An example of constructive defining:

From a GOLDEN RECTANGLE to GOLDEN QUADRILATERALS and Beyond Part 2

MICHAEL DE VILLIERS

This article continues the investigation started by the author in the March 2017 issue of At Right Angles, available at: http://teachersofindia.org/en/ebook/golden-rectangle-golden-quadrilaterals-and-beyond-1 The focus of the paper is on constructively defining various golden quadrilaterals analogously to the famous golden rectangle so that they exhibit some aspects of the golden ratio phi. Constructive defining refers to the defining of new objects by modifying or extending known definitions or properties of existing objects. In the first part of the paper in De Villiers (2017), different possible definitions were proposed for the golden rectangle, golden rhombus and golden parallelogram, and they were compared in terms of their properties as well as 'visual appeal'.

In this part of the paper, we shall first look at possible definitions for a golden isosceles trapezium as well as a golden kite, and later, at a possible definition for a golden hexagon.

Constructively Defining a 'Golden Isosceles Trapezium'

How can we constructively define a 'golden' isosceles trapezium? Again, there are several possible options. It seems natural though, to first consider constructing a golden isosceles trapezium *ABCD* in two different ways from a golden parallelogram (*ABXD* in the 1st case, and *AXCD* in

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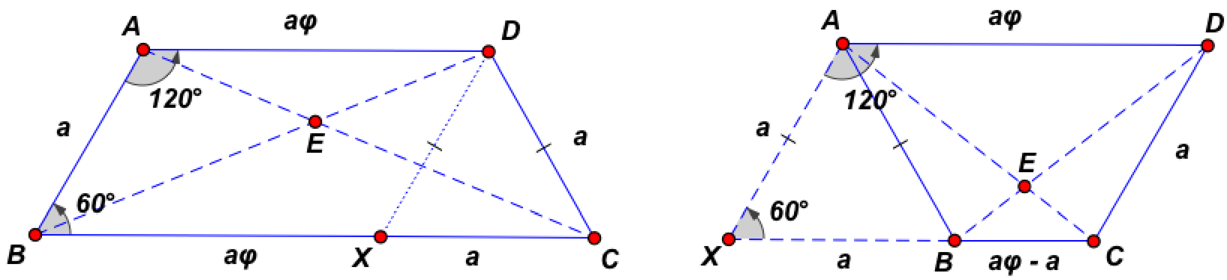

Figure 8. Constructing a golden isosceles trapezium in two ways

the 2nd case) with an acute angle of 60° as shown in Figure 8. In the first construction shown, this amounts to defining a golden isosceles trapezium as an isosceles trapezium *ABCD* with *AD // BC*, angle *ABC* = 60°, and the (shorter parallel) side *AD* and 'leg' *AB* in the golden ratio phi.

From the first construction, it follows that triangle *DXC* is equilateral, and therefore *XC* = *a*. Hence, *BC/AD* = (phi + 1)/phi, which is well known to also equal phi[1] . This result together with the similarity of isosceles triangles *AED* and *CEB*, further implies that *CE/EA* = *BE/ED* = phi. In other words, not only are the parallel sides in the golden ratio, but the diagonals also divide each other in the golden ratio. Quite nice!

In the second case, however, *AD/BC* = phi/(phi – 1) = (phi + 1) = phi squared. Also note in the second case, in contrast to the first, it is the longer parallel side *AD* that is in the golden ratio to the 'leg' *AB*, and the 'leg' *AB* is in the golden ratio with the shorter side *BC*. So the sides of this golden isosceles trapezium form a geometric progression from the shortest to the longest side, which is quite nice too!

Subdividing the golden isosceles trapezium in the first case in Figure 8, like the golden parallelogram in Figure 5, by respectively constructing a rhombus or two equilateral triangles at the ends, clearly does not produce an isosceles trapezium similar to the original. In this case the parallel sides (longest/shortest) of the obtained isosceles trapezium are also in the ratio (phi + 1), and is therefore in the shape of the second type in Figure 8. The rhombus formed by the midpoints of the sides of the first golden isosceles trapezium is also not any of the previously defined 'golden' rhombi.

