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Teaching Concepts of CALCULUS

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The topic of calculus is an integral part of the senior secondary mathematics curriculum. The concepts of limits and derivatives, which form the foundation of Calculus, are often hard to teach. In this article, we suggest a dynamic way of teaching the concept of the derivative through practical examples which may be easily explored through GeoGebra applets available through the Internet.

Introduction

Calculus may be described as the mathematical study of change. It provides us with the tools to study and analyse change of specific phenomena in the real world. The change may be with respect to time or with respect to another quantity. Let us consider the example of a liquid being poured into a cylindrical can. The volume of the liquid in the can varies with respect to time. The height (level) of the liquid also varies with respect to time. Further, the volume and the height of the liquid also vary with respect to each other. Thus, calculus enables us to describe change represented by variables which vary with respect to each other. Calculus may be described as the branch of mathematics used to study any phenomena involving change, which may involve any pair of dimensions such as time, force, mass, length, displacement, temperature, etc.

Much of the calculus taught in school is procedural in nature, as a great deal of emphasis is laid on learning the manipulative skills for computing limits and derivatives of different functions. Often, students believe that calculus is mainly about applying formulae and rules, and computing the correct symbolic expressions for derivatives and integrals. There is a strong need to help students see the relevance of calculus to daily life by highlighting appropriate examples.

One way to start a lesson in calculus is to connect it to something the students are familiar with. Riding a bicycle, travelling in a car or skiing down a mountain slope may be situations which students can easily relate to and may be used to build on their intuitive idea of change in velocity. These experiences probably

Keywords: Variables, change, graphs, GeoGebra, slope, tangent, derivative

occur long before they encounter the concepts of calculus in upper secondary school. Thompson (1994) argued that

> *The development of images of rate starts with children's image of change in some quantity (e.g., displacement of position, increase in volume), progresses to a loosely coordinated image of two quantities (e.g., displacement of position and duration of displacement), which progresses to an image of the covariation of two quantities so that their measures remain in constant ratio. (p. 128)*

Thus teaching of calculus could begin with a discussion about change of velocity. Changes may be represented graphically as a function. Nevertheless, to understand the relation between a function and its derivative through graphs is by no means trivial. In *The development of students' graphical understanding of the derivative* (Asiala, Cottrill, Dubinsky & Schwingendorf, 1997, p. 402), the authors write: "many authors discuss specific problems students have with the graphical interpretation of the derivative."

Classroom activity: A daily life situation

Let us consider a classroom activity where students are required to investigate a motorcycle in motion. This may be introduced graphically as shown in Figure 1. The motorcycle's motion starts at time $t = 0$ and spans a period of 4 hours.

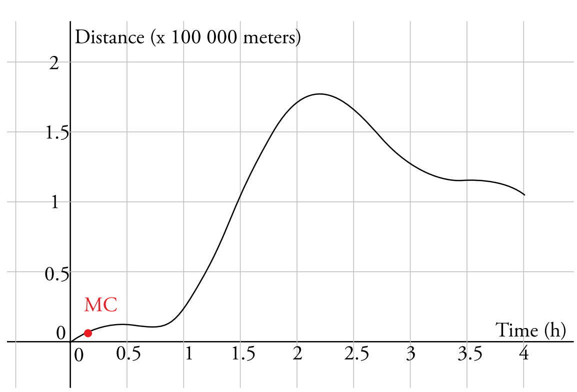

Figure 1. A motorcycle and its motion illustrated by GeoGebra.

Figure 1 shows the graphical representation of the motorcycle's displacement from its position at time t. Students may explore the dynamic version of the activity through the GeoGebra file https://www.geogebra.org/m/SVDrYnvQ

Using this applet the teacher may facilitate a discussion by moving the red point labeled MC (which represents the position of the motorcycle) and asking students to explain what change in displacement means (they should also discuss the displacement at any instant of time and understand the difference between the displacement at an instant and the change in displacement over an interval).

> *Students should be encouraged to express their ideas in their own words before a formal explanation is provided.*

Usually, it is difficult for students to translate the mental image of a motorcycle into a point on a graph. In addition, they should be able to visualize that movement away from the starting point (the *origin*) is related to a positive velocity while movement back to the starting point is related to a negative velocity. We measure the displacement of the motorcycle by a vertical line for a time t_1 and then we identify where that line meets the graph.

