G-Numbers

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Notation. Given two digits a and b, by $\overline{ab}$ we mean the number $10a + b$.

Definition. Consider an n-digit number N. Suppose it happens that N is equal to *the sum of all permutations of* $(n-1)$*-digit numbers whose digits are taken from the original number*; then we call N a **G-number**. (Here we take $n > 1$, as the definition becomes meaningless for $n = 1$.)

For example, consider a three-digit number $\overline{abc}$. Suppose it is equal to the sum $\overline{ab} + \overline{ba} + \overline{ac} + \overline{ca} + \overline{bc} + \overline{cb}$; then it is a G-number.

Two-digit G-numbers

It is easy to check that there can be no 2-digit G-number. For, if $\overline{ab} = 10a + b$ were such a number, where a, b are digits and $a > 0$, then the definition leads to the following relation:

$$10a + b = a + b, \quad \therefore\ 10a = a. \tag{1}$$

But this clearly has no solution with $a > 0$. Hence there are no 2-digit G-numbers.

Three-digit G-numbers

We next check whether there are any 3-digit G-numbers. Let $\overline{abc} = 100a + 10b + c$ be such a number; here a, b, c are digits and $a > 0$. Applying the definition, we see that

$$\overline{abc} = \overline{ab} + \overline{ba} + \overline{ac} + \overline{ca} + \overline{bc} + \overline{cb},$$

$$\therefore\ 100a + 10b + c = 11(a+b) + 11(a+c) + 11(b+c),$$

$$\therefore\ 100a + 10b + c = 22(a+b+c),$$

$$\therefore\ 78a = 12b + 21c,$$

$$\therefore\ 26a = 4b + 7c. \tag{2}$$

Keywords: Place value, digits, selections, permutations, powers of ten

The solutions of this equation can be worked out by hand, as follows.

(i) Suppose that $a = 1$. Then we have $4b + 7c = 26$. Since $4b$ and 26 are even numbers, it follows that c is even. Also, we must have $c < 4$ (since $7 \times 4 > 26$); so the only feasible value for c is 2. And indeed this yields a solution: $b = 3$, $c = 2$. So we have our first example of such a number: 132. To verify that this is correct, note that $132 = 12 + 21 + 13 + 31 + 23 + 32$.

Another way of writing this is

$$132 = (12 + 32) + (13 + 31) + (21 + 23) = 3 \times 44.$$

(ii) Suppose that $a = 2$. Then we get $4b + 7c = 52$. Since $4b$ and 52 are multiples of 4, it follows that c is a multiple of 4 as well. Since we must have $c < 8$, the only feasible value for c is 4. And this yields a solution: $b = 6$, $c = 4$. This yields the number 264; note that $264 = 2 \times 132$.

Another way of writing this is

$$264 = (24 + 64) + (26 + 62) + (42 + 46) = 3 \times 88.$$

(iii) Suppose that $a = 3$; then we end up getting the number 396 which happens to be equal to 3×132. Note that

$$396 = 3 \times 132 = (36 + 96) + (39 + 93) + (63 + 69).$$

(iv) If $a \geq 4$, then $26a \geq 104$. However, $4b + 7c$ cannot exceed $(4 \times 9) + (7 \times 9) = 99$. Hence there can be no solutions with $a \geq 4$.

It follows that there are precisely three such 3-digit numbers: 132, 264 and 396.

Four-digit G-numbers

We next check whether there are any 4-digit G-numbers. Let $\overline{abcd}$ be such a number; here a, b, c, d are digits, $a > 0$ and $\overline{abcd} = 1000a + 100b + 10c + d$. Applying the definition, we see that

$$\overline{abcd} = \overline{abc} + \overline{acb} + \cdots + \overline{bcd} \qquad \text{(24 such terms)}.$$

(The number 24 comes from $4 \times 3! = 4!$. To understand why, note that there are 3! terms corresponding to the selection $\{a, b, c\}$; another 3! terms corresponding to the selection $\{b, c, d\}$, etc.; and there are 4 such selections.) Now we have:

$$\overline{abc} + \overline{acb} + \overline{bac} + \overline{bca} + \overline{cab} + \overline{cba} = 222(a + b + c).$$

The sum of 4 such quantities is:

$$222(a + b + c) + 222(a + b + d) + 222(a + c + d) + 222(b + c + d),$$

which equals $666(a + b + c + d)$. Hence we get:

$$1000a + 100b + 10c + d = 666(a + b + c + d),$$

$$\therefore\ 334a = 566b + 656c + 665d. \qquad (3)$$

We must solve this equation in digits a, b, c, d; i.e., $a, b, c, d \in \{0, 1, 2, 3, \ldots, 8, 9\}$. It turns out that there are no solutions at all. This means that there are no 4-digit G-numbers. (Editor's note: You could try finding a proof of this claim on your own. In case you give up, a proof has been given in Box 1, by CoMaC.)

Five-digit G-numbers

Now we check whether there are any such numbers having 5 digits. Let

$$\overline{abcde} = 10^4 a + 10^3 b + 10^2 c + 10d + e$$

be such a number; here a, b, c, d, e are digits and $a > 0$. Applying the definition, we see that

$$\overline{abcde} = \overline{abcd} + \overline{abdc} + \cdots + \overline{bcde} + \cdots \qquad \text{(120 such terms)}.$$

(The number 120 comes from $5 \times 4! = 5!$, the explanation being as earlier.) Now we have:

$$\overline{abcd} + \overline{abdc} + \overline{acbd} + \cdots + \overline{dcba} = 6 \times 1111(a+b+c+d).$$

The sum of 5 such quantities is:

$$\begin{aligned} &6666(a+b+c+d) + 6666(a+b+c+e) + 6666(a+c+d+e) + \cdots \\ &= 4 \times 6666(a+b+c+d+e) = 24 \times 1111(a+b+c+d+e). \end{aligned}$$

Hence we get:

$$10000a + 1000b + 100c + 10d + e = 24 \times 1111(a+b+c+d+e) \qquad (4)$$

We must solve this equation in digits a, b, c, d, e. But it will be immediately obvious that there can be no solution in positive integers, as the coefficients on the right side uniformly exceed the corresponding coefficients on the left side ($24 \times 1111 > 10000$ and so on). Hence there are no 5-digit G-numbers.

