Review

Transcending the unbearable heaviness in the Teaching of Mathematics !

A Review of THE CARTOON GUIDE TO ALGEBRA and THE CARTOON GUIDE TO CALCULUS

By Larry Gonick

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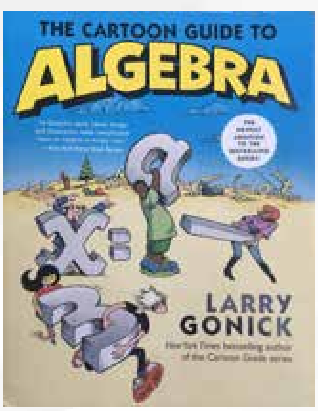

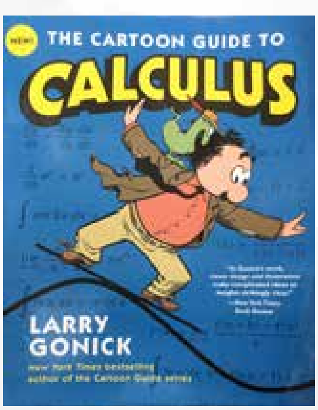

I don't know what your experience of learning mathematics was—but for me, till I went to graduate school and except for a few courses as an undergraduate, it was a very heavy affair. First of all, there was an overwhelming sense of being weighed down that was associated with 'knowledge'. There was so much to learn, so much to remember and so much to be tested on! There was no sense of lightness associated with learning, no sense of play or joy in discovering and understanding. I remember once, wandering around a huge library with this feeling, and coming upon a lovely poem by Justin Richardson (Punch, 1952), which gave me immense relief. It goes like this:

For years a secret shame destroyed my peace-

I'd not read Eliot, Auden or MacNeice.

But then I had a thought that brought me hope-

Neither had Chaucer, Shakespeare, Milton, Pope.

While this sense of gravitas can be associated with any area of knowledge, in mathematics it can get further compounded. This is especially so while learning courses like calculus when it is taught without any graphs and graphics, and one is expected

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to learn a huge bunch of formulae, and solve innumerable integrals, with no clue why you are learning what you are learning. A few years ago, a student of mine bitterly complained about how he was taught an undergraduate course in calculus and the teacher did not draw a single graph in the entire course!

I feel our relationship to mathematical knowledge and how we teach it has to change. We need to bring in a certain lightness (this is not the opposite of rigour), humour and sense of play in our teaching and also use graphical illustrations to make concepts, formulae and equations come to life!

The *Cartoon Guides* to algebra and calculus do just that. I will share some common features before going into the specifics of each book. Larry Gonick is a well known cartoonist, who studied and taught mathematics in Harvard. He has authored several cartoon guides in a wide variety of topics from history to physics. Both books referred to above are actually very rigorous introductions to the subject in hand. There is no sloppy hand-waving involved. Concepts are introduced very carefully, several examples and illustrations given from real life, and there are exercises at the end of each chapter.

What then is the difference between these cartoon guides and regular text books on these subjects? Given that it is a cartoon book the main tool for exposition and explanation is graphics. Every concept, in fact every page, is illustrated using graphics. These (see Figure 1) could be illustrations or pictorial proofs (for example, to show $a^2 - b^2 = (a + b)(a - b)$, to illustrate completing the square, the concept of a function and its inverse, the chain rule, and so on).

There is a great emphasis on using mathematical graphs to explain and illustrate concepts. Paraphrasing a famous dictum, I believe a graph is worth a thousand equations! Graphs give students an intuitive understanding of how functions work, why some results make sense and also serve as models for conjecturing results (the mean value theorem, for example). It is a real tragedy that teachers do not use them more in their classrooms.

Of course the most important feature is humour. There is a running narrative where cartoon characters (human and non-human) interact with each other while learning the various concepts. So we have, in the Algebra book, Al-Khwarizmi and Bhaskara making an appearance and in the calculus book, Newton and Leibnitz. This is refreshing from the history of mathematics point of view. It would have been really nice if the contribution from the Kerala school was mentioned in the calculus book and Omar Khayyam in the algebra book. The conversations between the various characters are a constant humorous banter, often using ‘puns’ to evoke laughter. Newton and Leibnitz, as is to be expected, are at constant loggerheads with each other (see Figure 2)!

