Mathematics Olympiads in India

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The history of Mathematical Olympiad (MO) activity in India is not available anywhere. Hence, while recording this history, we also mention the people who initiated and nurtured this activity. However, before we talk about it, we first highlight a few aspects of the International Mathematical Olympiad (IMO).

The International Mathematical Olympiad – an overview

The International Mathematical Olympiad (IMO for short) is a major event in the world mathematical scene today, featuring close to 100 different countries, but it began on a very small note. The first IMO was held in 1959 when Romania invited a small number of countries from what was then known as the Eastern Bloc to participate in the event. The following seven countries participated in this IMO: Romania, Hungary, Czechoslovakia, Bulgaria, Poland, Union of Soviet Socialist Republics (USSR, more commonly known as Soviet Union; the Union ceased to exist in 1991) and the German Democratic Republic (more commonly known as East Germany; note that West Germany and East Germany united in 1989).

Here is a quote from the official IMO website, [5]: "The International Mathematical Olympiad (IMO) is the World Championship Mathematics Competition for High School students and is held annually in a different country. The first IMO was held in 1959 in Romania, with 7 countries participating. It has gradually expanded to over 100 countries from 5 continents. The IMO Board ensures that the competition takes place each year and that each host country observes the regulations and traditions of the IMO."

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And here is a quote from the Wikipedia website, [9]: "The IMO examination consists of six problems. Each problem is worth seven points, so the maximum total score is 42 points. No calculators are allowed. The examination is held over two consecutive days; each day the contestants have four-and-a-half hours to solve three problems. The problems chosen are from various areas of secondary school mathematics, broadly classifiable as geometry, number theory, algebra, and combinatorics. They require no knowledge of higher mathematics such as calculus and analysis, and solutions are often short and elementary. However, they are usually disguised so as to make the solutions difficult. Prominently featured are algebraic inequalities, complex numbers, and construction-oriented geometrical problems, though in recent years the latter has not been as popular as before."

One of the extremely attractive traditions of the IMO is the procedure for selection of the examination problems. Participating countries are requested to send problem proposals to the host country a few months in advance of the IMO. It is expected that these proposals are original and have been kept confidential; in particular, it is expected that they have not been seen by the students participating from that country. Generally, 100 or more proposals are received by the host country, which then prepares from this large collection a shortlist of 30 problems. This is done by the Problem Committee set up by the host country. Needless to say, the task of preparing the shortlist is complex and requires the problem committee to intensively study the proposals that have been received (and to find their own solutions to these problems – no mean task).

Each participating team consists of a Team Leader, a Deputy Team Leader and up to 6 student participants. The team leaders arrive at the IMO venue a few days in advance of the IMO and all leaders together form the Jury, which selects six problems from the shortlist after a long discussion spread over two full days. After the problems have been selected, the Jury further subdivides them into two groups of three problems each (each set of three problems to be done at one sitting; two sittings are required). The wordings of the problems are examined carefully to avoid ambiguities and misinterpretations. Translations into numerous languages are also done at this point in time. By tradition, the three problems on each day include one that is regarded as 'fairly easy', one that is 'moderately difficult', and one that is 'extremely difficult.' (Of course, these terms are relative! It has happened at times that the Jury has completely underestimated the level of difficulty of a problem.)

The Deputy Leader arrives at the IMO venue later with the students and he/she and the participants stay separately from the Jury, without any communication with the leaders. This is important from the standpoint of security and confidentiality.

There is an elaborate but well-worked out procedure for evaluation of the scripts and award of grades. Needless to say, this is an exhausting period of time for the Team Leaders and the Deputy Team Leaders. By tradition, each problem is created out of 7 points, so the maximum score possible for any individual participant is 42 points.

Officially, the IMO is not a team event (though unofficial tallies of team scores are maintained; in this article, we shall not refer to these unofficial team rankings at all), and participants are ranked based on their individual scores. Medals are awarded to the highest ranked participants; slightly fewer than half of them receive a medal. The cutoffs (minimum scores required to receive a gold, silver or bronze medal respectively) are chosen so that the numbers of gold, silver and bronze medals awarded are approximately in the ratio 1 : 2 : 3. Participants who do not win a medal but who score 7 points on at least one problem receive an 'Honourable Mention.' Special prizes may be awarded for solutions of outstanding elegance or involving good generalisations of a problem. The details are available at [2].

