. Unpublished doctoral dissertation submitted at University of Wisconsin.
2. Galebach, B. L. (2002). *N-uniform Tilings* at http://probabilitysports.com/tilings.html.
3. Grünbaum, B. and Shephard, G.C. (1986). *Tiling and Pattern*. W.H. Freeman and Company. New York.
4. www.maa.org/sites/default/files/pdf/upload_library/22/Allendoerfer/1978/0025570x.di021102.02p0230f.pdf
5. www.math.nus.edu.sg/aslaksen/papers/Demiregular.pdf

Acknowledgements:

Images of tessellations have been taken from open-sources:

1. Images of all tessellations are from: Weisstein, Eric W. "Tessellation." From MathWorld--A Wolfram Web Resource. http://mathworld.wolfram.com/Tessellation.html
2. The images of non-edge-to-edge tessellations are from: https://en.wikipedia.org/wiki/Euclidean_tilings_by_convex_regular_polygons#Tilings_that_are_not_edge-to-edge
3. The repeating units were made on Geogebra.

Glossary of terms

- 1-uniform vertex-homogeneous tiling: a tessellation in which the way in which the polygons are arranged around each vertex is the same for all vertices
- *k*-uniform ($k \geq 2$) vertex tessellation: a tessellation in which there are *k* different ways in which the polygons are arranged around the vertices
- Edge-to-edge tessellating pattern: a tessellation which has the feature that the polygons which touch each other do so along complete edges
- Regular tessellation: a tessellation in which the component polygons all have the same number of edges
- Semi-regular tessellation: a tessellation which has two or more kinds of component polygons, but the way in which the polygons are arranged around the vertices is the same for all vertices
- Demi-regular tessellation: a tessellation which has two or more kinds of component polygons, but the way in which the polygons are arranged around the vertices is not the same for all vertices

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