While Gonick does have his witty and clever moments, I must confess, I found many of the jokes to be of the ‘poor joke’ or ‘sick joke’ variety! So you might well ask, then what is the point? I think humour (poor or rich) has a way of keeping interactions with students light, and also bringing a playful aspect to learning a concept. Sometimes a bad joke gets associated with a concept and may serve as an aid to memory! Here are examples of jokes that I crack in my classes, often to the sound of my students groaning. For example, while teaching complex numbers: a negative number goes to see a psychiatrist. The psychiatrist asks the negative number: “Why are you always so negative?” To which the negative number replies “That is because I am the product of two complex entities!” Another example of a joke (suitable while teaching infinite geometric series), which is actually quite deep, goes like this. An infinite number of mathematicians go to a bar. The first one orders half a mug of beer, the second one-fourth, the third one-eighth, and so on. The bartender gives them a mug full of beer and says “Know your limit guys!”

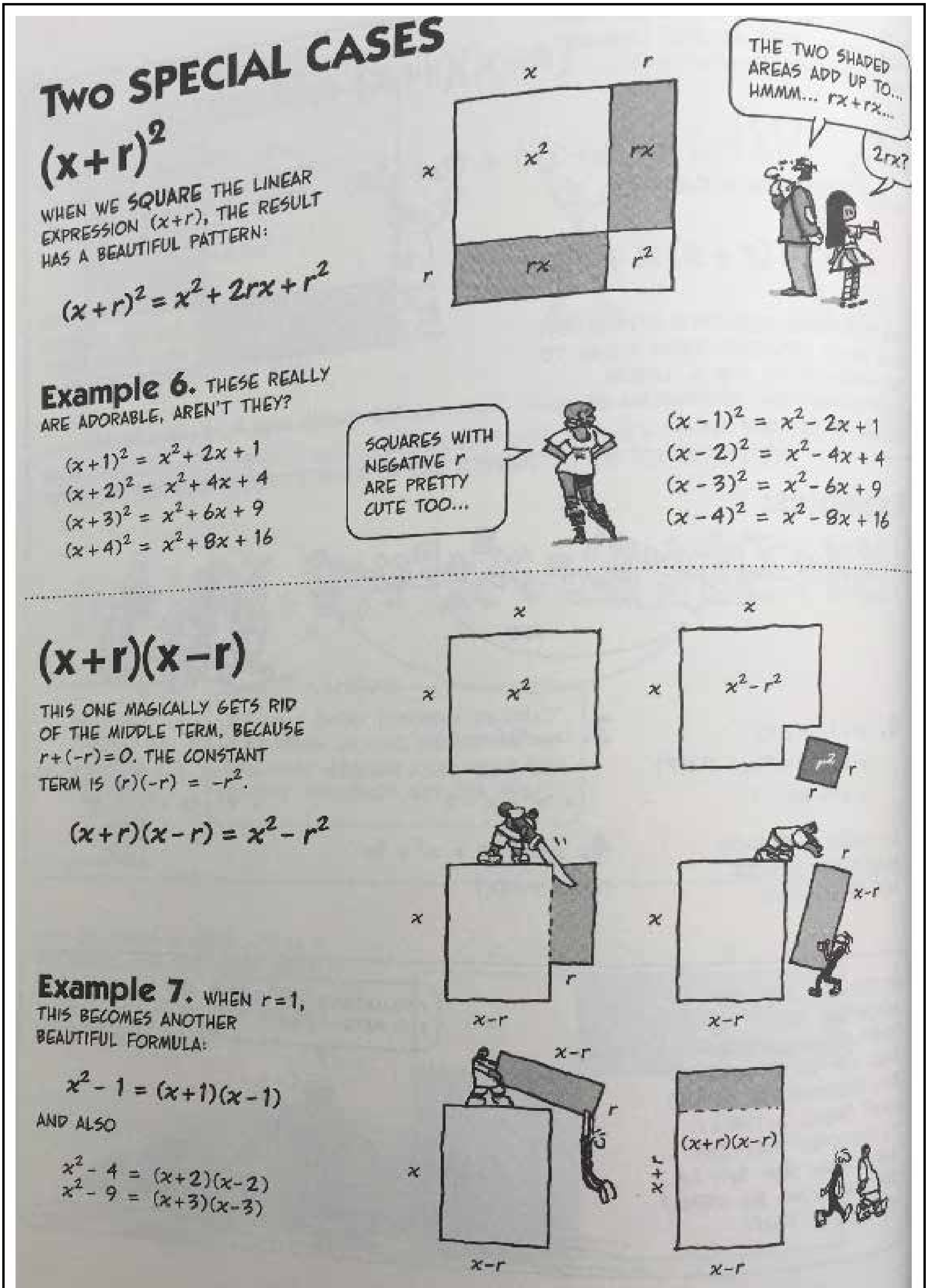
TWO SPECIAL CASES
$(x+r)^2$
WHEN WE SQUARE THE LINEAR EXPRESSION $(x+r)$, THE RESULT HAS A BEAUTIFUL PATTERN:
$(x+r)^2 = x^2 + 2rx + r^2$
x
r
x
r
x^2
rx
rx
r^2
THE TWO SHADED AREAS ADD UP TO... HMMM... rx + rx...
2rx?
Example 6. THESE REALLY ARE ADORABLE, AREN'T THEY?
$(x+1)^2 = x^2 + 2x + 1$
$(x+2)^2 = x^2 + 4x + 4$
$(x+3)^2 = x^2 + 6x + 9$
$(x+4)^2 = x^2 + 8x + 16$
SQUARES WITH NEGATIVE r ARE PRETTY CUTE TOO...
$(x-1)^2 = x^2 - 2x + 1$
$(x-2)^2 = x^2 - 4x + 4$
$(x-3)^2 = x^2 - 6x + 9$
$(x-4)^2 = x^2 - 8x + 16$
$(x+r)(x-r)$
THIS ONE MAGICALLY GETS RID OF THE MIDDLE TERM, BECAUSE $r+(-r)=0$. THE CONSTANT TERM IS $(r)(-r) = -r^2$.
$(x+r)(x-r) = x^2 - r^2$
x
x
x^2
x
x
$x^2 - r^2$
r^2
r
r
x
r
x-r
x
r
x-r
x-r
Example 7. WHEN r=1, THIS BECOMES ANOTHER BEAUTIFUL FORMULA:
$x^2 - 1 = (x+1)(x-1)$
AND ALSO
$x^2 - 4 = (x+2)(x-2)$
$x^2 - 9 = (x+3)(x-3)$
x-r
r
x
x-r
x+r
$(x+r)(x-r)$
x-r

