Problem Corner

HARMONIC TRIANGULAR TRIPLES

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The equation $1/u + 1/v = 1/w$, where u, v, w are permitted to take positive integral values only, has been studied earlier in this magazine, on more than one occasion. Triples (u, v, w) which serve as solutions of the equation are known as *harmonic triples*. We have seen that there is a simple algorithmic way of listing all such triples. (See the July 2013 issue of *At Right Angles*.)

It is of interest to consider a variation of this equation in which u, v, w are permitted to take values only in some specified set of numbers. Depending on what that set is, interesting mathematics can be uncovered. For example: Can there be any solution in which u, v, w are all odd integers? A simple parity argument shows that such solutions cannot exist. (Please fill in the details for yourself.) More interesting: Can there be any solution in which u, v, w are all prime numbers? It is easy to show that there cannot be any solutions under this condition either. For, the equation leads to the equality $w(u + v) = uv$, showing that w is a divisor of uv. As u, v, w are all prime, this implies that w is a divisor of at least one of the primes u, v and hence is equal to one of them. In turn, this leads to one of u, v being zero. But this is absurd. Hence there can be no solution under the imposed condition.

In this article, we consider the case where u, v, w must all be *triangular numbers*. Recall that the sequence of triangular numbers is $1, 3, 6, 10, 15, \ldots$. The n-th triangular number T_n for any positive integer n is the sum

$$1 + 2 + 3 + \cdots + n.$$

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It is well-known that $T_n = \frac{1}{2}n(n+1)$. So we seek positive integral values for a, b, c such that $1/T_a + 1/T_b = 1/T_c$, i.e.,

$$\frac{1}{a(a+1)} + \frac{1}{b(b+1)} = \frac{1}{c(c+1)}. \tag{1}$$

This problem was first studied and solved by R. Venkatachalam Iyer in *The American Mathematical Monthly*, Vol. 67, No. 10 (Dec., 1960), pp. 1034-1035. As we shall see below, it gives rise to rich mathematics.

The equation is well-suited for computer-assisted investigation, using a general-purpose language like *Python* (which is gaining in power and popularity these days), or a computer algebra software like *Mathematica*. As the equation is symmetric in a and b, and we do not wish to count two solutions as different if a and b have merely swapped places, let us impose the condition that $a \leq b$. Writing the appropriate code and running the program, we find that the following are the only solutions in which a, b, c do not exceed 100:

$$(a, b, c) = (3, 3, 2), \ (4, 5, 3), \ (20, 20, 14), \ (28, 29, 20).$$

(Thus we have:

$$\frac{1}{6} + \frac{1}{6} = \frac{1}{3}, \quad \frac{1}{10} + \frac{1}{15} = \frac{1}{6}, \quad \frac{1}{210} + \frac{1}{210} = \frac{1}{105}, \quad \frac{1}{406} + \frac{1}{435} = \frac{1}{210}.$$

The last relation may come as a surprise.)

We note right away that in each case, *a and b are either equal to each other or differ by* 1. This is a striking observation, and it turns out to be true in general. That is, we have the following result:

Theorem (Iyer). *If a, b, c are positive integers such that*

$$\frac{1}{T_a} + \frac{1}{T_b} = \frac{1}{T_c},$$

then either $a = b$ or $a = b \pm 1$.

For the proof, we shall exploit the well-known connection between the triangular numbers and the square numbers: $8T_n + 1 = (2n+1)^2$ for any positive integer n. (We had given a visual proof of this identity on page 57 of the March 2014 issue of *At Right Angles*. For the reader's convenience, we display the figure in Box 1, at the end of the article.) The given equation implies that

$$\frac{1}{(2a+1)^2 - 1} + \frac{1}{(2b+1)^2 - 1} = \frac{1}{(2c+1)^2 - 1}. \tag{2}$$

Let $x = 2a+1$, $y = 2b+1$, $z = 2c+1$. Then we have:

$$\frac{1}{x^2 - 1} + \frac{1}{y^2 - 1} = \frac{1}{z^2 - 1}. \tag{3}$$

This yields on cross multiplication and simplification:

$$\begin{aligned} \left(z^2 - 1\right)\left(x^2 + y^2 - 2\right) &= \left(x^2 - 1\right)\left(y^2 - 1\right), \\ &= x^2y^2 - 1 - \left(x^2 + y^2 - 2\right), \\ \therefore \ z^2\left(x^2 + y^2 - 2\right) &= x^2y^2 - 1. \end{aligned} \tag{4}$$

