Problem Corner

Problems for the MIDDLE SCHOOL

Problem Editor: SNEHA TITUS

Our focus this time is a step forward from parity (our theme for the last issue). In this issue, our problems are all based on multiples and factors; I'm sure you'll enjoy verifying and proving these number facts, and playing these games which improve your understanding of factorization and the divisibility rules.

The following basic rule will help you:

If the prime factorization of a number N is given by

$N = p_1^{b_1} \times \ldots \times p_k^{b_k}$, then the number has $(b_1+1)(b_2+1)(b_3+1)\ldots\ldots\ldots\ldots\ldots\ldots\ldots(b_k+1)$ factors.

For example, $75 = 3 \times 5^2$ has $(1+1)(2+1) = 6$ factors, viz., $1, 3, 5, 15, 25, 75$.

This is pretty easy to reason out: the factors can have no 3s or one 3, no 5s or one 5 or two 5s. So there are 2 ways in which 3 can be a factor and 3 ways in which 5 can be a factor and so $3 \times 2 = 6$ possible factors.

Problem VI-2-M.1 **http://nrich.maths.org/4989**

A certain number has exactly eight factors, including 1 *and itself. Two of its factors are* 21 *and* 35*. What is the number?*

Keywords: Multiples, factors, divisibility, factorization, primes, place value, number of factors

Problem VI-2-M.2 From http://nrich.maths.org/480

How can I find a number with exactly 14 *factors? How can I find the smallest such number?*

How can I find a number with exactly 18 *factors? How can I find the smallest such number?*

Which numbers have an odd number of factors?

Extension: *What is the smallest number with exactly* 100 *factors?*

Which number less than 1000 *has the most factors?*

Problem VI-2-M.3 From http://nrich.maths.org/524

Choose any 3 *digits and make a* 6 *digit number by repeating the* 3 *digits in the same order (e.g.* 594594*).*

Whatever digits you choose, the number will always be divisible by 7 *and by* 11 *and by* 13*, without a remainder.*

Can you explain why?

Problem VI-2-M.4

Faded documents from Adventures in Problem Solving by Shailesh Shirali

In this type of problem, some digits have 'faded away' and restoration is based on the digits that remain legible. It is usual to place an 'x' in place of the unreadable digits.

The Lonely '8'. A division problem with a solitary '8' . . .. see Figure 1. Can you find the unreadable digits?

$$\begin{array}{ccccccccc}
 & & & & x & x & 8 & x & x \\ \cline{3-9}
x & \multicolumn{1}{c|}{x} & x & x & x & x & x & x & x \\ \cline{1-2}
 & & x & x & x & & & & \\ \cline{3-5}
 & & & & & x & x & & \\
 & & & & & x & x & & \\ \cline{6-7}
 & & & & & & x & x & x \\
 & & & & & & x & x & x \\ \cline{7-9}
 & & & & & & & & 0 \\ \cline{7-9}
\end{array}$$

Figure 1

Solutions

Problem VI-2-M.1 http://nrich.maths.org/4989

A certain number has exactly eight factors including 1 *and itself. Two of its factors are* 21 *and* 35*. What is the number?*

We use the given property, i.e., if the prime factorization of a number N is given by

$N = p_1^{b_1} \times \ldots \times p_k^{b_k}$, then the number has $(b_1 + 1)(b_2 + 1)(b_3 + 1) \ldots\ldots\ldots\ldots\ldots\ldots\ldots (b_k + 1)$ factors.

If the number has 8 factors then
$(b_1 + 1)(b_2 + 1)(b_3 + 1) \ldots\ldots\ldots\ldots\ldots\ldots\ldots (b_k + 1) = 8 = 1 \times 8 = 2 \times 2 \times 2 = 4 \times 2.$

We know that this unknown number (call it N) has at least 3 distinct primes factors because 21 and 35 are both its factors and $21 = 3 \times 7$ and $35 = 5 \times 7$; so 3, 5 and 7 are definitely factors of the number. So the number of factors is given by $2 \times 2 \times 2$. (This also shows that the number has exactly three distinct prime factors.)

$(b_1+1)(b_2+1)(b_3+1) = 2 \times 2 \times 2$ implies that $b_1 = b_2 = b_3 = 1$

So $N = 3^1 \times 5^1 \times 7^1 = 105$

Teacher's Note:

This problem is easy enough to encourage the novice problem solver but at the same time it includes some delicate problem solving skills. Note how the options $8 = 1 \times 8 = 4 \times 2$ are rejected and how the terms 'at least' and 'exactly' are arrived at logically. A question which may need to be discussed is why $8 = 1 \times 2 \times 2 \times 2$ is not considered. It would also be interesting to ask students if this amount of data, i.e., number of factors and any two factors is sufficient to find N in all cases!