A good deal of research has been conducted to investigate students' alternative conceptions about graphical concepts (Elby, 2000; Hammer, 2000; McDermott & Schaffer, 2005; Lingefjärd & Farahani, 2017). Research indicates that it is often quite difficult for students to make interpretations of a graph in a coordinate system, especially if the graph describes a real situation in some way. Many students have alternative conceptions that compete with and interact with scientific interpretations. (We prefer to use the term 'alternative conception' instead of 'misconception' which is associated with inaccuracy and mistakes.)

Learning from graphical representations requires students to understand the syntax of each representation. The format of a graph includes attributes such as labels, number of axes, and line shapes. Interpretations require finding the gradients of lines, minima and maxima, as well as intercepts (Ainsworth, Bibby & Wood, 1997). Students must also understand which parts of the domain of the function are represented in the graph.

If we look carefully at the graphical representation in Figure 1, we may observe that the motorcycle is running rather slowly at first since it only covers a distance of 25 kilometres in the first hour. It also appears as if the motorcycle partly turns around and moves back between 25 minutes and 45 minutes after the start of the journey. Thereafter it changes its course once again.

The teacher may ask students to reason about why the motorcycle rider is steering the motorcycle in that way. Perhaps the rider was searching for the correct road. A motorcycle that is in motion for 4 hours will also have a velocity. How can we view velocity in a displacement time graph? What is velocity? We believe that students have a general understanding of velocity, which they interpret as a measurement of how fast they are able to move from position A to position B. "The faster I am moving from position A to position B, the higher is my velocity." At this point of the lesson it might be suitable to introduce the concept of the tangent line based on the slope of the graph.

Tangent Line

For visualizing the derivative it is important to introduce the concept of slope of a curve at a specific point. Is the motorcycle going forward or backward or changing its direction at a specific point on the graph? How do we translate that into common behavior when steering a motorcycle? As it passes through the point where the tangent line and the curve coincide, called the point of tangency, the tangent line is going in the same direction as the curve, and is thus the best straight-line approximation to the curve at that point. Such an example enables us to develop the concept image (Tall & Vinner, 1981) of derivative in the context of graphical representation; see Figure 2.

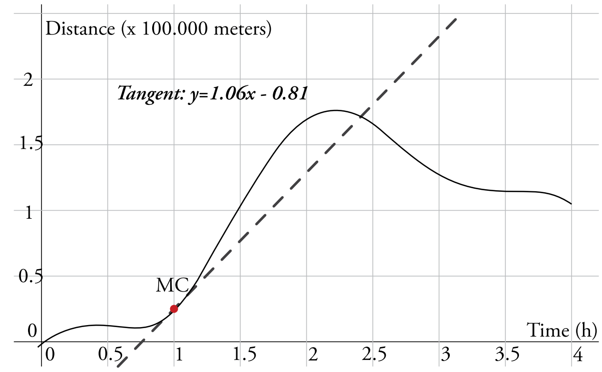

Figure 2. Introducing the concept of tangent line illustrated by GeoGebra.

While teaching the concept of a tangent line in relation to the motion, and velocity, of a motorcycle, it could be beneficial for students to explore the dynamic version of Figure 2. The GeoGebra applet available at https://www.geogebra.org/m/gnzPK9QH allows the user to move the point MC, and observe the tangent line as it moves along the curve. The symbolic representation (equation) of the tangent gets updated as the tangent moves. We argue that this dynamic approach is a more efficient way to explain and discuss the meaning of slope of the tangent than a pen and paper approach.

The teacher may move the point MC and ask students to explain the information given by the tangent line. They may be asked to interpret the positive and negative values of the slope of the tangent as it moves along the curve. They should be encouraged to express their ideas in their own words.

Velocity

The next step is to allow GeoGebra to illustrate how the velocity varies when the motorcycle is in motion. The velocity may be viewed as the trace of a point (h,k) where we define h as the x-value of the point MC and k is the slope of the tangent line at $x = h$ (in GeoGebra we enter $h = x(\text{MC})$ and k = Slope[Tangent] in the Input bar). If we furthermore allow the point velocity to be traced, we obtain the green curve as shown in Figure 3.

