INEQUALITIES in Algebra and Geometry Part 3

ClassRoom

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This article is the third in the 'Inequalities' series. In this part, we explore the validity and application of the AM-GM inequality for three numbers and four numbers respectively.

AM-GM inequality for three numbers and four numbers. In the previous two parts of this series, we studied the AM-GM inequality for two numbers in some detail. This is the statement that

$$\frac{a+b}{2} \geq \sqrt{ab} \qquad (1)$$

for any two nonnegative numbers a and b. Moreover, the equality sign holds if and only if $a = b$. Using this inequality, we proved various other inequalities in algebra as well as geometry.

Now we consider the extension of the AM-GM inequality to three nonnegative numbers and four nonnegative numbers, respectively. Rather oddly, the extension to four numbers is simpler than the extension to three numbers!

Theorem 1 (AM-GM inequality for four nonnegative numbers)**.** *Let a, b, c, d be any four nonnegative numbers. Then we have the following inequality:*

$$\frac{a+b+c+d}{4} \geq (abcd)^{1/4}. \qquad (2)$$

Moreover, the equality sign holds if and only if $a = b = c = d$.

Proof. We accomplish the desired result through three applications of the inequality (1) for two numbers. This takes place as follows. We have, firstly:

$$\left.\begin{aligned} \frac{a+b}{2} &\geq \sqrt{ab}, \\ \frac{c+d}{2} &\geq \sqrt{cd}, \end{aligned}\right\} \qquad (3)$$

and since

$$\frac{a+b+c+d}{4} = \frac{\frac{a+b}{2} + \frac{c+d}{2}}{2},$$

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it follows that

$$\frac{a+b+c+d}{4} \geq \frac{\sqrt{ab}+\sqrt{cd}}{2}. \tag{4}$$

We now apply the AM-GM inequality to the expression on the right side of (4):

$$\frac{\sqrt{ab}+\sqrt{cd}}{2} \geq \left(\sqrt{ab} \times \sqrt{cd}\right)^{1/2} = (abcd)^{1/4}. \tag{5}$$

The desired inequality now follows, i.e.:

$$\frac{a+b+c+d}{4} \geq (abcd)^{1/4}. \tag{6}$$

Moreover, for equality to hold in (6), the inequality signs in the preceding would *all* have to be replaced by equality signs. Starting at the top, we see that for equality to hold in (3), we would need to have $a = b$ and $c = d$. And for equality to hold in (5), we would need to have $ab = cd$. Combining these conditions, we see that equality will hold in (6) if and only if $a = b = c = d$. This proves Theorem 1. □

Remark. When we are studying the topic of inequalities, it is good practice to identify the *exact conditions* under which an equality sign can replace the inequality sign. We have consistently tried to do this in our study of inequalities, and we shall continue this practice.

We now take up the inequality for three nonnegative numbers. The curious thing is, in proving this result, we make use of the inequality for four nonnegative numbers!

Theorem 2 (AM-GM inequality for three nonnegative numbers)**.** *Let a, b, c be any three nonnegative numbers. Then we have the following inequality:*

$$\frac{a+b+c}{3} \geq (abc)^{1/3}. \tag{7}$$

Moreover, the equality sign holds in (7) if and only if $a = b = c$.

Proof. As already noted, we are going to make use of Theorem 1. As we are given only three nonnegative numbers a, b, c, we need to produce a suitable fourth number d before we can apply the theorem. Our choice for this number will be

$$d = \frac{a+b+c}{3}, \tag{8}$$

that is, d is the arithmetic mean of a, b, c. With this choice we obtain, on the left side of (6):

$$\frac{a+b+c+d}{4} = \frac{a+b+c+\frac{1}{3}(a+b+c)}{4} = \frac{a+b+c}{3} = d.$$

(This makes sense. Given a list of numbers, if you append the arithmetic mean of the numbers to the same list, the arithmetic mean will naturally remain unaltered.) On the right side of (6), we obtain:

$$(abcd)^{1/4} = (abc)^{1/4} \times d^{1/4}.$$

Therefore, the substitution $d = \frac{1}{3}(a+b+c)$ in Theorem 1 results in the following:

$$d \geq (abc)^{1/4} \times d^{1/4}.$$

This may be simplified to yield $d^{3/4} \geq (abc)^{1/4}$. Now raising both sides to power $4/3$, we get:

$$d \geq (abc)^{1/3},$$

i.e.,

$$\frac{a+b+c}{3} \geq (abc)^{1/3}. \tag{9}$$

We have proved the desired inequality. The conditions under which an equality sign can replace the inequality sign can be deduced exactly as earlier; the conditions are $a = b = c$. □

Remark. The path we have taken to prove Theorem 2 is indeed curious; we proved Theorem 1 first (the four-numbers case), and then came down to the three-numbers case. While this is not the only way to prove Theorem 2, it is one of the simplest.