Concerning IMO medalists, [10] notes the following: "A number of IMO medalists have gone on to become notable mathematicians. Some IMO participants have either received a Fields Medal, a Wolf Prize or a Clay Research Award, awards which recognise groundbreaking research in mathematics; a European Mathematical Society Prize, an award which recognizes young researchers; or one of the American Mathematical Society's awards (a Blumenthal Award in Pure Mathematics, Bôcher Memorial Prize in Analysis, Cole Prize in Algebra, Cole Prize in Number Theory or Veblen Prize in Geometry and Topology) recognizing research in specific mathematical fields."

A few IMO medalists have also gone on to become notable computer scientists, receiving prizes such as the Nevanlinna Prize, the Knuth Prize and so on (awards which recognise outstanding research in theoretical computer science).

Occasionally, the IMO features some exceptionally young participants. Just to mention two such individuals: Terence Tao (Australia), who won a gold medal in 1988, at age 13 years (he also won a bronze medal in 1986 and a silver medal in 1987, at age 11 years); Akshay Venkatesh (Australia) who won a bronze medal in 1994, at age 12 years. Both these mathematicians were awarded Fields medals (in 2006 and 2018 respectively).

Mention may be made of Ciprian Manolescu, the only person to achieve three perfect scores—at IMO 1995 (Canada), IMO 1996 (India) and IMO 1997 (Argentina). He is now a Professor of Mathematics at the University of California, Los Angeles.

To see how various countries have performed over the years, please see [6]. To see how India has performed over the years, please see [3].

IMO does not give much importance to the names of the leaders and deputy leaders at its site. However, the names of leaders and deputy leaders are available for any particular year; see [4]. By changing the year in this link, we can get information for other years.

Mathematical Olympiad activity in India

Now we come to Mathematics Olympiad activity in India. In 1968, only 12 countries participated in IMO, mostly from eastern Europe and just three other countries: Italy, Sweden and UK. But there was a visionary in India, P. L. Bhatnagar (at the Indian Institute of Science (IISc), Bangalore) or PLB as he was known, who organised the first ever Mathematical Olympiad (MO) in India for the students of Bangalore in January 1968 under the auspices of the Bangalore Mathematical Association. In the first two such MOs (the second was held in December 1968 at Bangalore, Mysore, Dharwar, Gulbarga and Mangalore), only one candidate qualified for an Olympiad prize. PLB left IISc in 1969 but as the President of the Association of the Mathematics Teachers of India (AMTI), he continued organising MO activities in various cities in India and AMTI continues to hold MO activities at the national level even today.

Some members (led by Phoolan Prasad, a Ph.D. student of PLB) of the Department of Applied Mathematics of IISc again started MO in 1978 for students of Bangalore city and then continued this activity. It became an official activity of IISc through its Centre of Continuing Education. A highlight of this activity was a one-day lecture program on the culture of mathematics for those who qualified in the MO examination. Many distinguished mathematicians and scientists from IISc and outside Bangalore (including S. Ramaseshan, Director of IISc, a physicist and T. Desiraju, a neuroscientist at NIMHANS) were invited to talk on mathematics and on mathematical sciences. Meanwhile, the first Ph.D. student of PLB, J. N. Kapur (then a member of the National Board of Higher Mathematics (NBHM), DAE, Govt. of India) proposed that NBHM should organise MO for participation in IMO. The MO lecture program of IISc had caught the attention of NBHM and NBHM requested IISc to conduct training program for the IMO, which started at IISc in 1986. Many enthusiastic mathematicians at IISc and some invitees like S. A. Shirali took part in the training program. M. S. Raghunathan (Chairman of NBHM), Phoolan Prasad (Coordinator, School Committee, NBHM), Izhar Husain, A. M. Vaidya and R. Subramanian played important roles in organising MO in India and IMO training program.

The IMO training camp in India started at IISc from 1986 and continued till 1993. In 1994 the training camp shifted to the Bhabha Atomic Research Centre (BARC) campus in Trombay, Mumbai. At present, all MO activities are organized by the Homi Bhabha Centre for Science Education (HBCSE) on behalf of NBHM; see [1].

India first participated in an IMO in 1989 at Braunschweig in West Germany (unification with East Germany had not yet happened) and the Indian team earned 4 silver medals and 1 Honourable Mention (HM). Till this time, IMO training camps at IISc were organised by mathematicians at IISc and a few others from outside (including C. R. Pranesachar, B. J. Venkatachala, both former students of IISc, and C. S. Yogananda) who were deeply interested in mathematical problem-solving. S. A. Shirali continued participating in the training camps. In 1990, NBHM felt a need to appoint permanent faculty for the work of organising MO activity and teaching at the IMO training camps. On request from NBHM, IISc set up a Mathematics Olympiad Cell in 1991 at the Department of Mathematics, which continues to function at IISc. Three NBHM Teacher Fellows were appointed in the Olympiad cell during 1991 and 1993.