What follows now is pure algebraic jugglery! We add/subtract the same quantity to both sides, twice. From the last line we get:

$$z^2\left(x^2+y^2-2xy-2\right)+1=x^2y^2-2xyz^2,$$
$$\therefore\ z^4+z^2\left(x^2+y^2-2xy-2\right)+1=x^2y^2-2xyz^2+z^4,$$
$$\therefore\ z^4+z^2\left((x-y)^2-2\right)+1=\left(xy-z^2\right)^2. \tag{5}$$

Now we invoke familiar logic concerning quadratic expressions. The expression on the right-hand side is identically a perfect square. Treating the expression on the left-hand side as a quadratic expression in z^2, we deduce that its discriminant must be 0. That is:

$$\left((x-y)^2-2\right)^2-4=0. \tag{6}$$

Hence we have: $(x-y)^2-2=\pm 2$, giving $x-y=0$ or ± 2. This yields $a-b=0$ or $a-b=\pm 1$. Thus a and b are either equal to each other or differ by 1, as had been claimed. The numerically observed pattern has been explained. □

Solving the equation

Now we shall show how to solve the given equation, $1/T_a+1/T_b=1/T_c$, given the result we have just found. We consider the two cases ($a=b$; $a-b=\pm 1$) separately. As we shall see, both cases lead to instances of a well-known equation which we have studied earlier (November 2014 issue): the Brahmagupta-Fermat equation. (It is referred to in most textbooks as the 'Pell equation' but as it is now known that this name represents a historical inaccuracy, we shall not use it.)

Case I, a = b: The condition yields $T_b=2T_c$. This amounts to finding pairs of triangular numbers in the ratio 2 : 1. From the relation $b(b+1)=2c(c+1)$, we get:

$$b^2+b=2\left(c^2+c\right),$$
$$\therefore\ \left(b+\frac{1}{2}\right)^2-\frac{1}{4}=2\left(c+\frac{1}{2}\right)^2-\frac{1}{2},$$
$$\therefore\ (2b+1)^2-1=2(2c+1)^2-2,$$
$$\therefore\ X^2-2Y^2=-1,\quad \text{where } X=2b+1,\ Y=2c+1. \tag{7}$$

Case II, a = b − 1: The condition yields

$$\frac{1}{T_{b-1}}+\frac{1}{T_b}=\frac{1}{T_c},$$
$$\therefore\ \frac{2}{(b-1)b}+\frac{2}{b(b+1)}=\frac{2}{c(c+1)},$$
$$\therefore\ \frac{2}{(b-1)(b+1)}=\frac{1}{c(c+1)},$$
$$\therefore\ 2b^2-2=4c^2+4c=(2c+1)^2-1,$$
$$\therefore\ (2c+1)^2-2b^2=-1, \tag{8}$$
$$\therefore\ X^2-2Y^2=-1,\quad \text{where } X=2c+1,\ Y=b. \tag{9}$$

We see that both cases lead to the same underlying equation, $X^2 - 2Y^2 = -1$. Hence each solution (X, Y) to the equation $X^2 - 2Y^2 = -1$ yields two solutions to the original equation, one of each type.

Now it is not hard to see that in any integral solution to this equation, both X and Y must be odd. For: $X^2 = 2Y^2 - 1$, from which we see that X^2 is odd, and hence X too. Now if Y were even, say $Y = 2m$ where m is an integer, then we would get $X^2 = 8m^2 - 1$. However, no square is of the form $8k - 1$. Hence Y must be odd.

A numerical illustration. We use the solution $(X, Y) = (7, 5)$ of the equation $X^2 - 2Y^2 = -1$ to demonstrate how each solution to this equation yields two solutions to the original problem, one of each type. Thus we have:

- A Type I solution with $(2b + 1, 2c + 1) = (7, 5)$, giving $b = 3$ and $c = 2$. From this we get the solution $(a, b, c) = (3, 3, 2)$, i.e., $1/T_3 + 1/T_3 = 1/T_2$.
- A Type II solution with $(2c + 1, b) = (7, 5)$, giving $c = 3$ and $b = 5$. From this we get the solution $(a, b, c) = (4, 5, 3)$, i.e., $1/T_4 + 1/T_5 = 1/T_3$.

Patterns in the solutions of the Brahmagupta-Fermat equation
Listing the solutions of the equation $X^2 - 2Y^2 = -1$ is an interesting as well as rewarding exercise because of the rich patterns to be found. A computer assisted search yields the following solution pairs:

X	1	7	41	239	1393	$\cdots$
Y	1	5	29	169	985	$\cdots$

Now consider the sequence of the X values alone:

$$1,\ 7,\ 41,\ 239,\ 1393,\ \ldots$$

This sequence obeys a Fibonacci-type recurrence relation! Namely: each term after the second one equals 6 times the one before it minus the term before that. That is, if the sequence is $X_1, X_2, X_3, \ldots$, then

$$X_n = 6X_{n-1} - X_{n-2}, \quad \text{for all } n > 2. \tag{10}$$

Remarkably, the sequence of Y values obeys the same recurrence; please check.