With reference to the first construction, we could define the golden isosceles trapezium without any reference to the 60° angle as an isosceles trapezium *ABCD* with *AD // BC*, and *AD/AB* = phi = *BC/AD* as shown in Figure 9.

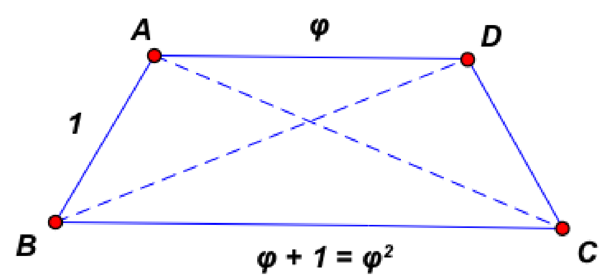

Figure 9. Alternative definition for first golden isosceles trapezium

However, this is clearly not as convenient a definition, as such a choice of definition requires again the use of the cosine formula to show that it implies that angle *ABC* = 60° (left to the reader to verify). As seen earlier, stating one of the angles and an appropriate golden ratio of sides or diagonals in the definition, substantially simplifies the deductive structure. This illustrates the important educational point that, generally, we choose our mathematical definitions for convenience and one of the criteria

[1] Keep in mind that phi is defined as the solution to the quadratic equation phi^2 – phi – 1 = 0. From this, it follows that phi = (phi + 1)/phi, phi = 1/(phi -1), phi/(phi +1) = phi + 1, or phi^2 = phi + 1.

for 'convenience' is the ease by which the other properties can be derived from it.

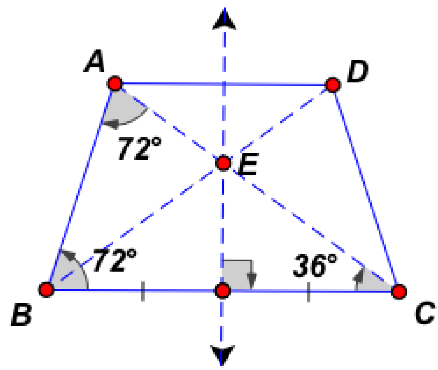

Figure 10. Third golden isosceles trapezium

Another completely different way to define and conceptualize a golden isosceles trapezium is to again use a golden triangle. As shown in Figure 10, by reflecting a golden triangle *ABC* in the perpendicular bisector of one of its 'legs' *BC*, produces a 'golden isosceles trapezium' where the ratio *BC/AB* is phi, and the acute 'base' angle is 72°. Moreover, since angle *BAD* = 108° and angle *ADB* = 36°, it follows that angle *ABD* is also 36°. Hence, *AD* = *AB* (= *DC*), and therefore the two parallel sides are also in the golden ratio, and as with the preceding case, the diagonals therefore also divide each other into the golden ratio. Of interest also, is to note that the diagonals *AC* and *DB* each respectively bisect the 'base' angles[2] at *C* and *B*.

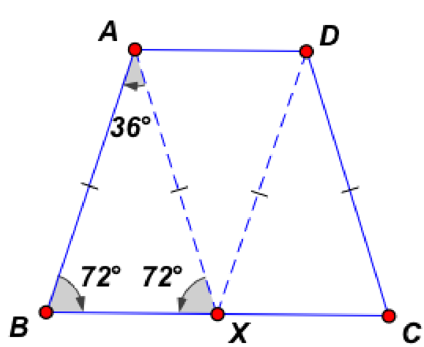

Figure 11. A fourth golden isosceles trapezium

A fourth way to define and conceptualize a golden isosceles trapezium could be to start again with a golden triangle *ABX*, but this time to translate it with the vector *BX* along its 'base' to produce a golden isosceles trapezium *ABCD* as shown in Figure 11. In this case, since the figure is made up of 3 congruent golden triangles, it follows that *AB/AD* = phi, and *BC* = 2*AD* (and therefore its diagonals also divide each other in the ratio 2 to 1).