Problem VI-2-M.2 From http://nrich.maths.org/480

(a) *How can I find a number with exactly* 14 *factors? How can I find the smallest such number?*

(b) *How can I find a number with exactly* 18 *factors? How can I find the smallest such number?*

(c) *Which numbers have an odd number of factors?*

(d) ***Extension:*** *What is the smallest number with exactly* 100 *factors?*

(e) *Which number less than* 1000 *has the most factors?*

(a) If the number has 14 factors then
$(b_1+1)(b_2+1)(b_3+1) \ldots\ldots\ldots\ldots\ldots\ldots (b_k+1) = 14 = 1 \times 14 = 2 \times 7$
So $b_1 = 13$ <u>or</u> $b_1 = 1$ and $b_2 = 6$
But this opens up an infinite number of possibilities, because the number could be $2^{13} = 8192$ or $3^1 \times 5^6 = 46875$, $3^6 \times 5^1 = 3645$; by varying the primes and their powers we could get many, many numbers which have 14 factors. The smallest such number is $2^6 \times 3^1 = 192$, whose 14 factors are $1, 2, 4, 8, 16, 32, 64, 3, 6, 12, 24, 48, 96$ and 192.

(b) If the number has 18 factors then $(b_1+1)(b_2+1)(b_3+1) \ldots\ldots\ldots\ldots\ldots\ldots (b_k+1) = 18$
$18 = 1 \times 18 = 3 \times 6 = 2 \times 9 = 2 \times 3 \times 3$
So $b_1 = 17$ <u>or</u> $b_1 = 2$ and $b_2 = 5$ <u>or</u> $b_1 = 1$ and $b_2 = 8$ <u>or</u> $b_1 = 1$ and $b_2 = 2$ and $b_3 = 2$
But this opens up an infinite number of possibilities, because the number could be $3^{17} = 129140163$ or $3^8 \times 5^1 = 32805$, or $2^2 \times 5^5 = 12500$, and so on. The smallest such number is $2^2 \times 3^2 \times 5 = 180$.

(c) If a number has an odd number of factors, then
$(b_1+1), (b_2+1), (b_3+1) \ldots\ldots\ldots\ldots\ldots\ldots (b_k+1)$ are all odd which means that $b_1, b_2, b_3 \ldots\ldots\ldots\ldots\ldots\ldots b_k$ are all even and this means that $N = p_1^{b_1} \times \ldots \times p_k^{b_k}$ is a perfect square. This is an enormously useful fact which is used in many puzzles; you could find one such on http://teachersofindia.org/en/article/atria-dumble-door-rescue

(d) $100 = 2 \times 2 \times 5 \times 5$
so $2^4 \times 3^4 \times 5 \times 7$ is the smallest number with 100 divisors.

(e) The last question is quite challenging but armed with all the discoveries of the previous sub- parts, we could reason that since $2 \times 3 \times 5 \times 7 = 210$ (which is less than 1000) and 210×11 is greater than 1000, the number less than 1000 with the most factors should have only 2, 3, 5 and 7 as the prime factors.

Now we need to increase the powers of these primes to the maximum possible without letting the product exceed 1000. By trial and error, we arrive at the conclusion that $3 \times 5 \times 7 \times 8$ is less than 1000 and has 64 factors. This seems to be the number with the highest number of factors.

Teacher's Note:
A step up from the previous problem and slightly more difficult. Logical reasoning is practised and each student can get a different, but correct, answer. It is also a good idea for them to list the factors of the numbers that they get as it helps them to understand the basic property that they are using. This also gives them some practice in using their understanding of exponents. Also, the difference between the different sub-parts of the problem clarifies their understanding of viable options in each case. The additional criterion of the smallest number helps them to practise comparison of numbers. And finally, trial and error is a good mathematical strategy which students must learn to use to validate their thinking.

Problem VI-2-M.3 From http://nrich.maths.org/524
Choose any 3 digits and make a 6 digit number by repeating the 3 digits in the same order (e.g. 523523).

Whatever digits you choose, the number will always be divisible by 7 and by 11 and by 13, without a remainder.

Can you explain why?

A lovely problem which simply uses place value to arrive at the conclusion that if a number is created in this way, then one of its factors is 1001, which is $7 \times 11 \times 13$ and which is why such a number will definitely have these three prime factors.

$$\begin{aligned}
\text{For example } 523523 &= 5 \times 10^5 + 2 \times 10^4 + 3 \times 10^3 + 5 \times 10^2 + 2 \times 10 + 3 \\
&= 5 \times 10^2(10^3 + 1) + 2 \times 10(10^3 + 1) + 3(10^3 + 1) \\
&= 523(10^3 + 1)
\end{aligned}$$

Teacher's Note:
Students can create variations of such problems to come up with interesting patterns; this is also a way for them to practise their divisibility rules while verifying the divisibility of the numbers that they have created.

