Problems for the SENIOR SCHOOL

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Problem VII-3-S.1

Let $f(x) = x^2 + bx + c$ where b is a negative integer and c is a real number. Suppose the sum of the roots of $f(f(x))$ is a prime number. Prove that $f(f(x))$ has no real root in the interval $(0, 1)$.

Problem VII-3-S.2

Let k be a given positive integer. Determine all real x, y, z such that $xyz \neq 0$ and

$$x^k + y^{k+1} = z^{k+2}, \quad x^{k+1} + y^{k+2} = z^{k+3}, \quad x^{k+2} + y^{k+3} = z^{k+4}.$$

Problem VII-3-S.3

A quadratic polynomial $f(x) = ax^2 + bx + c$ has no real roots. It is given that b is a rational number, and exactly one of c and $f(c)$ is a rational number. Is it possible for the discriminant of $f(x)$ to be a rational number? [Russian Mathematical Olympiad]

Problem VII-3-S.4

The sequence $\{a_n\}_{n \geq 0}$ is defined as follows:

$$a_0 = 1, \quad a_1 = 3, \quad a_{n+1} = a_n + a_{n-1} \text{ for all } n \geq 1.$$

Find all integers $n \geq 1$ for which $na_{n+1} + a_n$ and $na_n + a_{n-1}$ share a common factor greater than 1.

Problem VII-3-S.5

Consider the sequence $\{10^n\}_{n \geq 1}$. Prove that the sum of no two terms of the sequence is a perfect square.

Keywords: Quadratics, roots, functions, circles, triangles, equilateral

Solutions of Problems in Issue VII-2 (July 2018)

Solution to problem VII-2-S.1

Let AB be a fixed line segment in the plane. Let O and P be two points in the plane, on the same side of AB. If $\measuredangle AOB = 2\measuredangle APB$, does it necessarily follow that P lies on the circle with centre O and passing through A and B?

Not necessarily. Consider the circle passing through A, O and B. The magnitude of $\measuredangle AOB$ does not change if O moves on arc AB on the same side of AB as P. Thus there are infinitely many positions of O for which $\measuredangle AOB = 2\measuredangle APB$ but $OA \neq OB$. If we have the additional hypothesis that $OA = OB$, then the claim can be proven to be true.

Solution to problem VII-2-S.2

Let ABC be an equilateral triangle with centre O. A line through C meets the circumcircle of triangle AOB at points D and E. Prove that the points A, O and the midpoints of segments BD, BE are concyclic. [Tournament of Towns]

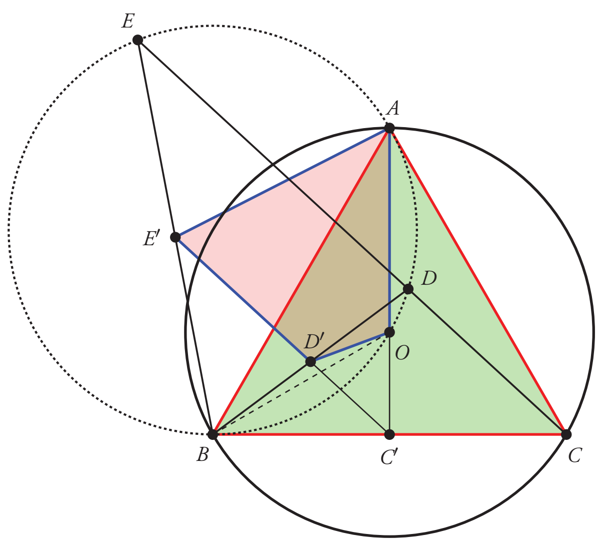

Figure 1.

Let D' and E' be the midpoints of BD and BE respectively. Extend $E'D'$ to meet BC at C'. Since $E'C'$ is parallel to EC and E' is the midpoint of EB, C' is the midpoint of BC. Therefore

$$C'D' \cdot C'E' = \left(\frac{1}{2}CD\right) \cdot \left(\frac{1}{2}CE\right) = \frac{1}{4}CD \cdot CE.$$

Observe that $\measuredangle OBC = \measuredangle OAB = 30^\circ$. Therefore, CB is tangent to the circumcircle of AOB and $CB^2 = CD \cdot CE$.