Under involvement from NBHM and faculty of IISc, MO activity in India and the IMO training program (with about 10 mathematicians joining as resource persons from various parts of India) grew in intensity and enthusiasm since 1989. Several medalists from India have gone on to become mathematicians and computer scientists across the world. Mention should be made here of K Soundararajan, who won a silver medal in IMO 1991 (Sweden) and who has gone on to do outstanding work in analytic number theory (see [7]); and Subhash Khot, who won silver medals in IMO 1994 (Hong Kong) and IMO 1995 (Canada) and who was awarded the Rolf Nevanlinna Prize in 2014 for outstanding work in theoretical computer science; see [8].

A notable event in the history of Olympiad activity in India was the hosting of IMO 1996 in Mumbai, under the chairmanship of A M Vaidya. It is important to point out that for the mathematicians who worked with dedication for the IMO activity in India, winning medals at the IMO was never the primary aim; rather, the aim was to draw attention to the need for quality mathematics education at the school level. This remains an urgent need today.

A look at how IMO problems have evolved over the years

As mentioned earlier, the first IMO was held in 1959. Looking back, the level of the problems posed in those early years seems impossibly low in comparison with the level of the problems posed these days. We list some problems for comparison.

IMO 1959, Problem 1. Prove that the fraction $\dfrac{21n+4}{14n+3}$ is irreducible for every natural number n.

IMO 1959, Problem 4. Construct a right triangle with given hypotenuse c such that the median drawn to the hypotenuse is the geometric mean of the two legs of the triangle.

IMO 1970, Problem 1. Let M be a point on the side AB of $\triangle ABC$. Let r_1, r_2 and r be the radii of the inscribed circles of triangles AMC, BMC and ABC. Let q_1, q_2 and q be the radii of the escribed circles of the same triangles that lie in the angle ACB. Prove that

$$\frac{r_1}{q_1}\cdot\frac{r_2}{q_2}=\frac{r}{q}.$$

IMO 1970, Problem 4. Find the set of all positive integers n with the property that the set $\{n\ :\ n,\ n+1,\ n+2,\ n+3,\ n+4,\ n+5\}$ can be partitioned into two sets such that the product of the numbers in one set equals the product of the numbers in the other set.

IMO 2010, Problem 1. Determine all functions $f\colon \mathbb{R}\to\mathbb{R}$ such that the equality

$$f(\lfloor x\rfloor\cdot y)=f(x)\cdot\lfloor f(y)\rfloor$$

holds for all $x, y\in\mathbb{R}$. (Here $\lfloor z\rfloor$ denotes the greatest integer less than or equal to z.)

IMO 2010, Problem 4. Let P be a point inside the triangle ABC. The lines AP, BP and CP intersect the circumcircle Γ of triangle ABC again at the points K, L and M respectively. The tangent to Γ at C intersects the line AB at S. Suppose that $SC = SP$. Prove that $MK = ML$.

Remark. Contrast Problems 1 and 4 in IMO 1959 with Problems 1 and 4 in IMO 2010.

A few problems proposed for the IMO by India
Listed below are a few problem proposals from India that were shortlisted for consideration by the problem committee of the host country. In cases where the problem was selected for the IMO, we have made a note of this fact. (Please note that the list is not a complete one, we have only given a sampling of the problems.)

Shortlisted for IMO 1989; proposed by Shailesh Shirali. A bicentric quadrilateral is one that is both inscribable in and circumscribable about a circle. Show that for such a quadrilateral, the centers of the two associated circles are collinear with the point of intersection of the diagonals.

IMO 1990, Problem 1; proposed by C R Pranesachar. Given a circle with two chords AB, CD that meet at E, let M be a point of chord AB other than E. Draw the circle through D, E, and M. The tangent line to the circle DEM at E meets the lines BC, AC at F, G, respectively. Given $AM/AB = \lambda$, find GE/EF.

Shortlisted for IMO 1992; proposed by Shailesh Shirali. Two circles $G1$ and $G2$ are inscribed in a segment of a circle G and touch each other externally at a point W. Let A be a point of intersection of a common internal tangent to $G1$ and $G2$ with the arc of the segment, and let B and C be the endpoints of the chord. Prove that W is the incentre of the triangle ABC.