Figure 1

Figure 2

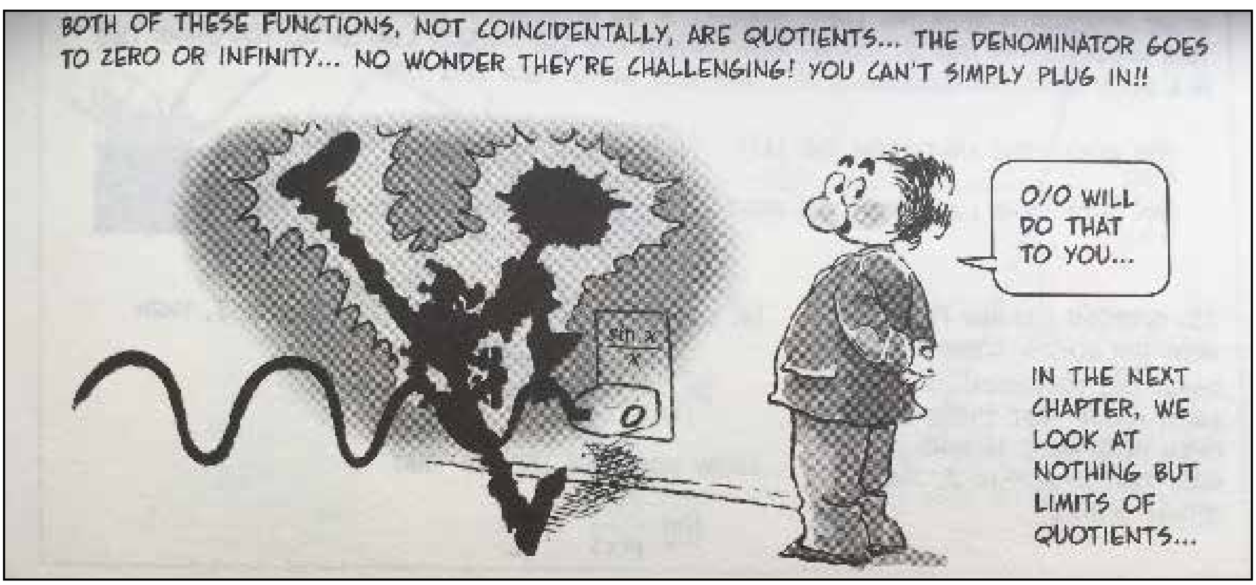

Figure 3

The Cartoon Guide to Algebra

The Cartoon Guide To Algebra has 17 chapters and starts off with very basic arithmetic, working its way up to solving quadratic equations. Along the way it covers standard topics in elementary algebra and coordinate geometry.

