ADVENTURES IN PROBLEM SOLVING

A Math Olympiad Problem … and a few cousins

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In this edition of *Adventures in Problem Solving*, we study in detail a problem which first appeared in the 1987 USA Mathematics Olympiad. However, we adopt a different strategy this time. Faced with the given problem which looks quite challenging, we tweak it in different ways and obtain related problems which are simpler than the original one. Solving this collection of problems turns out to be a fun activity and demonstrates yet one more time the importance and utility of quadratic functions and quadratic equations.

As usual, we state the problems first, so that you have an opportunity to tackle them before seeing the solutions.

The problem is to find all integer-valued solutions of the following equation:

$$\left(a^2+b\right)\left(a+b^2\right)=(a-b)^3.$$

We shall study not just this equation but others obtained by tweaking it. As noted above, we shall find that knowledge of quadratic equations and quadratic functions comes to our aid repeatedly. Those who have read earlier issues of this magazine will know that quadratic equations and quadratic functions have been studied many times in these pages. What is striking is that they invariably enter the scene in a completely natural manner, without fanfare or announcement. The manner in which this happens is worthy of close study. (For more such instances, please see [1].)

We start by studying three simpler variants of the USAMO problem (1–3 below). They are superficially similar to the original problem. In each case, our interest lies only in solutions where a and b are both nonzero (this may be captured in a compact way by the statement $ab \neq 0$). Problem 4 is the same as the one stated above. Please try solving all of them; you will find that one of them is particularly easy, but we won't tell you which one!

Problem 1: Find all nonzero integer-valued solutions of the following equation:

$$\left(a^2+b\right)\left(a+b^2\right)=(a+b)^3. \tag{1}$$

Problem 2: Find all nonzero integer-valued solutions of the following equation:

$$(a+b)\left(a^2+b^2\right)=(a+b)^3. \tag{2}$$

Problem 3: Find all nonzero integer-valued solutions of the following equation:

$$(a+b)\left(a^2+b^2\right)=a^3+kb^3. \tag{3}$$

Here k is an integer-valued parameter; the question here is, for which k do solutions exist?

Problem 4: Find all nonzero integer-valued solutions of the following equation:

$$\left(a^2+b\right)\left(a+b^2\right)=(a-b)^3. \tag{4}$$

Solutions

Solution to Problem 1. The pairs $(a,0)$ and $(0,b)$ work for any integer values of a and b; however, we have explicitly excluded such solutions from consideration.

The natural thing to do is to simplify the expressions involved and see where it leads us. We have:

$$\begin{aligned}\left(a^2+b\right)\left(a+b^2\right)-(a+b)^3 &= ab+a^2b^2-3ab^2-3a^2b\\ &= ab\,(ab-3a-3b+1)\,.\end{aligned}$$

As we have announced in advance that we will not be looking at solutions in which either of the variables is 0, we may assume that ab is nonzero. Hence equation 1 implies that

$$ab-3a-3b+1=0. \tag{5}$$

We have encountered this kind of equation on several occasions. They are solved using a standard artifice, using factorisation. We only need to observe that

$$(a-3)(b-3)=ab-3a-3b+9,$$

which is almost the same as $ab-3a-3b+1$; only the constant term is different. We therefore proceed as follows.

$$\begin{aligned}ab-3a-3b+1&=0,\\ \therefore\ ab-3a-3b+9&=8,\\ \therefore\ (a-3)(b-3)&=8.\end{aligned}$$

Therefore, $a-3$, $b-3$ are a pair of complementary factors of 8 (i.e., their product is 8). Here we permit the presence of negative factors. It therefore follows that

$$(a-3,b-3)\in\{(-8,-1),\ (-4,-2),\ (-2,-4),\ (-1,-8),\ (1,8),\ (2,4),\ (4,2),\ (8,1)\},$$

giving:

$$(a,b)\in\{(-5,2),\ (-1,1),\ (1,-1),\ (2,-5),\ (4,11),\ (5,7),\ (7,5),\ (11,4)\}. \tag{6}$$

These are thus the desired solutions; there are eight such integer pairs.

Solution to Problem 2. Once again, we begin by simplifying the expressions involved. We have:

$$(a+b)\left(a^2+b^2\right)-(a+b)^3=-2ab(a+b). \tag{7}$$

Hence the nonzero solutions to equation 2 all have $a+b=0$, i.e., they are all of the form $(t,-t)$ for some integer t. So equation 2 has infinitely many nonzero integer solutions.

This should not come as a surprise, for the equation is *homogeneous*; if we expand out the terms in the equation, then every term has degree 3. This implies that if a, b solves the equation, then so does ka, kb for every integer k.

Problem 2 has turned out to be rather too simple!

Solution to Problem 3. This problem turns out to be quite interesting. Note that as in Problem 2, the equation is homogeneous; every term has degree 3.

On opening out the brackets and simplifying, we obtain:

$$\begin{aligned}(a+b)\left(a^2+b^2\right)&=a^3+kb^3,\\ \therefore\ \ a^3+a^2b+ab^2+b^3&=a^3+kb^3,\\ \therefore\ \ (k-1)b^3-ab^2-a^2b&=0.\end{aligned}$$