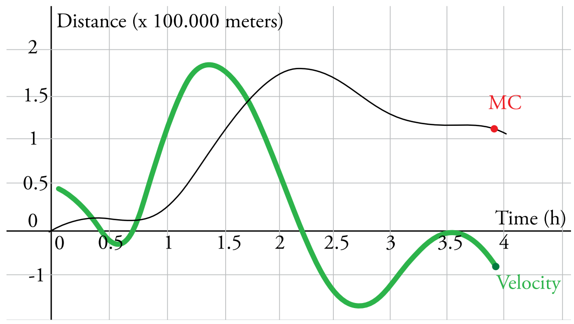

Figure 3. GeoGebra illustrates how the motorcycle's velocity is related to its position.

The GeoGebra file can be found at https://www.geogebra.org/m/pJqtPZyM. While using this applet, we can move the point MC to show how velocity is changing with respect to the motion of the motorcycle.

Students should now be given the challenge to interpret how the graphical representation of the velocity is related to the motorcycle's position. They may attempt this task alone at first, then discuss in pairs and finally participate in a classroom discussion. Maintaining students' written records would give the teacher an insight into their thinking. Students' interpretations of graphs and other representations can shed empirical light on a longstanding theoretical debate about learning of scientific concepts. Students' intuitive knowledge about science concepts may consist of unarticulated, loosely-connected knowledge elements, the activation of which depends sensitively on context.

Learning is not a matter of replacing bad mini-generalizations with good ones. Instead, it's partly a matter of tweaking those mini-generalizations into a more articulate, unified, coherent structure.

The green velocity graph (as shown in Figure 3) is the derivative of the blue graph of the motorcycle's position over time. So velocity can be interpreted as derivative of displacement with respect to time. The significant details of one representation can be used to interpret a different representation. This may result in interpretations such as viewing a graph as a picture (see Lingefjärd & Farahani, 2017). Students should be given ample time to understand the relationship between the displacement graph and the velocity graph.

Acceleration

Finally we can let GeoGebra illustrate a graphical representation of how the acceleration is changing over time when the motorcycle is in motion.

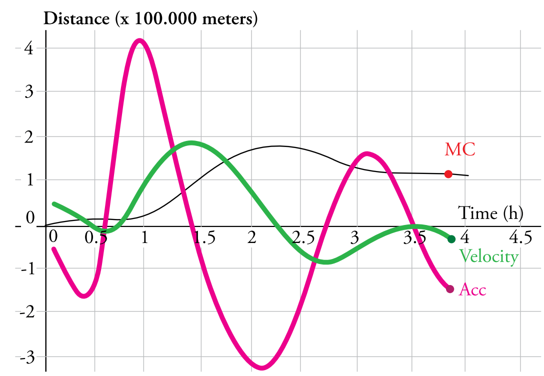

Figure 4. GeoGebra shows how the motorcycle's displacement, velocity and acceleration change over time.

The GeoGebra file can be found at https://www.geogebra.org/m/u56mCPFy.

While using this applet, we can once again move the point MC and show how velocity and acceleration change with respect to the motorcycle's motion. It is important to help students to understand the relationship between the original function (the displacement graph shown in blue), its derivative (the velocity graph shown in green) and the second derivative (the acceleration graph shown in red).

There must be scaffolding to translate between the graphical and symbolic representations, of velocity, acceleration, and displacement. Students need sufficient time to understand that velocity is the rate of change of displacement (symbolically written as $v(t) = d'(t)$) , and that acceleration is the rate of change of velocity ($a(t) = v'(t)$). Acceleration is therefore defined as the second derivative of displacement which is expressed as $a(t) = d''(t)$.

Conclusion

In this article we have highlighted the fact that a dynamic geometry software like GeoGebra can enable students to visualize and explore the concept of the derivative dynamically through graphical and symbolic representations. Research has shown that students face major difficulties in understanding graphical representations even in a dynamic environment. However we believe that with proper facilitation, the GeoGebra applets of the motorcycle in motion can be used to help the students develop a more coherent concept image of a function, slope at a point and derivative.

Students need an understanding of time as a variable in a changing process. They also need to develop an understanding of the slope of a curve in a graphical representation, something that is not always explicitly clear. Dynamic representations of realistic examples, such as that of the motion of a motorcycle or other similar phenomena can help students visualise concepts better than static iconic interpretations.

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