We are introduced to the unknown variable via arithmetic. We are then taught basic operations in mathematics, including how to deal with negative quantities, fractions, the number line, ratio and proportion, introduction to expressions, solving equations, introduction to coordinate geometry, linear and simultaneous equations and finally, quadratic equations.

There are many takeaways for a mathematics teacher. The books offer many ideas to introduce and teach concepts. For example, the author uses two 'models' to deal with negative numbers and operations with negative numbers. One is to treat real numbers as directed lengths and then have rules to show how these lengths combine with each other. The other is to treat negative numbers as debts. I liked how he illustrates the relationship between the slopes of perpendicular lines, his introduction of $a^{-n} = 1/a^n$ and his idea of checking roots of quadratic equations.

I was not too happy with the way 'zero' was treated. I feel the author could have spent more time discussing the concept as both a place holder and a number in its own right, along with its properties. The author does, however, illustrate (quite graphically!) why dividing by zero is not such a good idea (see Figure 3)!

While most of the chapters cover standard material, I found two chapters quite unusual. One is to do with rates, where Gonick introduces the 'All-purpose rate equation'

$$U = U_0 + r_U(t - t_0),$$

where U is any given quantity at time t, U_0 is the initial amount, r_U is the rate of change of U over time, and t_0 is initial time. He explains how a graph of such an equation would be a line with y-intercept U_0 and slope r_U. He then applies this equation in a variety of situations, ranging from speed and velocity to water flowing in and out of a tank.

The other unusual chapter is about weighted averages. We encounter a real-life example of computing the average monthly electricity bill, where the rate of consumption varies month by month. This chapter has been written with a great deal of humour!

The Cartoon Guide to Calculus

The Cartoon Guide To Calculus has 14 chapters and covers basic differential and integral calculus. Gonick begins by explaining the idea of instantaneous velocity and has the clever idea of introducing a so-called 'velocimeter' which is just like a speedometer, except that it assigns a negative sign to the speed of a car in reverse. Using the velocimeter he is able to introduce the notion that instantaneous velocity is closely approximated by the change in distance by change in time.
He then moves on to introducing the idea of a 'function,' via an animal which has an input and output (with predictable humour associated with such a model!). He has a nice idea of drawing two parallel lines (see Figure 4), where the input is marked on one line and the output on the other, and the function animal stands between the two, pointing with its hand to the input and its tail to the output. He then moves on to introducing us to all the standard functions, each of whose properties are very well illustrated using graphs.

Gonick spends a lot of time developing the notion of limits, and teachers will definitely benefit from using his ideas to illustrate this rather difficult concept. He then introduces the derivative using the idea of rate of change. His emphasis here is on showing that calculus is mainly concerned with understanding change and he uses speed, velocity and acceleration as his main tools to illustrate this idea. The book also covers standard applications of differentiation.

One thing that Gonick does not spend a lot of time on (except here and there) is to show how a tangent line is locally the best approximation to a curve, and consequently how calculus is extremely useful in curve sketching. I guess he had to make a choice between what to keep and what to leave out.

As in *The Cartoon Guide to Algebra*, most of the material is standard fare, and experienced teachers will recognize many techniques that one has picked up over the years to explain concepts. However, once again, I would like to emphasise that there is a lot one can learn on how to explain and illustrate difficult topics (like the Fundamental Theorem of Calculus). What I found unusual in *The Cartoon Guide to Calculus* is his treatment of the 'chain rule.' He introduces two new ideas. We need some notation here to explain these ideas.