Since we have assumed that $b\neq 0$, we get:

$$(k-1)b^2-ab-a^2=0. \tag{8}$$

We regard this as a quadratic equation with b as the unknown. If the equation is to have integer solutions, then its discriminant D must be a perfect square. Hence the quantity

$$a^2+4(k-1)a^2=a^2\,(4k-3)$$

must be a perfect square. For this to happen, $4k-3$ must itself be a perfect square. Since $4k-3$ is an odd number, we must have

$$4k-3=(2n+1)^2$$

for some integer n; i.e., $4k-3=4n^2+4n+1$ for some integer n. Hence

$$k=n^2+n+1 \tag{9}$$

for some integer n. It follows that equation 3 has nonzero integer solutions if and only if k is of the form n^2+n+1 for some integer n, i.e., iff $k\in\{1,3,7,13,21,31,43,\ldots\}$. When k is of this form, equation 8 yields:

$$\begin{aligned}&\left(n^2+n\right)b^2-ab-a^2=0,\\ \therefore\ \ &b=\frac{a\pm\sqrt{a^2\left(4n^2+4n+1\right)}}{2\left(n^2+n\right)}=\frac{a(1\pm(2n+1))}{2n(n+1)},\\ \therefore\ \ &b=\frac{a}{n},\quad b=-\frac{a}{n+1}.\end{aligned}$$

Hence the solutions to equation 3, with k having the form n^2+n+1, are the following:

$$(a,b)=(nt,t),\quad (a,b)=(-(n+1)t,t), \tag{10}$$

where t is an arbitrary integer. Thus equation 3 too has infinitely many nonzero integer solutions; indeed, two families of them.

Solution to Problem 4. Now we are ready to take up the equation posed at the start:

$$\left(a^2+b\right)\left(a+b^2\right)=(a-b)^3.$$

Our interest as earlier will be in the nonzero integer solutions. On opening the brackets and simplifying, we get:

$$\left(a^2+b\right)\left(a+b^2\right)-(a-b)^3=b\left(a^2b+3a^2-3ab+a+2b^2\right).$$

Since $b \neq 0$, the given equation yields:

$$a^2b+3a^2-3ab+a+2b^2=0. \tag{11}$$

We observe the following about the polynomial $a^2b+3a^2-3ab+a+2b^2$:

- it is quadratic in a individually;
- it is quadratic in b individually;
- it is a cubic expression when a, b are both treated as variables (by virtue of the term a^2b).

To make progress, we exploit the fact that the polynomial is quadratic in each of a and b individually; but which one should we use? It turns out that if we regard the expression as a quadratic in b, less work is involved; the expressions obtained are more manageable. Here is how the analysis proceeds. We write the equation $a^2b+3a^2-3ab+a+2b^2=0$ as

$$2b^2+a(a-3)b+a(3a+1)=0. \tag{12}$$

For this quadratic equation to have integer solutions, the discriminant must be a perfect square. Hence the quantity D given by

$$D=a^2(a-3)^2-8a(3a+1) \tag{13}$$

must be a perfect square. We now need to identify the integer values of a for which D is a perfect square.

We first simplify this expression; we get:

$$\begin{aligned} D &= a^2(a-3)^2-8a(3a+1)\\ &= a\left(a^3-6a^2-15a-8\right), \end{aligned}$$

on simplifying. Progress is achieved when we notice that $-1-6+15-8=0$, which tells us that $a+1$ is a factor of $a^3-6a^2-15a-8$. (Remember the factor theorem!) We have:

$$a^3-6a^2-15a-8=(a+1)\left(a^2-7a-8\right).$$

Next, we notice that $1+7-8=0$, which tells us that $a+1$ is a factor of a^2-7a-8:

$$a^2-7a-8=(a+1)(a-8).$$

Hence $a+1$ is a repeated factor (i.e., it occurs twice), and we have:

$$a\left(a^3-6a^2-15a-8\right)=a(a+1)^2(a-8). \tag{14}$$

It follows that D is a perfect square precisely when $a(a-8)$ is a perfect square. We must therefore identify all integers a for which $a(a-8)$ is a perfect square. This question brings us back to familiar ground; we have tackled such questions earlier — many times! We only need the tried and trusted completing-the-square procedure and the difference-of-two-squares formula. Let $a(a-8)=c^2$. Then we have:

$$\begin{aligned} a^2 - 8a &= c^2, \\ \therefore\ a^2 - 8a + 16 &= c^2 + 16, \\ \therefore\ (a-4)^2 - c^2 &= 16, \\ \therefore\ (a-4-c)(a-4+c) &= 16. \end{aligned}$$

The ordered integer pairs (u, v) whose product is 16 are the following:

$$\begin{matrix} (-16,-1), & (-8,-2), & (-4,-4), & (-2,-8), & (-1,-16), \\ (1,16), & (2,8), & (4,4), & (8,2), & (16,1). \end{matrix}$$

We can equate the pair $(a-4-c, a-4+c)$ with each of these pairs and solve for a and c; and from a we then solve for b. The pairs $(\pm 16, \pm 1)$ and $(\pm 1, \pm 16)$ yield fractional values of a, and so need not be considered. (This happens because 1 and 16 have opposite parity.) That leaves six pairs. Solving for c and a and then for b, we get the following:

$(a-4-c, a-4+c)$	c	a	b
$(-8,-2)$	3	-1	-1
$(-4,-4)$	0	0	0
$(-2,-8)$	-3	-1	-1
$(2,8)$	3	9	$-21, -6$
$(4,4)$	0	8	-10
$(8,2)$	-3	9	$-21, -6$

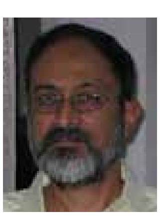
It follows that the nonzero integer pairs which satisfy equation 4 are the following:

$$(a,b) = (-1,-1),\ (9,-21),\ (9,-6),\ (8,-10). \tag{15}$$

Exercise. Write the equation that we had obtained, $a^2b + 3a^2 - 3ab + a + 2b^2 = 0$, as a quadratic in a and check that this approach leads to the same solutions listed above.

References

1. Chris Budd and Chris Sangwin, "101 uses of a quadratic equation" from +plus Magazine, https://plus.maths.org/content/101-uses-quadratic-equation



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