ClassRoom

THREE ELEGANT PROOFS of mathematical properties

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Elegant proofs of mathematical statements present the beauty of mathematics and enhance our learning pleasure. This is particularly so for 'proofs without words' which significantly improve the visual proof capability. In this article, we present three largely visual proofs which carry a great deal of elegance. In a strict sense they are not pure proofs without words because of the mathematical expressions and formulas that appear in them. Nevertheless, they carry a lot of appeal.

Many important statements have been made about the value of aesthetics and beauty in mathematics. Various studies in mathematics education have dwelt on their significance in the teaching of the subject. Here are a few relevant quotes:

- "*Mathematics is one of the greatest cultural and intellectual achievements of human-kind and citizens should develop an appreciation and understanding of that achievement, including its aesthetics and even recreational aspects*"[1].
- "*The mathematician's patterns, like the painter's or the poet's, must be beautiful; the ideas, like the colors or the words, must fit together in a harmonious way. Beauty is the first test: there is no permanent place in this world for ugly mathematics*"[2].
- "*Although it seems to us obvious that the aesthetics is relevant in mathematics education, the aesthetics also seems to be elusive when attempting to purposefully incorporate it in the mathematical experience*"[3].
- "*Mathematical beauty is the feature of the mathematical enterprise that gives mathematics a unique standing among the sciences*"[4].
- "*Mathematicians who successfully solve problems say that the experience of having done so contributes to an appreciation for the power and beauty of mathematics*"[5].

Among mathematicians and mathematics educators, many believe that aesthetics should be an integral part of the mathematics class.

In this article, we shall feature three elegant proofs which highlight the beauty of mathematics.

Keywords: Proof, proofs without words, pictures, areas, inscribed circles, right-angled triangles, hypotenuse, similar triangles

Products of four consecutive natural numbers

The product of four consecutive natural numbers plus 1 is the square of a natural number.

$$n(n+1)(n+2)(n+3)+1=(n^2+3n)(n^2+3n+2)+1=$$

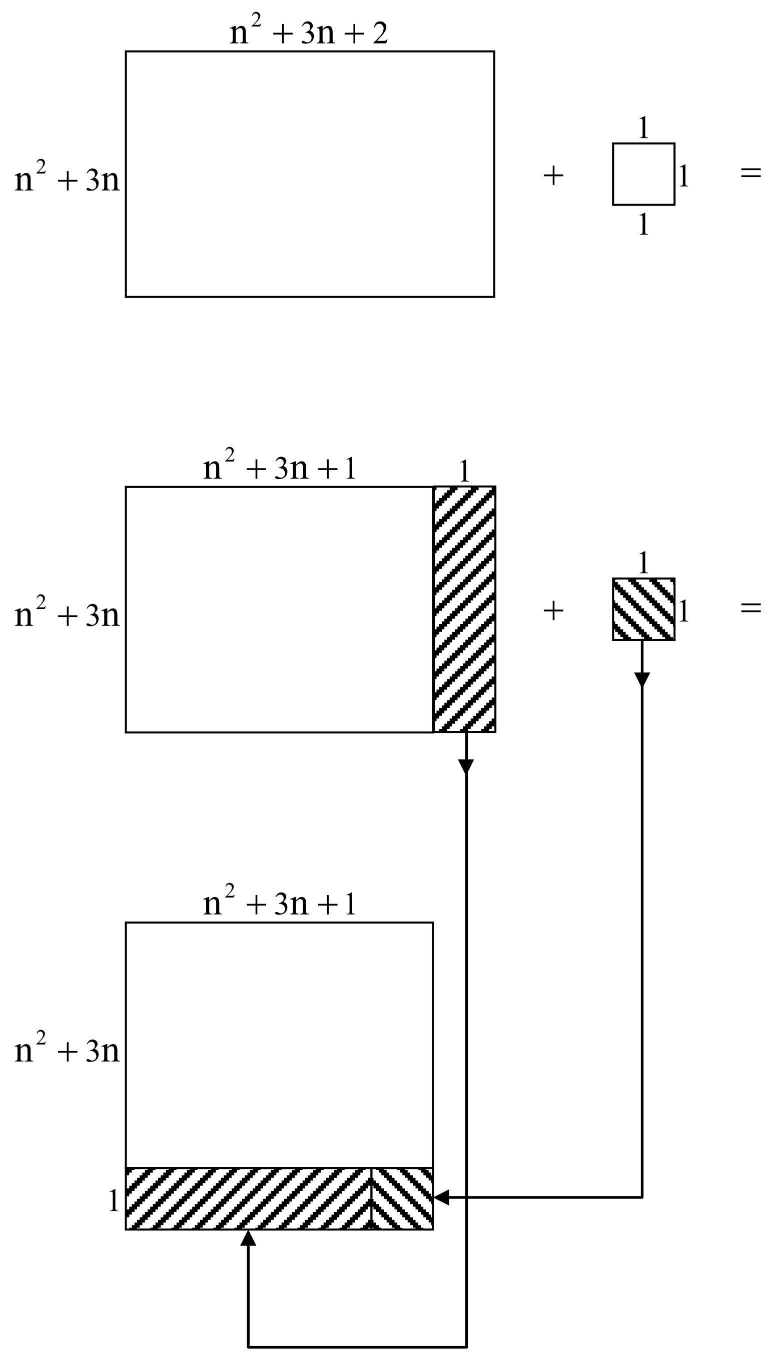


Figure 1

The mathematics of the above derivation is the following:

$$n(n+1)(n+2)(n+3)+1=(n^2+3n+1)^2.$$

Note 1: The proof is based on the following property: *The product of two natural numbers differing by* 2 *is* 1 *less than the square of the number in-between.* (For example, $5\times 7=35=6^2-1$.)

When four natural numbers are multiplied, the difference between the product of the two extreme numbers and the product of the two middle numbers is 2; the property noted above applies, and so the product of the products plus 1 is the square of a natural number. (For example, consider the four consecutive numbers 4, 5, 6, 7. The product of the numbers at the extremes is 28, and the product of the two middle numbers is 30. Observe that the difference between the two products is 2.)

Note 2: The above identity elegantly generalises to the case of an arithmetic progression: if a is the first term of the AP, and d is the common difference, then:

$$a(a+d)(a+2d)(a+3d)+d^4=(a^2+3ad+d^2)^2.$$

That is, *the product of four consecutive terms of the progression plus the fourth power of the common difference is a perfect square*.

Radii of three inscribed circles of a right angled triangle

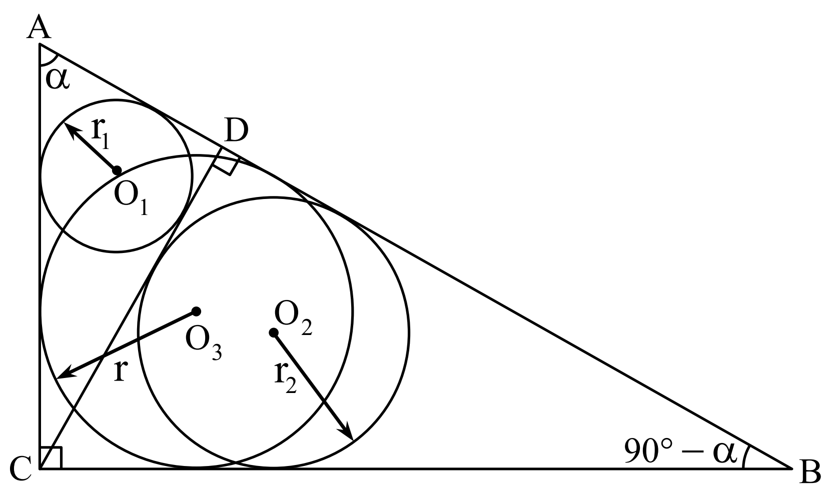


Figure 2

Let $[X]$ denote area of a figure X. Then:

$$\triangle ABC\sim\triangle ACD\sim\triangle CBD,$$

$$\therefore\ \frac{[\triangle ACD]}{[\triangle ABC]}=\frac{r_1^2}{r^2},\quad \frac{[\triangle CBD]}{[\triangle ABC]}=\frac{r_2^2}{r^2}.$$

Since $[\triangle ACD]+[\triangle CBD]=[\triangle ABC]$, we get

$$r_1^2+r_2^2=r^2.$$

Corollary. If $AC=BC$, then

$$r_1=r_2=\frac{r}{\sqrt{2}}.$$

Notes.

(1) In a similar way, if R denotes radius of the circumscribed circle of a triangle; h denotes the corresponding height (altitude) of the triangle; l denotes the length of the bisector of the corresponding angle; and m denotes the length of the corresponding median of the triangle, then:

$$R_1^2 + R_2^2 = R^2,$$

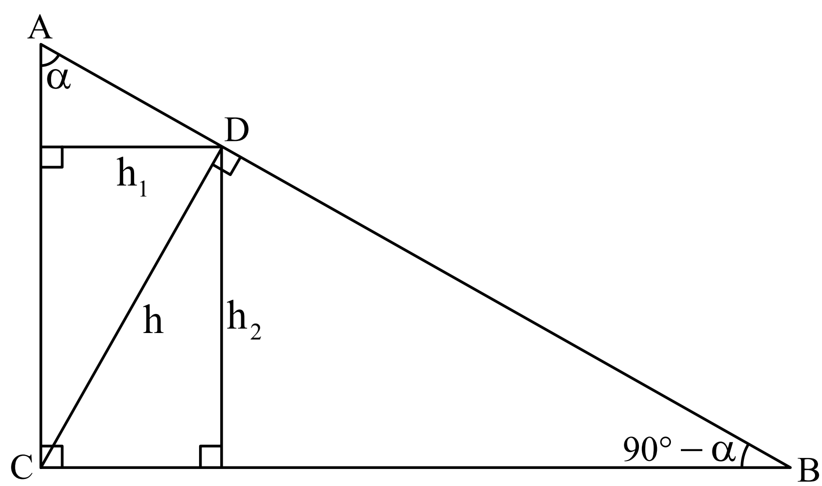
$$h_1^2 + h_2^2 = h^2,$$

$$l_1^2 + l_2^2 = l^2,$$

$$m_1^2 + m_2^2 = m^2.$$

Note: For a pair of similar triangles, the proportional relationship that holds between the lengths of corresponding sides also holds between the lengths of any two corresponding segments in the two triangles; for example, the corresponding heights, the corresponding angle bisectors, the corresponding medians; and so on.

(2) The statement that $h_1^2 + h_2^2 = h^2$ is actually another way of stating the Pythagorean theorem. So this yields another proof of that famous theorem.



More relationships in a right angled triangle

Let point D lie on the hypotenuse of a right-angled triangle ABC. Then:

$$AB^2 \cdot DC^2 + AC^2 \cdot BD^2 = AD^2 \cdot BC^2$$

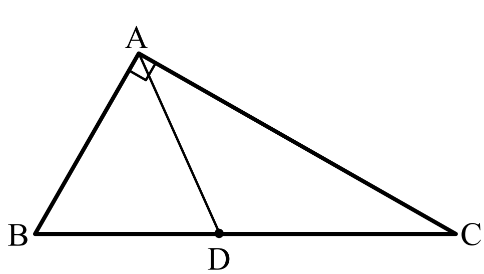


Figure 3.1

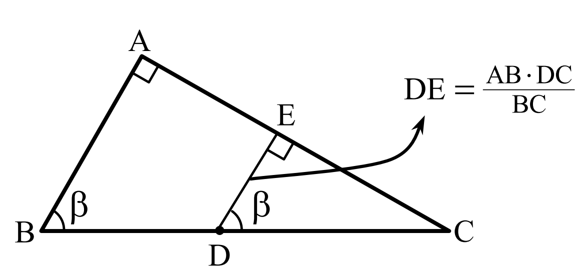
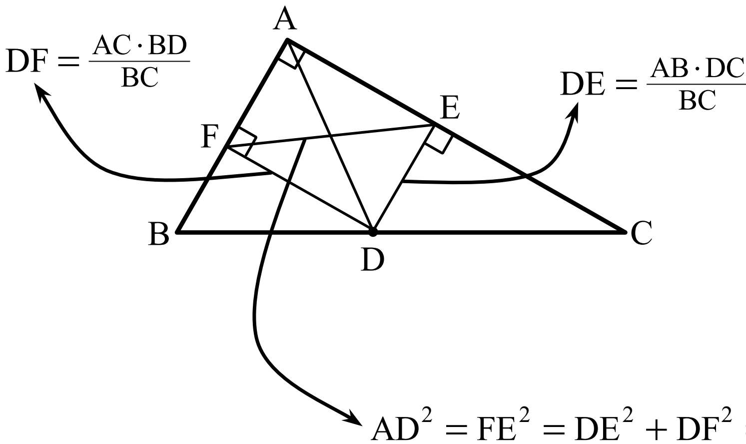


Figure 3.2

This is derived from the similarities of the triangles *ABC* and *EDC* (as shown in Figure 3.2). In the same manner, triangles *ABC* and *FBD* are similar (see Figure 3.3)



$$AD^2 = FE^2 = DE^2 + DF^2 = \frac{AB^2 \cdot DC^2 + AC^2 \cdot BD^2}{BC^2}$$

$$\Downarrow$$

$$AB^2 \cdot DC^2 + AC^2 \cdot BD^2 = AD^2 \cdot BC^2$$

Figure 3.3

If *DF* is parallel to side *AC*, then *AFDE* is a rectangle and hence its diagonals are equal (as shown in Fig. 3.3).

Note: If *D* is the midpoint of *BC* then, using the above, the theorem 'The median to the hypotenuse of a right triangle equals half the hypotenuse' can be easily proved.

References

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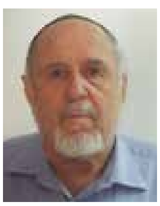
Classroom usage

These three proofs can be used in various ways in the classroom.

1. Giving just the pictures and asking students to find relationships in each of them.
2. Giving the students the pictures and relationships and asking them to explain 'why' the relationships exist; i.e., why they are true.
3. Asking them to prove the new results given at the end of each proof.
4. Asking them to suggest new relationships (example: trigonometric ratios in the case of proof II when the triangle is right-angled and isosceles).



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