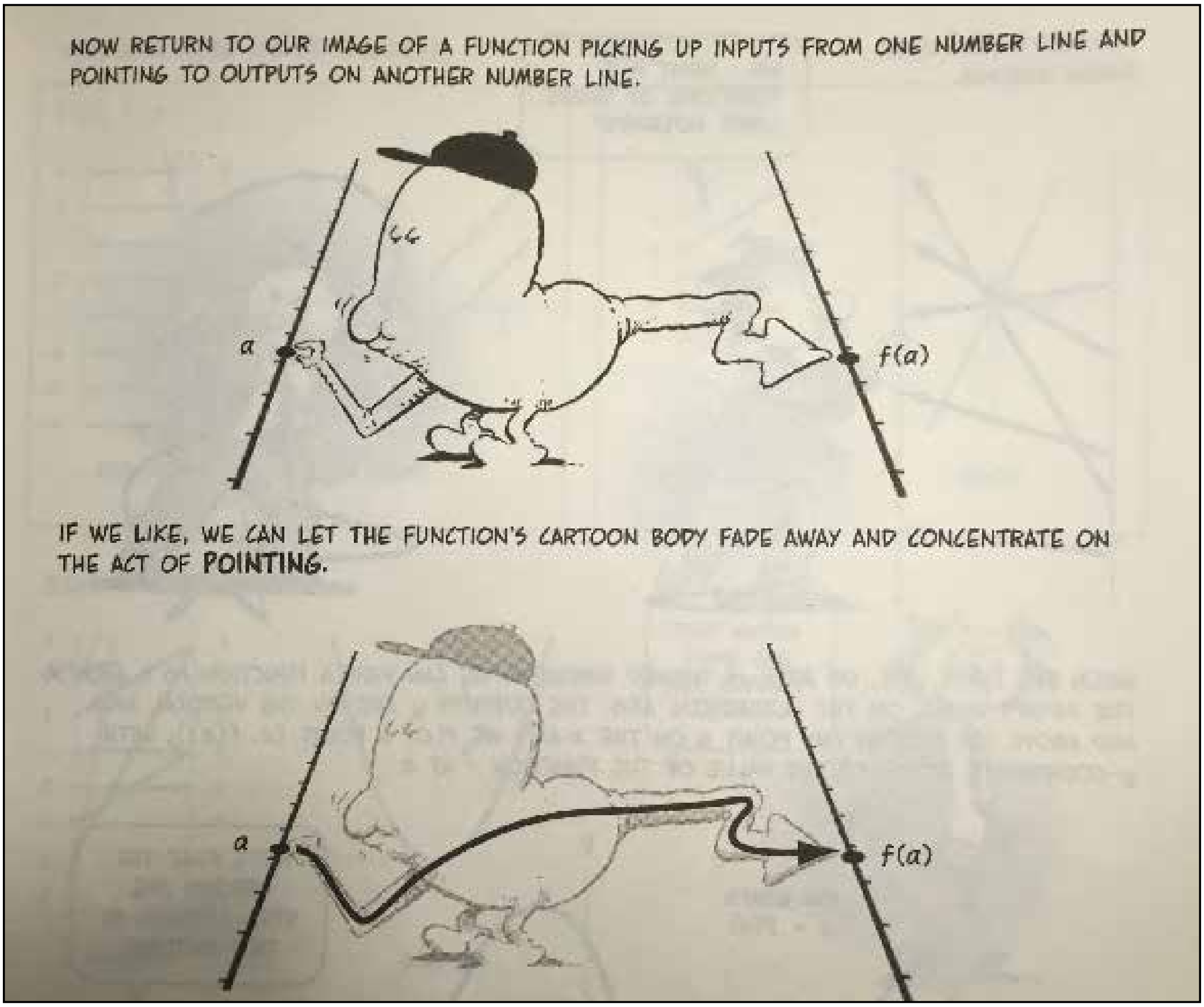

Figure 4

Let $f(x)$ be a continuous function; its derivative is $f'(x)$. As usual h is a non-zero real number and $\Delta f = f(x + h) - f(x)$. Gonick, in his irreverent manner, defines a concept called the 'Flea' to be a mathematical quantity that satisfies

$$\lim_{h\to 0} \frac{\text{Flea}}{h} = 0.$$

He shows how $\frac{\Delta f}{h}$ is the scaling factor, which when multiplied by h gives Δf. He goes on to derive what he calls the Fundamental equation of calculus,

$$\Delta f = hf'(x) + \text{Flea},$$

from this equation and the definition of Flea; we can then replace the scaling factor $\frac{\Delta f}{h}$ by $f'(x)$.

The scaling factor idea then helps us understand both the chain rule and the derivative of $f^{-1}(x)$ in terms of $f(x)$. (See Figure 5.)

The book does cover integration although the bulk of it is used to introduce basic precalculus notions and differentiation.