Though one could maybe argue that the first case of a golden isosceles trapezium in Figure 8 is too 'broad' and the one in Figure 11 is too 'tall' to be visually appealing, there is little visually different between the one in Figure 10 and the second case in Figure 8. However, all four cases or types have interesting mathematical properties, and deserve to be known.

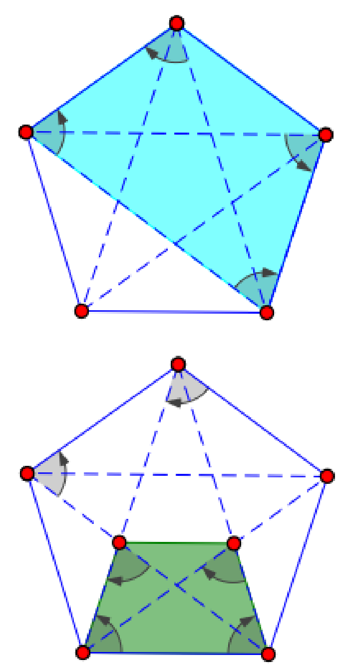

Figure 12. Golden isosceles trapezia of type 3

One more argument towards perhaps slightly favoring the golden isosceles trapezium, defined and constructed in Figure 10, might be that it appears in both the regular convex pentagon as well as the regular star pentagon as illustrated in Figure 12.

[2] In De Villiers (2009, p. 154-155; 207) a general isosceles trapezium with three adjacent sides equal is called a trilateral trapezium, and the property that a pair of adjacent, congruent angles are bisected by the diagonals is also mentioned. Also see: http://dynamicmathematicslearning.com/quad-tree-new-web.html

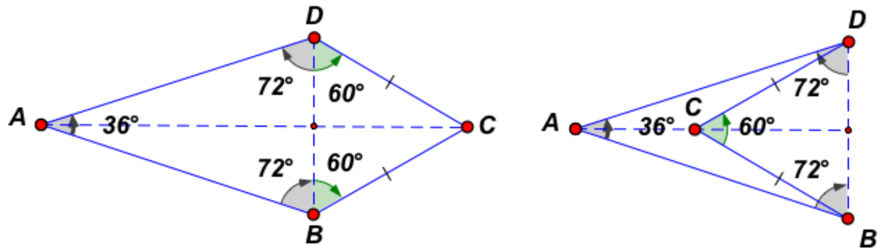

Figure 13. First case of golden kite

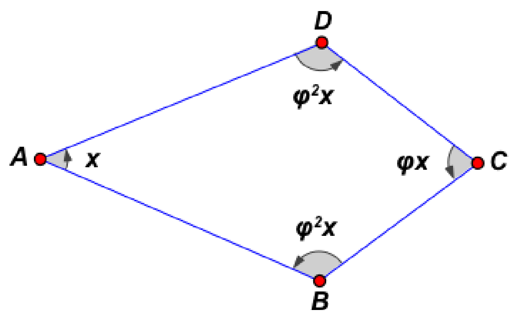

Figure 14. Second case of golden kite

Constructively Defining a 'Golden Kite'

Again there are several possible ways in which to constructively define the concept of a 'golden kite'. An easy way of constructing (and defining) one might be to again start with a golden triangle and construct an equilateral triangle on its base as shown in Figure 13. Since *AB*/*BD* = phi, it follows immediately that since *BD* = *BC* by construction, *AB* to *BC* is also in the golden ratio. Notice that the same construction applies to the concave case, but is probably not as 'visually pleasing' as the convex case.

Another way might be again to define the pairs of angles in the golden kite to be in the golden ratio as shown in Figure 14. Determining *x* from this geometric progression, rounded off to two decimals, gives:

$$x = \frac{360°}{1+\phi+2\phi^2} = 45.84°$$

Of special interest is that the angles at *B* and *D* work out to be precisely equal to 120°. This golden kite looks a little 'fatter' than the preceding convex one, and is therefore perhaps a little more visually pleasing. This observation, of course, also relates to the ratio of the diagonals, which in the first case is 2.40 (rounded off to 2 decimals) while in the case in Figure 14, it is 1.84 (rounded off to 2 decimals), and hence the latter is closer to the golden ratio phi.

