ClassRoom

Course Correction
WHAT TO DO WHEN ONE GETS TWO DIFFERENT ANSWERS TO A COUNTING PROBLEM

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This short article narrates a real-life classroom episode: a situation where two different answers were obtained to a counting problem and the class was nonplussed for a while. Ultimately, good sense prevailed and we were able to discover the error.

The problem studied was this:

> *Eight tennis players wish to split up into four pairs to play four singles games. In how many ways can they do this?*

Note that the pairs do not have any identifying names; it does not matter which pair plays on which court. What is of interest is only who gets paired with whom, and in how many different ways this pairing can be done.

We present three different approaches, just as it happened in the classroom.

First approach. Recall, firstly, that the number of ways that a group of $2n$ individuals can be partitioned into two subgroups of n individuals each is equal to

$$\frac{1}{2} \times \binom{2n}{n}.$$

Keywords: Counting, selection, combination, binomial coefficient

Division by 2 is needed to avoid over-counting. (The division is needed precisely because the subgroups are not identified by a name. What is of interest is only the manner in which the full group is split up. If we assign names to the subgroups to distinguish them from each other, then this step is not needed.)

Partition the group of 8 players into two subgroups of 4 players each. The number of ways in which this can be done is

$$\frac{1}{2} \times \binom{8}{4} = 35.$$

Now partition each group of 4 players into two subgroups of 2 players each. Each such partition can be achieved in $\frac{1}{2} \times \binom{4}{2} = 3$ ways. That is, each group of 4 players can be subdivided into two pairs in 3 different ways. As the two partitions are done independently, the number of possibilities is 3 × 3. Hence the desired number of ways (i.e., of subdividing a set of eight objects into four subsets with two objects each) is

$$35 \times 3 \times 3 = 315.$$

Second approach. Choose any 2 players from the set of 8 players; this can be done in $\binom{8}{2}$ ways. Remove this pair from consideration. Choose any 2 players from the remaining set of 6 players; this can be done in $\binom{6}{2}$ ways. Remove this pair too from consideration. Choose any 2 players from the remaining set of 4 players; this can be done in $\binom{4}{2}$ ways. As earlier, remove this pair from consideration. The remaining two players now automatically partner each other. These selections can be done in $\binom{8}{2} \times \binom{6}{2} \times \binom{4}{2}$ ways. However, there is a great deal of over-counting which has happened, so the number obtained needs to be divided by a suitable divisor. There are 4 pairs which have emerged at the end, but the same pairs could have been selected in a different order. As there are 4 pairs, this could have happened in 4! different ways. So the required divisor is 4!. Hence the number of ways in which the pairings can be made is equal to

$$\frac{1}{4!} \times \binom{8}{2} \times \binom{6}{2} \times \binom{4}{2}$$

$$= \frac{1}{24} \times \frac{8 \times 7}{1 \times 2} \times \frac{6 \times 5}{1 \times 2} \times \frac{4 \times 3}{1 \times 2} = 105.$$

Third approach. Arrange the eight players in a line. Start with the leftmost player. We must assign a partner to him; this can be done in 7 ways, as there are 7 players to choose from. Make a selection and then let the two players who have been 'processed' leave the line. Once again start with the leftmost player. We must assign a partner to him; this can be done in 5 ways. Make a selection and then (as earlier) let the two players who have been processed leave the line. There are now 4 players left.

Once again start with the leftmost player and assign him a partner; this can be done in 3 ways. Make a selection. The remaining two players automatically partner each other. It follows from this description that the number of ways the selections can be done is 7 × 5 × 3 = 105.

Remarks. As can be seen, the second and third approaches yield the same answer (105), whereas the first approach yields a different answer (315, which interestingly is 3 × 105). Which answer is correct?

If you think about it carefully, you will realise that the correct answer is 105 and not 315; we have inadvertently over-counted in the first approach. Here is why this happens.

Call the eight players A, B, C, D, E, F, G, H. (Not very imaginative names, I admit.) Suppose that the first partitioning into two groups of four each results in $\{A, B, C, D\}$ in one subgroup and $\{E, F, G, H\}$ in the other subgroup. Suppose next that when these two subgroups are subdivided into pairs, we obtain the pairs $\{A, B\}$, $\{C, D\}$, $\{E, F\}$ and $\{G, H\}$.

Now note that the same final pairing could have been obtained if the initial sub- division into two subgroups of four each had been $\{A, B, E, F\}$, $\{C, D, G, H\}$. This observation immediately shows that over-counting has happened, and that the actual answer must therefore be less than 315. A closer look reveals that the actual answer is 1/3 of 315, because the pair $\{A,B\}$ can be combined with three different pairs, namely: $\{C, D\}$, $\{E, F\}$ and $\{G, H\}$. That is to say, the following three different ways of doing the initial subdivision could all result in the same final pairings (i.e., the pairs $\{A, B\}$, $\{C, D\}$, $\{E, F\}$ and $\{G, H\}$) as listed above:

$\{A, B, C, D\}$ and $\{E, F, G, H\}$,

$\{A, B, E, F\}$ and $\{C, D, G, H\}$,

$\{A, B, G, H\}$ and $\{C, D, E, F\}$.

Hence to obtain the correct answer, we must divide 315 by 3, which yields 105, as it should.

The contradiction has thus been resolved. We have uncovered where the extra factor of 3 entered into the reckoning in the first approach.

The COMMUNITY MATHEMATICS CENTRE (CoMaC) is an outreach arm of Rishi Valley Education Centre (AP) and Sahyadri School (KFI). It holds workshops in the teaching of mathematics and undertakes preparation of teaching materials for State Governments and NGOs. CoMaC may be contacted at shailesh.shirali@gmail.com.