It is a nice exercise in mathematical induction to show that if the first two (X, Y) pairs satisfy the equation $X^2 - 2Y^2 = -1$, and we generate the X- and Y-sequences as per the recurrence noted above, then *all* the (X, Y) pairs satisfy the equation. That is, if the following are true:

$$\left.\begin{aligned} &X_1^2 - 2Y_1^2 = -1, \quad X_2^2 - 2Y_2^2 = -1, \\ &X_n = 6X_{n-1} - X_{n-2}, \quad \text{for all } n > 2, \\ &Y_n = 6Y_{n-1} - Y_{n-2}, \quad \text{for all } n > 2, \end{aligned}\right\} \tag{11}$$

then $X_n^2 - 2Y_n^2 = -1$ for all $n > 2$. We leave the algebraic verification to you.

An irrational way of generating the solutions of the Brahmagupta-Fermat equation
Here is another curious way, baffling at first encounter, of generating the positive integer solutions of the equation $X^2 - 2Y^2 = -1$.

Take the irrational quantity $\alpha = 1 + \sqrt{2}$. If we raise α to any integral power n, we necessarily get a quantity of the type $a + b\sqrt{2}$ where a and b are integers whose values depend on n. If we raise α to an *odd* integral power greater than 1 and extract the coefficients a and b from the resulting expression, we find exactly what we want! Here are the data for the first five such exponents n:

n	$(1+\sqrt{2})^n$	(a, b)
3	$7 + 5\sqrt{2}$	$(7, 5)$
5	$41 + 29\sqrt{2}$	$(41, 29)$
7	$239 + 169\sqrt{2}$	$(239, 169)$
9	$1393 + 985\sqrt{2}$	$(1393, 985)$
11	$8119 + 5741\sqrt{2}$	$(8119, 5741)$

Have a look at the third column!

This discovery enables us to find a fresh recurrence relation for the solutions, one which turns out to be more convenient than the one shown above. Suppose that

$$\left(1+\sqrt{2}\right)^n = a + b\sqrt{2} \tag{12}$$

for some pair (a, b) of integers. Then we have:

$$\begin{aligned} \left(1+\sqrt{2}\right)^{n+2} &= \left(a+b\sqrt{2}\right)\cdot\left(1+\sqrt{2}\right)^2 \\ &= \left(a+b\sqrt{2}\right)\cdot\left(3+2\sqrt{2}\right) \\ &= (3a+4b)+(2a+3b)\sqrt{2}. \end{aligned} \tag{13}$$

This yields the following recurrence relation connecting (X_{n+1}, Y_{n+1}) with (X_n, Y_n):

$$\left.\begin{aligned} X_{n+1} &= 3X_n + 4Y_n, \\ Y_{n+1} &= 2X_n + 3Y_n. \end{aligned}\right\} \tag{14}$$

Note that this is a single-step recurrence relation, as distinct from the two-step recurrence relation we had found earlier. It is more convenient to use this relation for computational purposes than the earlier one.

The following important question may have occurred to you after reading the above paragraphs. It is easy to justify that the procedures described do indeed generate infinitely many solutions to the equation $X^2 - 2Y^2 = -1$. (The simplest proof, as suggested above, is via the use of mathematical induction.) But it is far from clear whether they generate *every possible solution*. The question left unresolved by this approach is: Do some solutions get left out? The answer is: the two procedures do indeed generate every possible solution; nothing gets left out! But the full proof of this is rather intricate, and we shall not go into the details at this point.

Back to harmonic triangular triples

Let us now list the triples which are solutions to the original problem proposed: the triples (a, b, c) with $a \le b$, for which

$$\frac{1}{T_a} + \frac{1}{T_b} = \frac{1}{T_c}.$$

We have shown above that:

- Each such triple derives from a solution (X, Y) to the Brahmagupta-Fermat equation $X^2 - 2Y^2 = -1$;
- From a solution $(X, Y) = (2b+1, 2c+1)$ to the equation $X^2 - 2Y^2 = -1$, we get a Type I solution (b, b, c) as well as a Type II solution $(2c, 2c+1, b)$.

Here are the triples generated by the solution pairs (X, Y) listed above. We omit the solution $(X, Y) = (1, 1)$ as it leads to a trivial solution for (a, b, c).