To define a golden kite that is hopefully even more visually appealing than the previous two, I next thought of defining a 'golden kite' as shown in Figure 15, namely, as a (convex[3]) kite with both its sides and diagonals in the golden ratio.

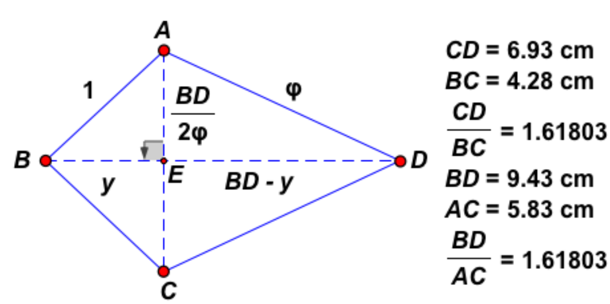

Figure 15. Third case: Golden kite with sides and diagonals in the golden ratio

[3] For the sake of brevity we shall disregard the concave case here.

Though one can drag a dynamically constructed kite in dynamic geometry with sides constructed in the golden ratio so that its diagonals are approximately also in the golden ratio, making an accurate construction required the calculation of one of the angles. At first I again tried to use the cosine rule, since it had proved effective in the case of one golden parallelogram as well as one isosceles trapezium case, but with no success. Eventually switching strategies, and assuming $AB = 1$, applying the theorem of Pythagoras to the right triangles ABE and ADE gave the following:

$$y^2 + \frac{BD^2}{4\phi^2} = 1$$

$$\frac{BD^2}{4\phi^2} + (BD - y)^2 = \phi^2.$$

Solving for y in the first equation and substituting into the second one gave the following equation in terms of BD:

$$BD^2 - 2BD\sqrt{1 - \frac{BD^2}{4\phi^2}} - \phi + 1 = 0.$$

This is a complex function involving both a quadratic function as well as a square root function of BD. To solve this equation, the easiest way as shown in Figure 16 was to use my dynamic geometry software (*Sketchpad*) to quickly graph the function and find the solution for $x = BD = 2.20$ (rounded off to 2 decimals). From there one could easily use Pythagoras to determine BE, and use the trigonometric ratios to find all the angles, giving, for example, angle $BAD = 112.28°$. So as expected, this golden kite is slightly 'fatter' and more evenly proportionate than the previous two cases. One could therefore argue that it might be visually more pleasing also.

In addition, the midpoint rectangle of the third golden kite in Figure 15, since its diagonals are in the golden ratio, is a golden rectangle.

On that note, jumping back to the previous section, this reminded me that a fifth way in which we could define a golden isosceles trapezoid might be to define it as an isosceles trapezium with its mid-segments KM and LN in the golden ratio as shown in Figure 17, since its midpoint rhombus would then be a golden rhombus (with diagonals in golden ratio). However, in general, such an isosceles trapezium is dynamic and can change shape, and we need to add a further property to fix its shape. For example, in the 1st case shown in Figure 17 we could impose the condition that BC/AD = phi, or as in the 2nd case, we can have $AB = AD = DC$ (so the base angles at B and C would respectively be bisected by the diagonals DB and AC). As can be seen, it is very difficult to visually distinguish between these two