Shortlisted for IMO 1992; proposed by C R Pranesachar. Show that in the plane there exists a convex polygon of 1992 sides satisfying the following conditions:

(i) its side lengths are $1, 2, 3, \ldots, 1992$ in some order;

(ii) the polygon is circumscribable about a circle.

IMO 1992, Problem 2; proposed by B J Venkatachala. Let $\mathbb{R}$ denote the set of all real numbers. Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that for all $x, y \in \mathbb{R}$,

$$f\left(x^2 + f(y)\right) = y + (f(x))^2 .$$

IMO 2002, Problem 5; proposed by B J Venkatachala. Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that for all $x, y, u, v \in \mathbb{R}$,

$$(f(x) + f(y))\cdot(f(u) + f(v)) = f(xu-yv)+f(xv+yu).$$

IMO 1998, Problem 2; proposed by R B Bapat. In a contest, there are m candidates and n judges, where $n \geq 3$ is an odd integer. Each candidate is evaluated by each judge as either pass or fail. Suppose that each pair of judges agrees on at most k candidates. Prove that

$$\frac{k}{m} \geq \frac{n-1}{2n}.$$

Shortlisted for IMO 2014; proposed by N V Tejaswi. There are n circles drawn on a piece of paper in such a way that any two circles intersect in two points, and no three circles pass through the same point. Turbo the snail slides along the circles in the following fashion. Initially he moves on one of the circles in clockwise direction. Turbo always keeps sliding along the current circle until he reaches an intersection with another circle. Then he continues his journey on this new circle and also changes the direction of moving, i.e., from clockwise to anticlockwise or *vice versa*.

Suppose that Turbo's path entirely covers all circles. Prove that n must be odd.

Some memorable problems from the IMOs
The mechanism by which problems are posed for the IMOs is a unique and remarkable one, and over the years some truly beautiful and memorable problems have been created through this tradition. We mention a few here.

IMO 1988, Problem 6. Let a and b be two positive integers such that $ab + 1$ divides $a^2 + b^2$. Show that

$$\frac{a^2 + b^2}{ab + 1}$$

is a perfect square.

IMO 1990, Problem 3. Find all positive integers n having the property that

$$\frac{2^n + 1}{n^2}$$

is an integer.

IMO 1991, Problem 2. Let $n > 6$ and let $a_1 < a_2 < \cdots < a_k$ be all the natural numbers that are less than n and relatively prime to n. Show that if $a_1, a_2, \ldots, a_k$ is an arithmetic progression, then n is a prime number or a natural power of 2.

IMO 1993, Problem 5. Let $\mathbb{N} = \{1, 2, 3, \ldots\}$. Determine whether there exists a strictly increasing function $f\colon \mathbb{N} \to \mathbb{N}$ with the following properties:

(a) $f(1) = 2$;

(b) $f(f(n)) = f(n) + n \quad (n \in \mathbb{N})$.

IMO 1993, Problem 6. Let n be an integer greater than 1. In a circular arrangement of n lamps $L_0, L_1, \ldots, L_{n-1}$, each one can be either ON or OFF. We start with the situation where all lamps are ON, and then carry out a sequence of steps, Step_0, Step_1, ... If L_{j-1} (j is taken mod n) is ON, then Step_j changes the status of L_j (it goes from ON to OFF or from OFF to ON) but does not change the status of any of the other lamps. If L_{j-1} is OFF, then Step_j does not change anything at all. Show that:

(a) There is a positive integer $M(n)$ such that after $M(n)$ steps, all lamps are ON again.

(b) If n has the form 2^k, then all lamps are ON after $n^2 - 1$ steps.

(c) If n has the form $2^k + 1$, then all lamps are ON after $n^2 - n + 1$ steps.

IMO 1996, Problem 5. Let $ABCDEF$ be a convex hexagon such that AB is parallel to DE, BC is parallel to EF, and CD is parallel to AF. Let R_A, R_C, R_E be the circumradii of triangles FAB, BCD, DEF respectively, and let P denote the perimeter of the hexagon. Prove that

$$R_A + R_C + R_E \geq \frac{P}{2}.$$

Remark. Problem 6 of IMO 1988 was for many years regarded as "the most difficult problem ever posed in an IMO." But it seems likely that problem 5 of IMO 1996 now has that label.

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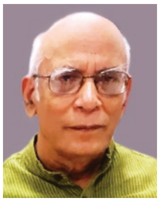

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