	(X, Y)	$(7, 5)$	$(41, 29)$	$(239, 169)$	$(1393, 985)$
Type I	(a, b, c)	$(3, 3, 2)$	$(20, 20, 14)$	$(119, 119, 84)$	$(696, 696, 492)$
Type II	(a, b, c)	$(4, 5, 3)$	$(28, 29, 20)$	$(168, 169, 119)$	$(984, 985, 696)$

The relationships between corresponding solutions of Types I and II are striking. Thus for example, we have:

$$\frac{1}{T_{20}} + \frac{1}{T_{20}} = \frac{1}{T_{14}},$$

$$\frac{1}{T_{28}} + \frac{1}{T_{29}} = \frac{1}{T_{20}},$$

which together imply that:

$$2\left(\frac{1}{T_{28}} + \frac{1}{T_{29}}\right) = \frac{1}{T_{14}}.$$

Similarly for the other cases.

Closing remark

To conclude this article, we mention a connection between harmonic triangular triples and the famous house number puzzle which too has been studied in an earlier issue of this magazine (March 2014, page 36). That puzzle essentially requires the determination of all pairs (m, n) of positive integers such that $m < n$ and

$$1 + 2 + 3 + \cdots + (m-1) = (m+1) + (m+2) + \cdots + n.$$

An example of such a pair is $(6, 8)$; for $1 + 2 + 3 + 4 + 5 = 7 + 8$, both sides being equal to 15. More generally, observe that the above equation gives rise to the following:

$$\frac{1}{2}(m-1)m = \frac{1}{2}n(n+1) - \frac{1}{2}m(m+1),$$

$$\therefore \quad m^2 = \frac{1}{2}n(n+1),$$

$$\therefore \quad 8m^2 = 4n^2 + 4n = (2n+1)^2 - 1,$$

$$\therefore \quad (2n+1)^2 - 2(2m)^2 = 1.$$

Let $X = 2n+1$ and $Y = 2m$. Then we have $X^2 - 2Y^2 = 1$, and we see yet again an instance of the Brahmagupta-Fermat equation (but note that the sign of the 1 on the right-hand side is positive rather than negative). Its solutions can be generated in a similar way: take the irrational quantity $\alpha = 1 + \sqrt{2}$ and raise it to an *even* integral power larger than 2. (Raising it to power 2 yield a trivial solution.) The coefficients in the resulting expression then yield a solution to our problem. Thus:

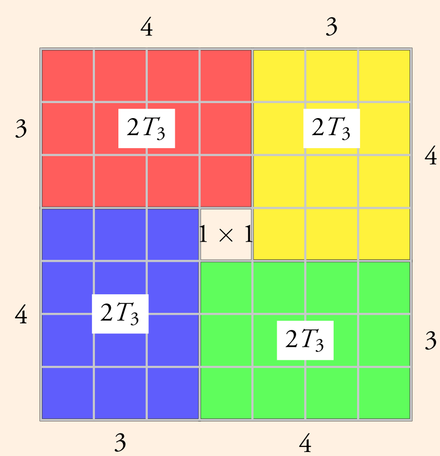
n	$(1+\sqrt{2})^n$	(a, b)	(m, n)	Corresponding equality
4	$17 + 12\sqrt{2}$	$(17, 12)$	$(8, 6)$	$1 + 2 + \cdots + 5 = 7 + 8$
6	$99 + 70\sqrt{2}$	$(99, 70)$	$(49, 35)$	$1 + 2 + \cdots + 34 = 36 + 37 + \cdots + 49$
8	$577 + 408\sqrt{2}$	$(577, 408)$	$(288, 204)$	$1 + 2 + \cdots + 203 = 205 + 206 + \cdots + 288$
10	$3363 + 2378\sqrt{2}$	$(3363, 2378)$	$(1681, 1189)$	$1 + 2 + \cdots + 1188 = 1190 + 1191 + \cdots + 1681$

The column of (m, n) values is obtained from the (a, b) column by using the relations $m = (a-1)/2$, $n = b/2$.

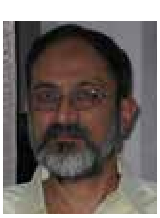
You may wonder whether it will always be the case that a is odd and b is even. The answer is Yes, but we leave the proof for you to complete. (It can be done using the principle of induction.)

Visual proof of an identity concerning the triangular numbers

The identity being referred to is the following: $8T_n + 1 = (2n+1)^2$ for any positive integer n. The following figure illustrates the identity for the case $n = 3$.



Box 1. Illustrating `why' $8T_3 + 1$ is a perfect square



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