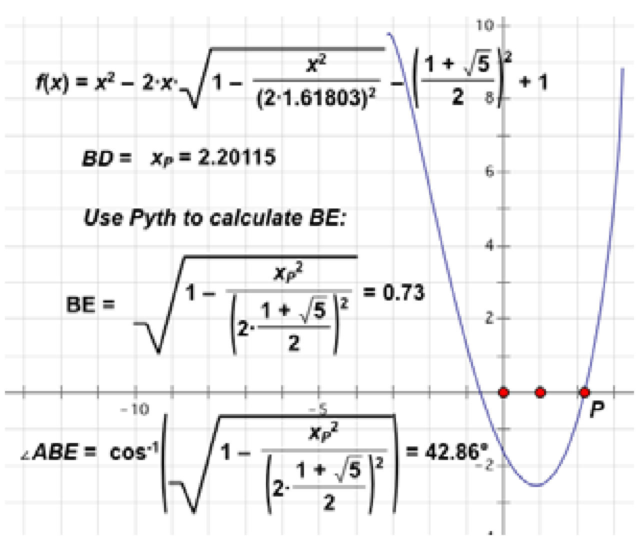

Figure 16. Solving for BD by graphing

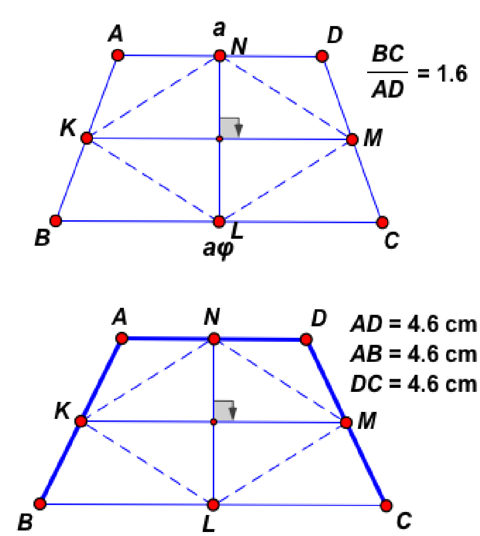

Figure 17. Fifth case: Golden isosceles trapezia via midsegments in golden ratio

cases since the angles only differ by a few degrees (as can be easily verified by calculation by the reader). Also note that for the construction in Figure 17, as we've already seen earlier, *AD* to *AB* will be in the golden ratio, if and only if, isosceles trapezium *ABCD* is a golden rectangle.

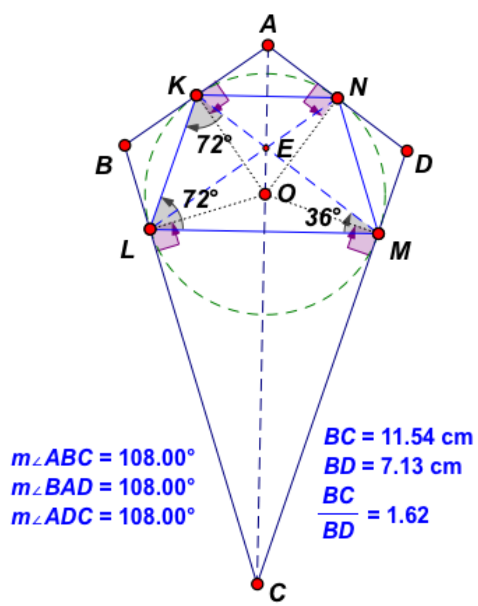

Figure 18. Constructing golden kite tangent to circumcircle of KLMN

Since all isosceles trapezia are cyclic (and all kites are circumscribed), another way to conceptualize and constructively define a 'golden kite' would be to also construct the 'dual' of each of the golden isosceles trapezia already discussed. For example, consider the golden isosceles trapezium *KLMN* defined in Figure 10, and its circumcircle as shown in Figure 18. As was the case for the golden rectangle, we can now similarly construct perpendiculars to the radii at each of the vertices to produce a corresponding dual 'golden kite' *ABCD*. It is now left to the reader to verify that *CBD* is a golden triangle (hence *BC/BD* = phi) and angle *ABC* = angle *BAD* = angle *ADC* = 108°. In addition *ABCD* has the dual property (to the angle bisection of two angles by diagonals in *KLMN*) of *K* and *N* being respective midpoints of *AB* and *AD* . The reader may also wish to verify that *AC/BD* = 1.90 (rounded off to 2 decimals), and since it is further from the golden ratio, explains the elongated, thinner shape in comparison with the golden kites in Figures 14 and 15.