Thus $C'D' \cdot C'E' = \frac{1}{4}CD \cdot CE = \frac{1}{4}CB^2 = C'B^2 = C'O \cdot C'A$. This shows that A, O, D' and E' are concyclic.

Solution to problem VII-2-S.3

Three nonzero real numbers are given. It is given that if they are written in any order as the coefficients of a quadratic trinomial, then each of these trinomials has a real root. Does it follow that each of these trinomials has a positive root? [Tournament of Towns]

Let the three nonzero real numbers be a, b, c. Since a, b and c are all non-zero, 0 is not a root of any of the six trinomials under consideration. Suppose that $ax^2 + bx + c$ has two negative roots $-r$ and $-s$, where r and s are positive numbers. Then

$$ax^2 + bx + c = a(x + r)(x + s),$$

so $b = a(r + s)$ and $c = ars$ both have the same sign as a. Hence we may assume that they are all positive. Since one of the roots is real, both are real, so that we have $b^2 \geq 4ac$. Similarly, $c^2 \geq 4ab$ and $a^2 \geq 4bc$. Multiplication yields $(abc)^2 \geq (8abc)^2$, which is a contradiction. It follows that each of the six trinomials has a positive root.

Solution to problem VII-2-S.4

D is the midpoint of the side BC of triangle ABC. E and F are points on CA and AB respectively, such that BE is perpendicular to CA and CF is perpendicular to AB. If DEF is an equilateral triangle, does it follow that ABC is equilateral? [Tournament of Towns]

We shall show by actually constructing a counterexample that triangle ABC is not necessarily equilateral. Start with an equilateral triangle DEF. (See Figure 2.) Draw a segment BC through D, such that D is the midpoint of BC and BC is perpendicular to DE (at D), with F closer to B than to C. Construct a semicircle with centre D and radius DE. Since $DF = DE$, point F lies on this semicircle. Extend BF and CE to meet at A.

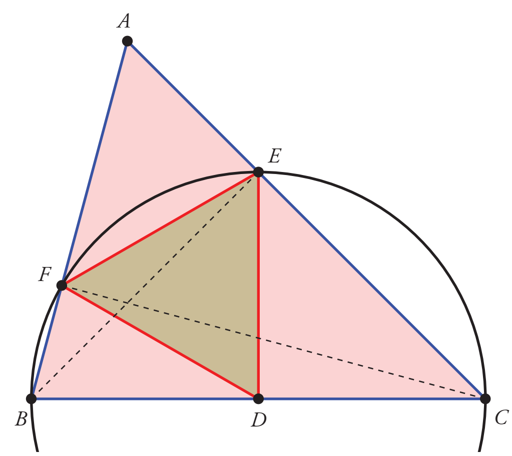

Figure 2.

Since $\measuredangle BEC = 90^\circ = \measuredangle BFC$, BE and CF are altitudes of triangle ABC. Since A lies on the extension of CE and DE is the perpendicular bisector of BC, $AB < AC$. Hence ABC is not equilateral.

Solution to problem VII-2-S.5

A boy computed the product of first n positive integers and his sister computed the product of the first m even positive integers where $m \geq 2$. Is it possible for them to get the same result?

The product of the first m even positive integers is $2^m \cdot m!$. Suppose that $n! = 2^m \cdot m!$ for some $m \geq 2$. Clearly $m \neq 2$; so $m > 2$. But then $n \geq 3$ in order for both $n!$ and $m!$ to be divisible by 3. In each product, every third factor is divisible by 3. For them to be divisible by the same power of 3, $n!$ can have at most two more terms than $m!$. Thus $n = m + 1$ or $n = m + 2$. If $n = m + 1$, then $m + 1 = 2^m$ which leads to $m = 1$. If $n = m + 2$, then $(m + 1)(m + 2) = 2^m$; but this has no solution in integers. (If $m > 2$, the product $(m + 1)(m + 2)$ has an odd factor > 3; but 2^m can have no odd factor > 1.) Thus we arrive at contradictions and we conclude that the two products cannot be the same.