Adventures in PROBLEM SOLVING

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In this edition of 'Adventures' we study a few miscellaneous problems, mostly from the Pre-Regional Mathematics Olympiad (PRMO; this year's PRMO was conducted on August 19 in centres all over the country). As usual, we pose the problems first and present solutions later.

Miscellaneous problems

Problem 1. Consider all 6-digit numbers of the form $abccba$ where b is odd. Determine the number of all such 6-digit numbers that are divisible by 7. (Problem 3 of PRMO 2018)

Problem 2. In a triangle ABC, the median from B to CA is perpendicular to the median from C to AB. If the median from A to BC is 30, determine $(BC^2 + CA^2 + AB^2)/100$. (Problem 10 of PRMO 2018)

Problem 3. If $a, b, c \geq 4$ are integers, not all equal, and $4abc = (a+3)(b+3)(c+3)$, then what is the value of $a+b+c$? (Problem 18 of PRMO 2018)

Problem 4. A positive integer k is said to be 'good' if there exists a partition of the set $\{1, 2, 3, \ldots, 20\}$ into disjoint proper subsets such that the sum of the numbers in each subset of the partition is k. How many good numbers are there? (Problem 22 of PRMO 2018)

Problem 5. Find all prime numbers p such that $\frac{1}{p}\left(2^{p-1} - 1\right)$ is a perfect square. (Problem posed on the Math Stack Exchange website)

Keywords: Place value, digits, divisibility, triangles, medians, subsets, partitions

Solutions to the problems

Solution to problem 1

Let n be a 6-digit number of the form *abccba*; then

$$n = 10^5 a + 10^4 b + 10^3 c + 10^2 c + 10b + a = 100001a + 10010b + 1100c.$$

Take remainders modulo 7. We get $n \equiv 6a + c \pmod 7 \equiv c - a \pmod 7$. So, for n to be a multiple of 7, we must have $a \equiv c \pmod 7$. Note that the value of b does not affect divisibility of n by 7. Moreover, we must have $a > 0$, as it is the leading digit of the number. We now list the possibilities.

- $a = 1$. Since $a \equiv c \pmod 7$, the possible values of c are $1, 8$ (2 choices). There are 5 choices for b (namely, $1, 3, 5, 7, 9$). This yields 10 possibilities.
- $a = 2$. The possible values of c are $2, 9$ (2 choices). With 5 choices for b, this yields 10 possibilities.
- $a = 3$. The only possible value of c is 3. This yields 5 possibilities.
- $a = 4$. The only possible value of c is 4. This yields 5 possibilities.
- $a = 5$. The only possible value of c is 5. This yields 5 possibilities.
- $a = 6$. The only possible value of c is 6. This yields 5 possibilities.
- $a = 7$. The possible values of c are $0, 7$ (2 choices). This yields 10 possibilities.
- $a = 8$. The possible values of c are $1, 8$ (2 choices). This yields 10 possibilities.
- $a = 9$. The possible values of c are $2, 9$ (2 choices). This yields 10 possibilities.

Hence the total number of possibilities is $5 \times 10 + 4 \times 5 = 70$.

Solution to problem 2

Here we make use of (i) the theorem of Pythagoras; (ii) the fact that the circumcentre of the right-angled triangle lies at the midpoint of its hypotenuse; (iii) the fact that the point of intersection of two medians of a triangle is a point of trisection of each median.

Let $BG = 2x$, $CG = 2y$; then $GE = x$, $GF = y$. Also

$$BC^2 = BG^2 + CG^2 = 4\left(x^2 + y^2\right).$$

We are told that $AD = 30$. Hence $GD = 10$. Since triangle BGC is right-angled at G, its circumcentre lies at the midpoint of its hypotenuse, i.e., at D. It follows that $DB = 10$, and therefore that $BC = 20$. Combining this fact with what we deduced above, we see that

$$x^2 + y^2 = 100.$$

From the right-angled triangles BGF and CGE, we obtain

$$BF^2 = BG^2 + FG^2 = 4x^2 + y^2,$$

$$CE^2 = CG^2 + EG^2 = x^2 + 4y^2.$$

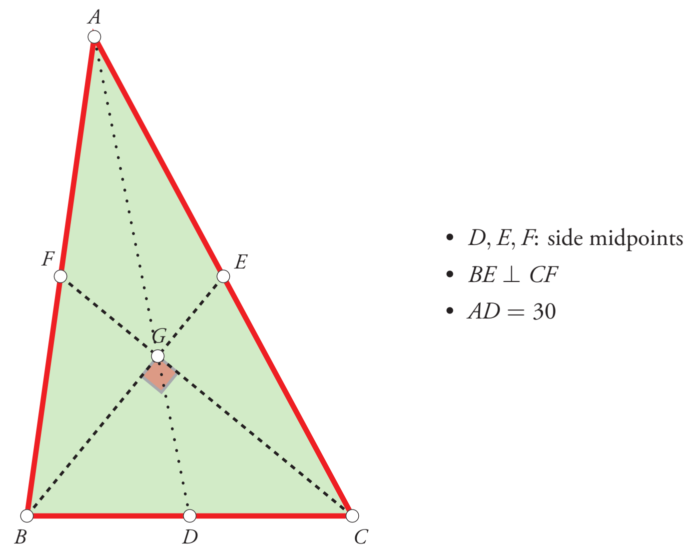


Figure 1.