Problem VI-2-M.4 Lonely 8
Figure 1 can be rewritten as

$$\begin{array}{cc|ccccccc}
 & & & & c & d & 8 & e & f \\ \hline
a & b & p & q & r & s & t & u & v \\ \hline
 & & x & x & x & & & & \\ \cline{3-5}
 & & & & & x & x & & \\
 & & & & & x & x & & \\ \cline{6-7}
 & & & & & & x & x & x \\
 & & & & & & x & x & x \\ \cline{7-9}
 & & & & & & & & 0 \\ \cline{7-9}
\end{array}$$

It must be understood that the number ab is a two digit number with a in the tens place and b in the units place, i.e., it is not the product of a and b. This notation will hold through the problem. Remember that x represents anything that is unknown.

Using our knowledge of long division, we can do more

$$\begin{array}{cc|ccccccc}
 & & & & c & 0 & 8 & 0 & f \\ \cline{3-9}
a & b & p & q & r & s & t & u & v \\ \cline{1-2}
 & & p & q & r & & & & \\ \cline{3-5}
 & & & & & s & t & & \\
 & & & & & x & x & & \\ \cline{6-7}
 & & & & & & x & u & v \\
 & & & & & & x & u & v \\ \cline{7-9}
 & & & & & & & & 0 \\ \cline{7-9}
\end{array}$$

As you can see, the number of unknowns has gone down significantly. We also know that the number *st* has 2 digits and that 8 times *ab* is again a 2 digit number which differs from the 2 digit number *st* by a single digit number.

Since the only two digit numbers which when multiplied by 8 give a 2 digit number are 10, 11 and 12, we get three possible options for *ab*. But *ab* when multiplied by c (a single digit number) gives a three digit number *pqr*. So *ab* can only be 12 and then $c = 9$ and $pqr = 108$

$$\begin{array}{cc|ccccccc}
 & & & & 9 & 0 & 8 & 0 & f \\ \cline{3-9}
1 & 2 & 1 & 0 & 8 & s & t & u & v \\ \cline{1-2}
 & & 1 & 0 & 8 & & & & \\ \cline{3-5}
 & & & & & s & t & & \\
 & & & & & 9 & 6 & & \\ \cline{6-7}
 & & & & & & g & u & v \\
 & & & & & & g & u & v \\ \cline{7-9}
 & & & & & & & & 0 \\ \cline{7-9}
\end{array}$$

We have reduced the number of unknowns even further!

Also, *st* must be 97, 98 or 99.

st	*g*	*gu* (should be $<$ 12)	*guv* (should be a multiple of 12)	*pqrstuv*
97	1	10	108	1089708
	1	11		
98	2	20	204	1089804
	2	21	216	1089816
	2	22	228	1089828
99	3	30	300	1089900
99	3	31	312	1089912
	3	32	324	1089924
	3	33	336	1089936
	3	34	348	1089948
	3	35		

Out of all the options for *pqrstuv*, only the first option gives a quotient which has zeroes in the tens and thousands place of the quotient. So our final answer is

$$
\begin{array}{cc|ccccccc}
 & \multicolumn{1}{c}{} & & & 9 & 0 & 8 & 0 & 9 \\ \cline{3-9}
1 & 2 & 1 & 0 & 8 & 9 & 7 & 0 & 8 \\ \cline{1-2}
 & \multicolumn{1}{c}{} & 1 & 0 & 8 & & & & \\ \cline{3-5}
 & \multicolumn{1}{c}{} & & & & 9 & 7 & & \\
 & \multicolumn{1}{c}{} & & & & 9 & 6 & & \\ \cline{6-7}
 & \multicolumn{1}{c}{} & & & & & 1 & 0 & 8 \\
 & \multicolumn{1}{c}{} & & & & & 1 & 0 & 8 \\ \cline{7-9}
 & \multicolumn{1}{c}{} & & & & & & & 0 \\ \cline{7-9}
\end{array}
$$

Clearly, a case of systematic reasoning and a clear understanding of the process of long division winning the day!

Teacher's Note: Getting the students to create their own faded document problems will be a tremendous learning experience for them.

Pedagogy: Problem solving strategies that have been imparted with this problem set are:

1. Systematic reasoning.
2. Listing of possibilities and elimination on a case by case basis.
3. Trial and error.
4. Learning to understand and apply given formulae.
5. Learning to use symbols appropriately.
6. Problem posing.