General treatment of numbers with 4 or more digits

The situation for numbers with still more digits remains exactly the same as for 5-digit numbers. Namely, we get an equation which has no solutions in positive integers. It is possible to express this argument symbolically as follows. For $n > 2$ let

$$\overline{a_n a_{n-1} a_{n-2} \ldots a_2 a_1} = 10^{n-1} a_n + 10^{n-2} a_{n-1} + \cdots + 10 a_2 + a_1 \qquad (5)$$

be an n-digit number with the stated property; here $a_n, a_{n-1}, \ldots, a_2, a_1$ are digits and $a_n > 0$. This n-digit number should be equal to the sum of $n!$ numbers each having $(n-1)$ digits which are taken from the digits of the original n-digit number:

$$\begin{aligned} &10^{n-1} a_n + 10^{n-2} a_{n-1} + \cdots + 10 a_2 + a_1 \\ &= \text{sum of } n! \text{ numbers each having } (n-1) \text{ digits, as noted above.} \end{aligned}$$

In these $n!$ numbers, each digit will appear exactly $(n-1)!$ times in each place (i.e., units, tens, hundreds, …). Hence the sum of all the $n!$ numbers is

$$(n-1)!\,(\underbrace{111\ldots11}_{(n-1)})\,(a_n + a_{n-1} + \cdots + a_2 + a_1)\,.$$

Hence the equation which we have to solve is:

$$\begin{aligned} &10^{n-1} a_n + 10^{n-2} a_{n-1} + \cdots + 10 a_2 + a_1 \\ &= (n-1)!\,(\underbrace{111\ldots11}_{(n-1)})\,(a_n + a_{n-1} + \cdots + a_2 + a_1)\,. \end{aligned}$$

This may be rewritten as follows:

$$\sum_{k=1}^{k=n} \left((n-1)!\,(\underbrace{111\ldots11}_{(n-1)}) - 10^{n-k} \right) a_{n-k+1} = 0. \qquad (6)$$

If a solution can be found for this equation, in digits, then $\overline{a_n a_{n-1} a_{n-2} \ldots a_2 a_1}$ is a G-Number.

But as we argued earlier, there are no solutions for $n \geq 4$. This is so simply because for $n \geq 4$, each bracketed coefficient is a positive number.

It follows that there are precisely three G-numbers, namely: 132, 264 and 396. It is curious and interesting that each one has just three digits.

Comment from the editors. Teachers may wonder about the pedagogical value of such an exploration (which some may find rather whimsical!). Note that it demands reasoning, rigour and an understanding of topics such as permutations, factorial notation and solution of equations with more than three variables. Note also the fact that an important aim in mathematics education is *nurturing the ability to think mathematically*. In that sense, embarking on such explorations seems a very worthy exercise.

Finding all 4-digit G-numbers

We are required to solve the equation

$$334a = 566b + 656c + 665d \tag{7}$$

in digits a, b, c, d; i.e., with $a, b, c, d \in \{0, 1, 2, 3, \ldots, 8, 9\}$. The exercise may look tedious at first sight, in view of the large numbers involved, but we can reduce the labour greatly by means of a few observations.

(i) It cannot be that both c and d are 0; for this leads to the equation $334a = 566b$, or $157a = 283b$. This cannot have any solution in digits since 157 and 283 are coprime. Hence $c + d > 0$.

(ii) From the relation $334a = 566b + 656c + 665d$, we get $334a > 566b + 566c + 566d$, which yields $a > b + c + d$.

(iii) Since $a \leq 9$, the above inequality implies that $b + c + d \leq 8$, and hence that $a + b + c + d \leq 17$.

(iv) Reading equation (7) modulo 2, we get: $0 \equiv d \pmod 2$. Hence d is even.

(v) Reading equation (7) modulo 5, we get: $-a \equiv b + c \pmod 5$, which implies that $a + b + c$ is a multiple of 5.

(vi) Reading equation (7) modulo 9, we get: $a \equiv -b - c - d \pmod 9$, which implies that $a + b + c + d$ is a multiple of 9. Combining this with observation (iii), we deduce that $a + b + c + d = 9$, and hence that $a + b + c = 5$ and $d = 4$.

(vii) Since $a + b + c + d = 9$ and $a > b + c + d$, we deduce that $2a > 9$, i.e., $a \geq 5$ and $b + c + d \leq 4$. Since $a + b + c = 5$, this leads to $b = 0 = c$ and $a = 5$.

(viii) Therefore the conditions lead to just one set of values for the variables, namely: $(a, b, c, d) = (5, 0, 0, 4)$. These values satisfy all the necessary conditions obtained above, but unfortunately do not satisfy the defining equation itself, because $5 \times 334 \neq 4 \times 665$. So they do not lead to a solution.

Therefore, equation (7) has no solution in digits. It follows that there is no 4-digit G-number.

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https://www.youtube.com/watch?v=Gh8h8MJFFdI&feature=share

Follow this link to find the following problem: 36 is a triangle-square number, i.e.,

$$36 = 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8,$$

and it is a square number as well, i.e.,

36=6^2.

Are there other triangle-square numbers and is there a formula to generate such numbers? Send in your answer to AtRiA.editor@apu.edu.in