Hence

$$AB^2 + AC^2 = 4\left(5x^2 + 5y^2\right) = 20\left(x^2 + y^2\right) = 2000,$$

and so

$$AB^2 + AC^2 + BC^2 = 2400.$$

Therefore the required answer is $2400/100 = 24$.

Solution to problem 3

Given that $a, b, c \geq 4$ *are integers and*

$$4abc = (a+3)(b+3)(c+3),$$

to find the value of $a + b + c$. The problem additionally requires that a, b, c are not all equal. However, this seems an unnecessary requirement; for if $a = b = c$, then we would obtain $4a^3 = (a+3)^3$, implying that the cube root of 4 is a rational number, which is not the case. Hence a, b, c cannot all be equal. Without any loss of generality, we may assume that $a \leq b \leq c$. From the given equation we obtain:

$$4 = \left(1 + \frac{3}{a}\right)\left(1 + \frac{3}{b}\right)\left(1 + \frac{3}{c}\right),$$

therefore

$$\left(1 + \frac{3}{c}\right)^3 \leq 4 \leq \left(1 + \frac{3}{a}\right)^3.$$

Solving these inequalities individually for a and c, we obtain

$$a \leq 5.1 \leq c,$$

so $a \leq 5$ and $c \geq 6$. Therefore $a \in \{1, 2, 3, 4, 5\}$. As we have also been told that $a \geq 4$, it follows that $a \in \{4, 5\}$. We consider both these possibilities.

- If $a = 4$, the given equation leads to $16bc = 7(b+3)(c+3)$. This may be rewritten as $9bc - 21(b+c) = 63$, which yields $(3b-7)(3c-7) = 112$. It follows that the pair $(3b-7, 3c-7)$ is one of the following possibilities:

$$(1, 112), \quad (2, 56), \quad (4, 28), \quad (7, 16), \quad (8, 14).$$

For b, c to assume integer values, both factors must be of the form 2 (mod 3), hence the pair $(3b-7, 3c-7)$ must be either $(2, 56)$ or $(8, 14)$. Hence the triple (a, b, c) is one of the following:

$$(4, 3, 21), \quad (4, 5, 7).$$

Of these, the first possibility need not be listed as we had supposed that $a \leq b$.

- If $a = 5$, the given equation leads to $bc - 2b - 2c - 6 = 0$. This may be rewritten as $(b-2)(c-2) = 10$. It follows that the pair $(b-2, c-2)$ is one of the following possibilities:

$$(1, 10), \quad (2, 5),$$

and therefore that the triple (a, b, c) is one of the following:

$$(5, 3, 12), \quad (5, 4, 7).$$

Neither of these possibilities needs to be listed as we had supposed that $a \leq b \leq 4$.

Hence the only triple (a, b, c) which satisfies the relations

$$4 \leq a \leq b \leq c, \quad 4abc = (a+3)(b+3)(c+3)$$

is $(4, 5, 7)$. This yields $a + b + c = 16$.

Solution to problem 4
A positive integer k is said to be 'good' if there exists a partition of the set $\{1, 2, 3, \ldots, 20\}$ into disjoint proper subsets such that the sum of the numbers in each subset of the partition is k; to find all the good numbers. Clearly, any such k must be a proper divisor of 210, and since 20 itself must belong to some subset, we must also have $k \geq 20$. These two requirements yield six possible values of k:

$$21, \ 30, \ 35, \ 42, \ 70, \ 105.$$

We examine each of these six possibilities for feasibility.