End Note

Obviously The Cartoon Guides cannot replace textbooks in the subjects. I think they serve as excellent supplementary material. They fulfill different purposes for the teacher and the student. Teachers will find, as I mentioned earlier, ideas to introduce concepts, illustrations that can be used in the classroom, ideas to lighten one's teaching and, perhaps, even examples and problems that can be used in class.

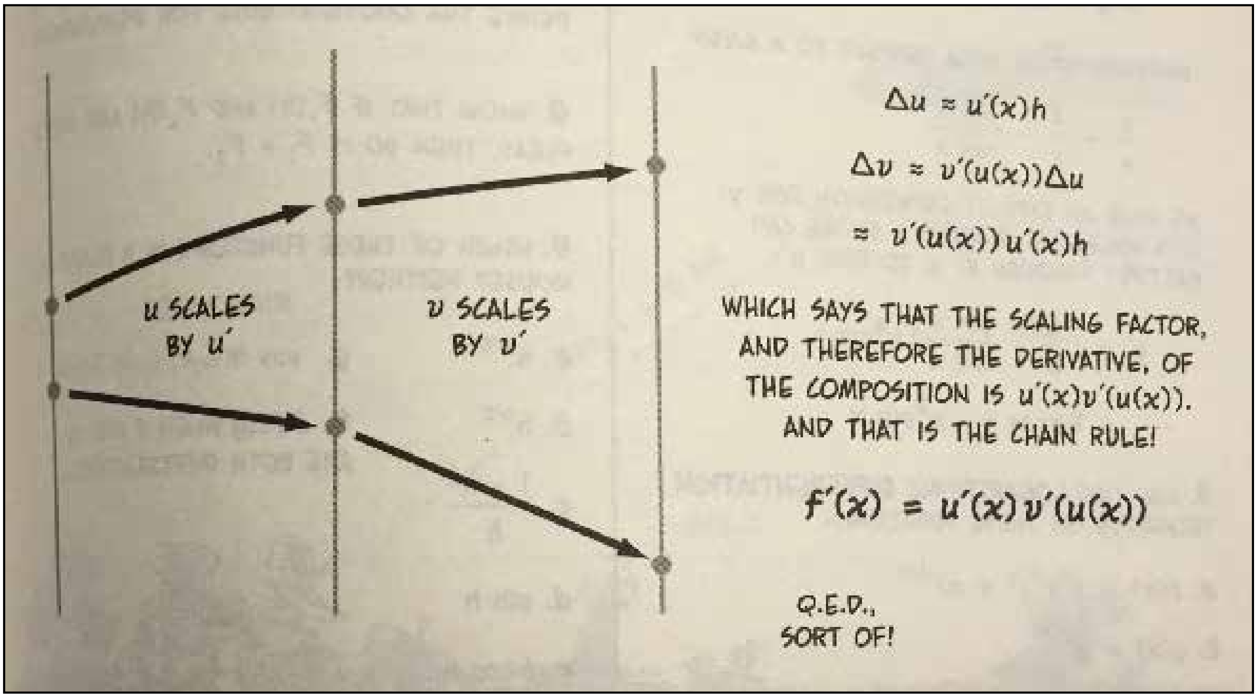

Figure 5

For students of either an algebra or a calculus course, I would use the material selectively. Perhaps ask them to read particular chapters after the material has been covered. The books will not only bring a smile onto their faces, its graphical approach will strengthen both intuitive and formal understanding. Once the material has been learned, the books will definitely serve as excellent resources for the reinforcement of concepts. They will enjoy the last chapter of both books which give previews of the more advanced topics that they will encounter, if they decide to learn higher mathematics.

I strongly recommend that school libraries stock these books. Before signing off, I would warn readers that, although the book is written with humour and wit and has some really amusing graphics, it is not a casual read. The mathematics has been treated very rigorously, so be prepared to learn and smile, and not merely be entertained!

Reference:

1. Gonick, Larry. *The Cartoon Guide to Algebra*. William Murrow, 2015.
2. Gonick, Larry. *The Cartoon Guide to Calculus*. William Murrow, 2012.

SHASHIDHAR JAGADEESHAN received his PhD from Syracuse in 1994. He is a teacher of mathematics with a belief that mathematics is a human endeavour; his interest lies in conveying the beauty of mathematics to students and looking for ways of creating environments where children enjoy learning. He may be contacted at jshashidhar@gmail.com