Last, but not least, one can also choose to define the famous Penrose kite and dart as 'golden kites', which are illustrated in Figure 19. As can be seen, they can be obtained from a rhombus with angles of 72° and 108° by dividing the long diagonal of the rhombus in the ratio of phi so that the 'symmetrical' diagonal of the Penrose kite is in the ratio phi to the 'symmetrical' diagonal of the dart. It is left to the reader to verify that from this construction it follows that both the Penrose kite and dart have their sides in the ratio of phi. Moreover, the Penrose kites and darts can be used to tile the plane non-periodically, and the ratio of the number of kites to darts tends towards phi as the number of tiles increase (Darvas, 2007: 204). Of additional interest, is that the 'fat' rhombus formed by the Penrose kite and dart as shown in Figure 19, also non-periodically tiles with the 'thin' rhombus given earlier by the second golden rhombus in Figure 7, and the ratio of the number of 'fat' rhombi to 'thin' rhombi similarly tends towards phi as the number of tiles increase (Darvas, 2007: 202). The interested reader will find various websites on the Internet giving examples of Penrose tiles of kites and darts as well as of the mentioned rhombi.

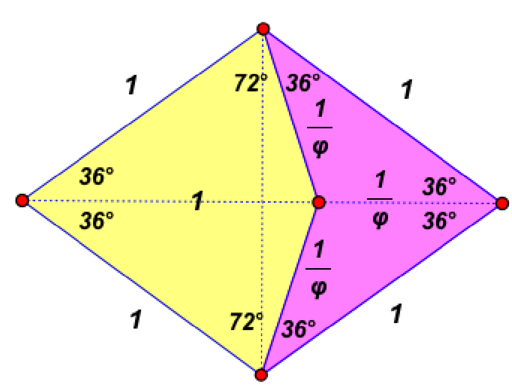

Figure 19. Penrose kite and dart

[4] In De Villiers (2009, p. 154-155; 207) , a general kite with three adjacent angles equal is called a triangular kite, and the property that a pair of adjacent, congruent sides are bisected by the tangent points of the incircle is also mentioned. The Penrose kite in Figure 19 is also an example of a triangular kite. Also see: http://dynamicmathematicslearning.com/quad-tree-new-web.html

Constructively Defining Other 'Golden Quadrilaterals'

This investigation has already become longer than I'd initially anticipated, and it is time to finish it off before I start boring the reader. Moreover, my main objective of showing constructive defining in action has hopefully been achieved by now.

However, I'd like to point out that there are several other types of quadrilaterals for which one can similarly explore ways to define 'golden quadrilaterals', e.g., cyclic quadrilaterals, circumscribed quadrilaterals, trapeziums[5], bi-centric quadrilaterals, orthodiagonal quadrilaterals, equidiagonal quadrilaterals, etc.

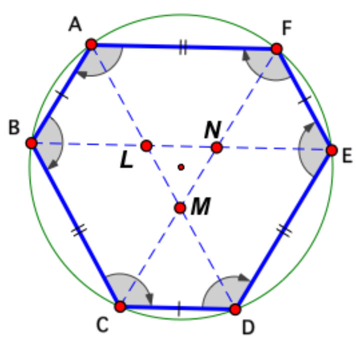

Figure 20. A golden hexagon with adjacent sides in golden ratio

Constructively Defining a 'Golden Cyclic Hexagon'

Before closing, I'd like to briefly tease the reader with considering defining hexagonal 'golden' analogues for at least some of the golden quadrilaterals discussed here. For example, the analogous equivalent of a rectangle is an equi-angled, cyclic hexagon[6] as pointed out in De Villiers (2011; 2016). Hence, one possible way to construct a hexagonal analogue for the golden rectangle is to impose the condition on an equi-angled, cyclic hexagon that all the pairs of adjacent sides as shown in Figure 19 are in the golden ratio; i.e., a 'golden (cyclic) hexagon'. It is left to the reader to verify that if *FA/AB* = phi, then *AL/LM* = phi[7], etc. In other words, the main diagonals divide each other into the golden ratio.

