ClassRoom

3 4 5...
STRIKES AGAIN!

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A few days back, I came across the following problem on Dan Meyer's blog **dy/dan** (entry dated October 27, 2016: "How I'm Learning to Step into Math Problems"; I have restated the problem in my own words): *In the figure shown* (Figure 1), *the circle touches the base BC of the square and passes through the two upper vertices, A and D. Find the ratio of the radius of the circle to the side of the square.*

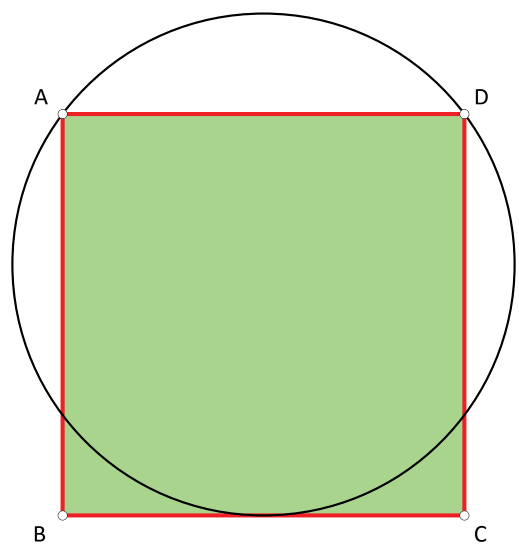

Figure 1

A related problem: *Find a way to construct such a figure*. (That is: given the square, show how to construct the circle; or: given the circle, show how to construct the square.)

Keywords: Pythagoras, problem solving, construction, circle, square

Solution. Denote the radius of the circle by r, and the side of the square by $2a$ (the '2' is only to avoid unnecessary fractions). See Figure 2.

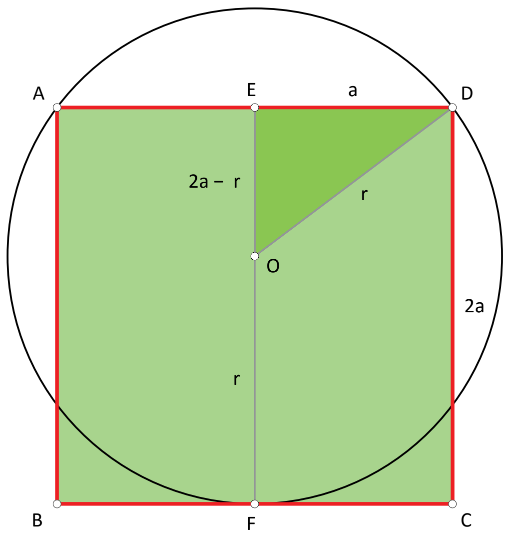

Figure 2

Let O be the centre of the circle, and let EF be the midline of the square, connecting midpoints E and F of AD and BC. (Note that O lies on EF. Why should this be so? In other words: *Given only the information that the circle touches the base BC and passes through the upper two vertices A and D, can we conclude that the midline EF of the square will be an axis of symmetry of the figure?* We leave this question for the reader.) Then we have $OF = r$, $OE = 2a - r$, $DE = a$, $OD = r$. Hence from ΔODE we get, using Pythagoras theorem:

$$a^2 + (2a - r)^2 = r^2, \quad \therefore\ 4ar = 5a^2, \quad \therefore\ \frac{a}{r} = \frac{4}{5}.$$

It follows that

$$2a - r : a : r = 3 : 4 : 5,$$

i.e., ΔODE is a 3–4–5 triangle!

Once we have deduced this, the construction procedure becomes clear. All we need to use is the fact that $EO : OF = 3 : 5$ and $FC : r = 4 : 5$.

Remark. A natural extension to the question studied above is obtained by replacing the word ***square*** throughout by ***regular hexagon***. The configuration is depicted in Figure 3. A similar question can now be asked: *Compute the ratio of the radius of the circle to the side of the hexagon.* We leave this question too for the reader to solve. Likewise for more extensions of this kind.

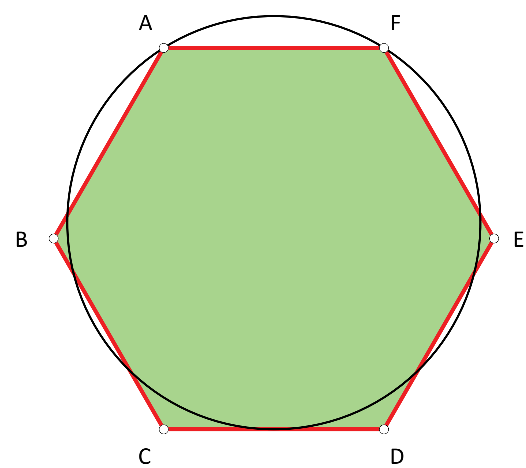

Figure 3

Pedagogical note

These are excellent GeoGebra exercises for students helping them to develop and practise skills of visualisation, logical sequencing, making connections and recalling theory.

The COMMUNITY MATHEMATICS CENTRE (CoMaC) is an outreach arm of Rishi Valley Education Centre (AP) and Sahyadri School (KFI). It holds workshops in the teaching of mathematics and undertakes preparation of teaching materials for State Governments and NGOs. CoMaC may be contacted at shailesh.shirali@gmail.com.