Some applications of the AM-GM inequality for three numbers

The inequality in Theorem 2 can also be stated in the following form: if a, b, c are any three nonnegative numbers, then

$$a + b + c \geq 3 \cdot (abc)^{1/3}. \tag{10}$$

This form of stating the inequality turns out to be more useful in problem-solving and in establishing various results.

We now consider a few problems and their solutions.

Problem 1: Find the positive number x which minimises the value of $x^2 + \dfrac{16}{x}$.

Solution. We use (10) with $a = x^2$, $b = \dfrac{8}{x}$, $c = \dfrac{8}{x}$. This yields:

$$x^2 + \frac{16}{x} = x^2 + \frac{8}{x} + \frac{8}{x} \geq 3 \times \left(x^2 \cdot \frac{8}{x} \cdot \frac{8}{x}\right)^{1/3} = 3 \times 64^{1/3} = 3 \times 4 = 12.$$

This proves that

$$x^2 + \frac{16}{x} \geq 12,$$

with equality if and only if

$$x^2 = \frac{8}{x}, \quad \text{i.e.,} \quad x = 2.$$

So the minimising number is $x = 2$, and the minimum value of the expression is 12. More generally, if k is a positive number, then the positive number x which minimises the value of

$$x^2 + \frac{k}{x}$$

is $x = \sqrt[3]{k/2}$. This result would usually be proved using derivatives.

Problem 2: Show that among all triangles that share the same perimeter, the one with the largest area is equilateral. (This is called the **isoperimetric problem** for triangles. Earlier, we had solved the isoperimetric problem for rectangles.)

Solution. Let a, b, c be the sides of the triangle, and let $s = \frac{1}{2}(a + b + c)$ be the semi-perimeter; then the area K is given by

$$K = \sqrt{s(s-a)(s-b)(s-c)}. \tag{11}$$

We are told that the perimeter $2s$ is a constant; this implies that $a + b + c$ is a constant. And since

$$(s-a) + (s-b) + (s-c) = 3s - (a + b + c) = s = \text{constant},$$

the sum of the numbers $s-a$, $s-b$ and $s-c$ is a constant as well. Note also that each of these three numbers is strictly positive; for, by the triangle inequality, we have:

$$b+c>a, \quad \therefore \ a+b+c>2a, \quad \therefore \ 2s>2a, \quad \therefore \ s-a>0,$$

and similarly for $s-b$ and $s-c$. So we are in a position to apply the AM-GM inequality to these three numbers. We obtain:

$$(s-a)+(s-b)+(s-c) \geq 3 \times ((s-a)(s-b)(s-c))^{1/3}.$$

This yields:

$$(s-a)(s-b)(s-c) \leq \frac{s^3}{27}.$$

From this we deduce that

$$s(s-a)(s-b)(s-c) \leq \frac{s^4}{27},$$

and hence, by taking square roots, that

$$K \leq \frac{s^2}{3\sqrt{3}}. \qquad (12)$$

For the upper bound to be attained, we must have $s-a=s-b=s-c$, i.e., $a=b=c$. Hence the largest area is attained by the equilateral triangle. This justifies the statement that was to be proved.

Problem 3: Given: a square sheet of paper, measuring 30 cm by 30 cm. From each corner we snip off a square shape (all squares of the same size), and then fold the remaining piece of paper into an open box with a square base (open at the top; we tape the sides at the corners). What should be the dimensions of the parts snipped off for the volume of the resulting box to be largest? (See Figure 1. In the literature, this is referred to as the **Box Problem**.)

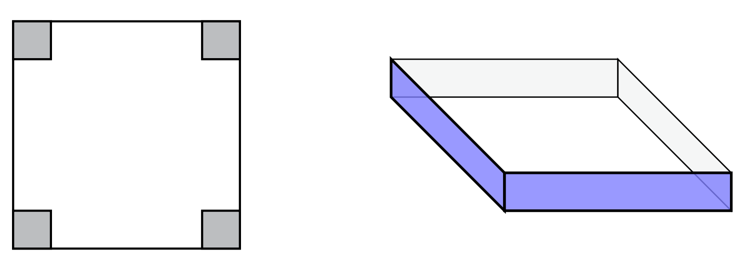


Figure 1.

Solution. The dimensions of the square are 30×30. Let the squares snipped away from the corners have dimensions $a \times a$. Then the base of the box has dimensions $(30-2a) \times (30-2a)$, and its height is a. Therefore, its volume V is given by

$$V = a(30-2a)^2. \qquad (13)$$

To maximise V we typically use derivatives; but we shall use Theorem 2 in the form given in (10). First, however, we need to make a few adjustments in the expression for V. We have:

$$\begin{aligned} a(30-2a)^2 &= 4a \times (15-a) \times (15-a) \\ &= 2 \times (2a \times (15-a) \times (15-a)). \end{aligned}$$

In order to maximise the expression $a(30-2a)^2$, we might as well maximise the expression $2a \times (15-a) \times (15-a)$, as the second expression is equal to the first expression divided by a constant (namely, 2). The second expression is in a form that allows us to apply Theorem 2. For, the quantities $2a$, $15-a$, $15-a$ are nonnegative, and their sum is 30. Therefore the following is true:

$$(2a \times (15-a) \times (15-a))^{1/3} \leq \frac{2a+(15-a)+(15-a)}{3} = \frac{30}{3} = 10,$$

and so:

$$2a \times (15-a) \times (15-a) \leq 10^3.$$

Hence the volume of the box cannot exceed 2000 cm^3. For this upper bound to be attained, we must have $2a = 15-a$, i.e., $a = 5$. So the squares snipped off must have dimensions 5×5.