The observant reader would also note that *ABEF*, *ABCD* and *CDEF*, are all three golden trapezia of the type constructed and defined in the first case in Figure 8. Moreover, *ALNF*, *ABCF*, etc., are golden trapezia of the second type constructed and defined in Figure 8.

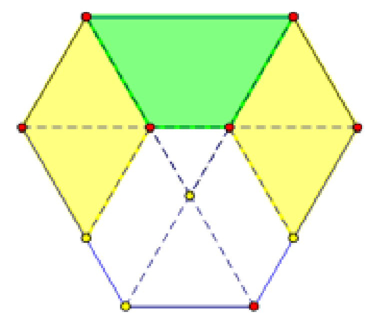

Figure 21. Cutting off two rhombi and a golden trapezium

By cutting off two rhombi and a golden isosceles trapezium as shown in Figure 21, we also obtain a similar golden cyclic hexagon. Lastly, it is also left to the reader to consider, define and investigate an analogous dual of a golden cyclic hexagon.

Concluding Remarks

Though most of the mathematical results discussed here are not novel, it is hoped that this little investigation has to some extent shown the productive process of constructive defining by illustrating how new mathematical objects can be defined and constructed from familiar definitions of known objects. In the process, several different possibilities may be explored and

[5] Olive (undated), for example, constructively defines two different, interesting types of golden trapezoids/trapeziums.

[6] This type of hexagon is also called a semi-regular angle-hexagon in the referenced papers.

[7] It was with surprised interest that in October 2016, I came upon Odom's construction at: http://demonstrations.wolfram.com/HexagonsAndTheGoldenRatio/, which is the converse of this result. With reference to the figure, Odom's construction involves extending the sides of the equilateral triangle LMN to construct three equilateral triangles ABL, CDM and EFN. If the extension is proportional to the golden ratio, then the outer vertices of these three triangles determine a (cyclic, equi-angled) hexagon with adjacent sides in the golden ratio.

compared in terms of the number of properties, ease of construction or of proof, and, in this particular case in relation to the golden ratio, perhaps also of visual appeal. Moreover, it was shown how some definitions of the same object might be more convenient than others in terms of the deductive derivation of other properties not contained in the definition.

The process of constructive defining also generally applies to the definition and exploration of different axiom systems in pure, mathematical research where quite often existing axiom systems are used as starting blocks which are then modified, adapted, generalized, etc., to create and explore new mathematical theories. So this little episode encapsulates at an elementary level some of the main research methodologies used by research mathematicians. In that sense, this investigation has hopefully also contributed a little bit to demystifying where definitions come from, and that they don't just pop out of the air into a mathematician's mind or suddenly magically appear in print in a school textbook.

In a classroom context, if a teacher were to ask students to suggest various possible definitions for golden quadrilaterals or golden hexagons of different types, it is likely that they would propose several of the examples discussed here, and perhaps even a few not explored here. Involving students in an activity like this would not only more realistically simulate actual mathematical research, but also provide students with a more personal sense of ownership over the mathematical content instead of being seen as something that is only the privilege of some select mathematically endowed individuals.

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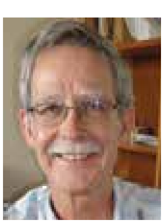

MICHAEL DE VILLIERS has worked as researcher, mathematics and science teacher at institutions across the world. He retired from the University of KwaZulu-Natal at the end of January 2016. He has subsequently been appointed Professor Extraordinaire in Mathematics Education at the University of Stellenbosch. He was editor of *Pythagoras*, the research journal of the Association of Mathematics Education of South Africa, and was vice-chair of the SA Mathematics Olympiad from 1997 -2015, but is still involved. His main research interests are Geometry, Proof, Applications and Modeling, Problem Solving, and the History of Mathematics. His home page is http://dynamicmathematics-learning.com/homepage4.html. He maintains a web page for dynamic geometry sketches at http://dynamic-mathematicslearning.com/JavaGSPLinks.htm. He may be contacted on profmd@mweb.co.za.