- $k = 21$ is feasible as we can form the following 10 two-element subsets: $\{1, 20\}$, $\{2, 19\}$, $\{3, 18\}$, $\{4, 17\}$, …, $\{8, 13\}$, $\{9, 12\}$ and $\{10, 11\}$, each with sum 21.
- $k = 30$ is feasible as we can form the following 7 subsets (note that they are not all of the same size): $\{10, 20\}$, $\{11, 19\}$, $\{12, 18\}$, $\{13, 17\}$, $\{14, 16\}$, $\{4, 5, 6, 15\}$ and $\{1, 2, 3, 7, 8, 9\}$, each with sum 30.
- $k = 35$ is feasible as we can form the following 6 subsets: $\{15, 20\}$, $\{16, 19\}$, $\{17, 18\}$, $\{1, 2, 3, 4, 5, 6, 14\}$, $\{7, 8, 9, 11\}$ and $\{10, 12, 13\}$, each with sum 35.
- $k = 42$ is feasible as we can form the following 5 subsets: $\{20, 19, 3\}$, $\{18, 17, 7\}$, $\{16, 15, 11\}$, $\{14, 13, 12, 2, 1\}$ and $\{10, 9, 8, 6, 5, 4\}$, each with sum 42.
- $k = 70$ is feasible as we can form the following: $\{20, 19, 18, 13\}$, $\{17, 16, 15, 14, 8\}$ and $\{12, 11, 10, 9, 7, 6, 5, 4, 3, 2, 1\}$, i.e., 3 subsets, each with sum 70.
- $k = 105$ is feasible: $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14\}$ and the complement set $\{15, 16, 17, 18, 19, 20\}$ each have sum 105.

We see that all six possibilities yield partitions of the required type. Note that we had to resort to ad hoc methods to find these partitions. In general, the problem of finding such partitions has no easy solution, requiring a great amount of computational work.

Solution to problem 5
To find all prime numbers p such that $\frac{1}{p}\left(2^{p-1}-1\right)$ is a perfect square. See [1].

The 'little theorem' of Fermat assures us that if p is any odd prime number, then the quantity

$$\frac{1}{p}\left(2^{p-1}-1\right)$$

is an integer. Here are the values taken by this expression for the first few odd primes p:

p	3	5	7	11	13	17	...
$(2^{p-1}-1)/p$	1	3	9	93	315	3855	...

The only square numbers we spot in the second row are 1 and 9. We shall show that these are in fact the only square numbers possible, i.e., the only primes p for which $\frac{1}{p}\left(2^{p-1}-1\right)$ is a perfect square are $p=3$ and $p=7$.

The stated condition implies that p is odd. Let $p=2k+1$, where k is a positive integer. Then $2^{p-1}-1=2^{2k}-1=\left(2^k-1\right)\left(2^k+1\right)$, so

$$\frac{2^{p-1}-1}{p}=\frac{\left(2^k-1\right)\left(2^k+1\right)}{p}.$$

Now p can be a divisor of only one of 2^k-1, 2^k+1, as these two quantities cannot share any common factor greater than 1. We consider both the possibilities.

- Suppose that p is a divisor of 2^k-1. Then the quantities $\left(2^k-1\right)/p$ and 2^k+1 are coprime (indeed, the quantities 2^k-1 and 2^k+1 themselves are coprime, being consecutive odd numbers), and as their product is a perfect square, each of them must be a perfect square. That is, we must have for some integers a, b,

  $$\frac{2^k-1}{p}=a^2, \quad 2^k+1=b^2.$$

  The second equality yields $b^2-1=2^k$, hence $(b-1)(b+1)=2^k$. This implies that $b-1$ and $b+1$ are both powers of 2. Moreover, we also have $(b+1)-(b-1)=2$. But the only two powers of 2 that differ by 2 are $2^2=4$ and $2^1=2$. Hence it must be that $b+1=4$, i.e., $b=3$, which yields $k=3$. Hence $2^k-1=7$, which tells us that $p=7$. So this possibility yields just one prime number, namely $p=7$.

- Suppose that p is a divisor of 2^k+1. Then the quantities $\left(2^k+1\right)/p$ and 2^k-1 are coprime, and as their product is a perfect square, each of them must be a perfect square. That is, we must have for some integers a, b,

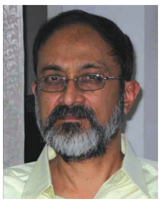
  $$\frac{2^k+1}{p}=a^2, \quad 2^k-1=b^2.$$

  The second equality yields $2^k=b^2+1$. As the quantity on the left side is even, b must be odd, hence $b^2\equiv 1 \pmod 4$, therefore $b^2+1\equiv 2 \pmod 4$. This implies that $2^k\equiv 2 \pmod 4$. The only positive integer k for which this is true is $k=1$. (If $k\geq 2$, then $2^k\equiv 0 \pmod 4$.) Hence $k=1$, so p is a divisor of $2^1+1=3$. Hence $p=3$. So this possibility too yields just one prime number, namely $p=3$.

So there are just two prime numbers for which the stated condition is true: 3 and 7.

References

1. Find all prime numbers p such that $\frac{1}{p}\left(2^{p-1}-1\right)$ is a perfect square, https://math.stackexchange.com/questions/161599/find-all-primes-p-such-that-dfrac2p-1-1p-is-a-perfect-square



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