Solutions to problems from July 2017 issue

(1) Let a, b, c be positive real numbers. Show that:

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a} \geq 3.$$

Solution. We use (10). We have:

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a} \geq 3 \times \left(\frac{a}{b} \cdot \frac{b}{c} \cdot \frac{c}{a}\right)^{1/3} = 3 \times 1 = 3.$$

Equality holds if and only if $\frac{a}{b} = \frac{b}{c} = \frac{c}{a}$, i.e., if and only if $a = b = c$.

(2) Let a, b, c be positive real numbers. Show that:

$$\frac{a^2}{bc}+\frac{b^2}{ca}+\frac{c^2}{ab} \geq 3,$$

with equality if and only if $a = b = c$.

Solution. We again use (10). We have:

$$\frac{a^2}{bc}+\frac{b^2}{ca}+\frac{c^2}{ab} \geq 3 \times \left(\frac{a^2}{bc} \cdot \frac{b^2}{ca} \cdot \frac{c^2}{ab}\right)^{1/3} = 3 \times 1 = 3.$$

Equality holds if and only if $\frac{a^2}{bc} = \frac{b^2}{ca} = \frac{c^2}{ab}$, i.e., if and only if $a = b = c$.

(3) Let a, b, c be positive real numbers. Show that:

$$\left(a^2 b + b^2 c + c^2 a\right) \cdot \left(ab^2 + bc^2 + ca^2\right) \geq 9a^2 b^2 c^2,$$

with equality if and only if $a = b = c$.

Solution. Yet again we use (10). We have:

$$a^2 b + b^2 c + c^2 a \geq 3 \times \left(a^2 b \cdot b^2 c \cdot c^2 a\right)^{1/3} = 3\left(a^3 b^3 c^3\right)^{1/3} = 3abc.$$

Similarly,

$$ab^2 + bc^2 + ca^2 \geq 3 \times \left(ab^2 \cdot bc^2 \cdot ca^2\right)^{1/3} = 3\left(a^3 b^3 c^3\right)^{1/3} = 3abc.$$

The desired result now follows by multiplication. Equality holds if and only if $a^2 b = b^2 c = c^2 a$ and $ab^2 = bc^2 = ca^2$, i.e., if and only if $a = b = c$.

(4) Let a, b, c be positive real numbers. Show that:

$$a^3 + b^3 + c^3 \geq a^2 b + b^2 c + c^2 a,$$

with equality if and only if $a = b = c$.

Solution. This is a very challenging problem! We are going to use a clever trick to solve it. We apply the AM-GM inequality to the following three lists of numbers: $\{a^3, a^3, b^3\}$; $\{b^3, b^3, c^3\}$; $\{c^3, c^3, a^3\}$. Note the symmetry in these lists. Invoking (10) for each list, we get:

$$a^3 + a^3 + b^3 \geq 3 \times \left(a^3 \cdot a^3 \cdot b^3\right)^{1/3} = 3a^2 b,$$

$$b^3 + b^3 + c^3 \geq 3 \times \left(b^3 \cdot b^3 \cdot c^3\right)^{1/3} = 3b^2 c,$$

$$c^3 + c^3 + a^3 \geq 3 \times \left(c^3 \cdot c^3 \cdot a^3\right)^{1/3} = 3c^2 a.$$

By adding the three inequalities, we immediately get the desired result. The condition for equality is easily deduced.

A small modification of the lists yields the following inequality: If a, b, c are three positive real numbers, then

$$a^3 + b^3 + c^3 \geq ab^2 + bc^2 + ca^2,$$

with equality if and only if $a = b = c$.

The problem and solution have been taken from [1].

Using exactly the same idea, we can prove results such as the following:

Theorem. *If a, b, c are three positive real numbers, then*

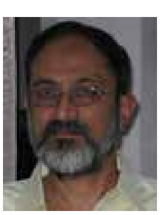
$$a^4 + b^4 + c^4 \geq a^3 b + b^3 c + c^3 a,$$

$$a^5 + b^5 + c^5 \geq a^3 b^2 + b^3 c^2 + c^3 a^2,$$

and so on, with equality in each case if and only if $a = b = c$.

References

1. Brilliant, "Applying the Arithmetic Mean Geometric Mean Inequality", https://brilliant.org/wiki/applying-the-arithmetic-mean-geometric-mean/. (Home page: https://brilliant.org/.)



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