A TRIANGLE PROBLEM

$\mathscr{C}\otimes\mathscr{M}\alpha\mathscr{C}$

A reader once sent me the following nice problem:

> *Two sides of a triangle have lengths* 6 *and* 10*, and the radius of the circumcircle of the triangle is* 12*. Find the length of the third side.*

Let the triangle be labelled ABC, with sides $a = BC = 6$ and $b = CA = 10$. Let the radius of the circumcircle be $R = 12$. We must find c, the length of side AB.

Attempts to solve the problem using trigonometry will probably prove challenging for most high school students. To get a handle on the problem, it is useful to study the situation using GeoGebra. Figure 1 depicts the situation. We draw a circle Γ with radius $R = 12$, centred at an arbitrary point O. Next, we select any point on Γ and label it B. Then we draw a circle ω_B centred at B, with radius $a = 6$. This circle intersects Γ at two points; we arbitrarily select one of them and label it C. (No essential difference will result if we were to select the other point of intersection. Please verify this for yourself.) Next, we draw a circle ω_C centred at C, with radius $b = 10$. Circle ω_C too intersects Γ at two points; call them A_1 and A_2. *Both these points serve as solutions to our problem. That is, we have* ***two*** *triangles* A_1BC *and* A_2BC*, both of which satisfy the given conditions.* The 'algebra' window of GeoGebra reveals the two possible values of c; namely, 4.23 and 15.14, respectively.

Let us now see if we can derive these figures using trigonometry.

The trigonometric approach for solving the problem is the following. We use the cosine rule twice, to find the angles subtended by the two given sides (a and b) at the centre O of the circumscribing circle Γ. Then we use the addition and subtraction formulas to find the cosines of the sum and difference of these two angles. Finally, we use the cosine rule again to find the third side.

Keywords: Triangle, circumradius, sine rule, cosine rule

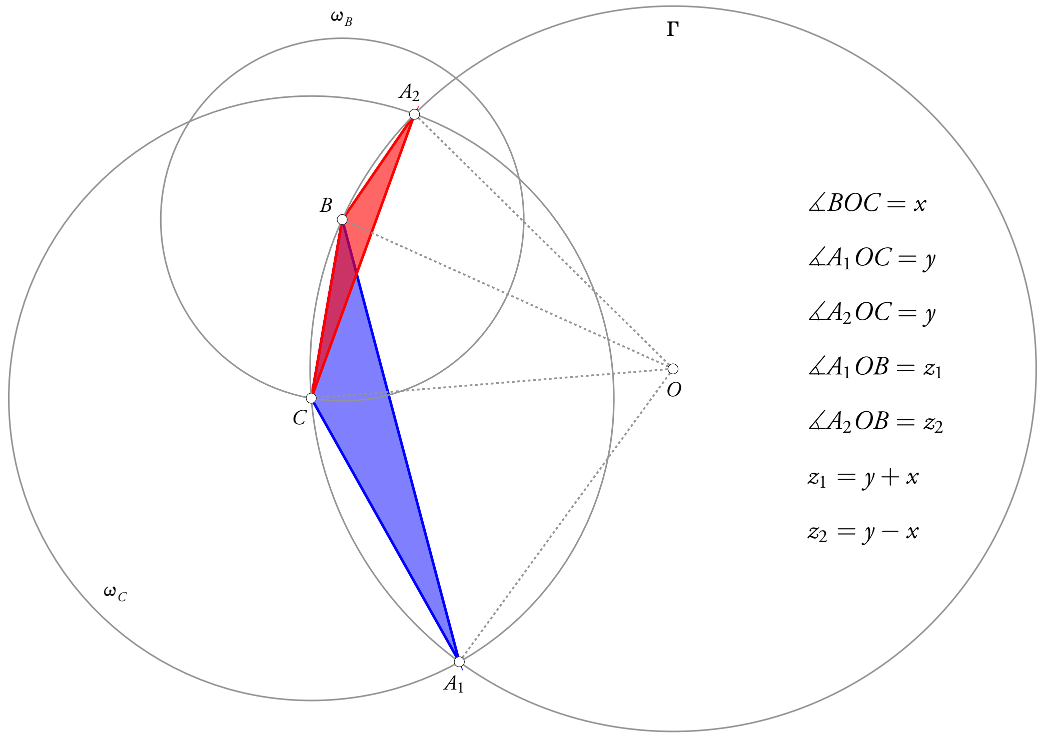

Figure 1.

Let $\measuredangle BOC = x$ and $\measuredangle A_1OC = y = \measuredangle A_2OC$. Using the cosine rule, we get:

$$\cos x = \frac{12^2 + 12^2 - 6^2}{2 \times 12 \times 12} = \frac{7}{8},$$
$$\cos y = \frac{12^2 + 12^2 - 10^2}{2 \times 12 \times 12} = \frac{47}{72}. \tag{1}$$

From this we get:

$$\sin x = \frac{\sqrt{8^2 - 7^2}}{8} = \frac{\sqrt{15}}{8},$$
$$\sin y = \frac{\sqrt{72^2 - 47^2}}{72} = \frac{5\sqrt{119}}{72}. \tag{2}$$

These in turn yield:

$$\cos(y + x) = \cos y \cos x - \sin y \sin x$$
$$= \frac{329 - 5 \cdot \sqrt{1785}}{576}, \tag{3}$$

$$\cos(y - x) = \cos y \cos x + \sin y \sin x$$
$$= \frac{329 + 5 \cdot \sqrt{1785}}{576}. \tag{4}$$

We need both these values, i.e., $\cos(y + x)$ and $\cos(y - x)$, precisely because of the two possibilities indicated in Figure 1. The first possibility arises when angles x and y do not overlap with each other (point A_1); the second possibility arises when they do overlap with each other (point A_2).

The first possibility yields, by using the cosine rule in $\triangle OA_1B$:

$$c^2 = 2R^2(1 - \cos(y + x))$$
$$\therefore\ c = \frac{5\sqrt{15} + \sqrt{119}}{2}, \quad \text{on simplification,}$$
$$\therefore\ c \approx 15.137.$$

The second possibility yields, by using the cosine rule in $\triangle OA_2B$:

$$c^2 = 2R^2(1 - \cos(y - x))$$
$$\therefore\ c = \frac{5\sqrt{15} - \sqrt{119}}{2}, \quad \text{on simplification,}$$
$$\therefore\ c \approx 4.228.$$

These values agree with what GeoGebra had told us earlier. Just as well!

Remark. It is an interesting commentary on how our minds work that when I initially solved the problem using trigonometry, I considered only the possibility where angles x and y do not overlap; so I only obtained the first possible value of c (the larger value). It is only when I drew the diagram using GeoGebra that I noticed the other possibility! Of course, I then back-checked the calculation and spotted the point at which I had missed a possibility. The results then tallied.

The **COMMUNITY MATHEMATICS CENTRE** (CoMaC) is an outreach arm of Rishi Valley Education Centre (AP) and Sahyadri School (KFI). It holds workshops in the teaching of mathematics and undertakes preparation of teaching materials for State Governments and NGOs. CoMaC may be contacted at shailesh.shirali@gmail.com.