Trisection of a 60 degree angle? Not Quite!

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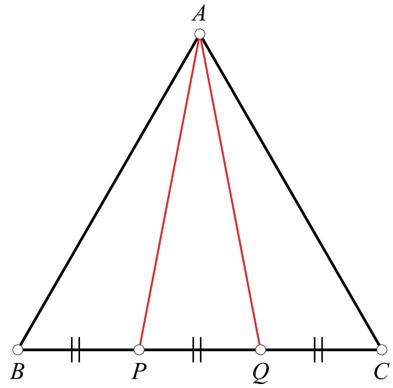
Many students on first hearing that "Trisection of a general angle is not possible using only compass and straight-edge" immediately set about trying to disprove this assertion! Curiously, many among them hit upon the following method (illustrated for a 60° angle).

In the figure, $\triangle ABC$ is equilateral, and P and Q are points of trisection of BC (so $BP = PQ = QC$). Segments AP and AQ are drawn. *Question*. Do these two segments trisect $\measuredangle BAC$? Many students believe that they do. How do we check whether they are right? Noting that $\measuredangle BAP = \measuredangle CAQ$ by symmetry, we only need to compare $\measuredangle BAP$ and $\measuredangle PAQ$.



We choose to make the comparison using coordinates. Let $B = (0, 0)$, $C = (6, 0)$, $A = (3, 3\sqrt{3})$, $P = (2, 0)$ and $Q = (4, 0)$. Then the slopes of AB, AP, AQ and AC are as follows:

$$\text{slope}(AB) = \tan 60^\circ = \sqrt{3},$$
$$\text{slope}(AC) = \tan 120^\circ = -\sqrt{3}.$$

$$\text{slope}(AP) = \frac{3\sqrt{3} - 0}{3 - 2} = 3\sqrt{3},$$
$$\text{slope}(AQ) = \frac{3\sqrt{3} - 0}{3 - 4} = -3\sqrt{3}.$$

Hence, using the 'angle between two lines' formula, we get, for $\measuredangle BAP$ and $\measuredangle PAQ$:

$$\tan \measuredangle BAP = \frac{3\sqrt{3} - \sqrt{3}}{1 + 3\sqrt{3} \cdot \sqrt{3}} = \frac{2\sqrt{3}}{10} = \frac{1}{5} \times \sqrt{3},$$

$$\tan \measuredangle PAQ = \frac{-3\sqrt{3} - 3\sqrt{3}}{1 - 3\sqrt{3} \cdot 3\sqrt{3}} = \frac{-6\sqrt{3}}{-26} = \frac{3}{13} \times \sqrt{3}.$$

We see right away that $\measuredangle BAP$ and $\measuredangle PAQ$ are unequal (since $1/5$ and $3/13$ are unequal). But we can say more: since $1/5 < 3/13$, it follows that $\measuredangle BAP < \measuredangle PAQ$. (Here we make implicit use of the fact that for acute angles x and y, if $x < y$ then $\tan x < \tan y$, and vice versa. Differently expressed, $\tan \theta$ is an increasing function of θ for $0 \le \theta < \pi/2$.)

Thus, $\measuredangle PAQ$ exceeds both $\measuredangle BAP$ and $\measuredangle QAC$. Here are the actual magnitudes of the angles:

$$\measuredangle BAP = \measuredangle QAC \approx 19.1066^\circ, \quad \measuredangle PAQ \approx 21.7868^\circ.$$

So $\measuredangle PAQ$ exceeds $\measuredangle BAP$ by a fair bit. The method doesn't quite work ....

Can we prove this without computation?

Is there a *non-computational way* of proving that $\measuredangle BAP < \measuredangle PAQ$? It is a nice challenge to find such a proof. Note that if we do find one, it will not tell us by how much the two angles differ.

Here is a possible approach. Consider $\triangle ABP$ and $\triangle APQ$. The two triangles have equal bases ($BP = PQ$) and the same altitude (namely: the altitude of $\triangle ABC$). So they have equal area.

Now we invoke another formula: *area of a triangle equals half the product of any two sides and the sine of the included angle*. Applying this to $\triangle ABP$ and $\triangle APQ$, which we know have equal area, we get:

$$\frac{1}{2} AB \times AP \times \sin \measuredangle BAP = \frac{1}{2} AP \times AQ \times \sin \measuredangle PAQ,$$
$$\therefore\ AB \times \sin \measuredangle BAP = AQ \times \sin \measuredangle PAQ.$$

Hence $AB/AQ = \sin \measuredangle PAQ / \sin \measuredangle BAP$. Now which is greater, AB or AQ? Clearly, it is AB which is larger. This can be seen from $\triangle ABQ$, in which $\measuredangle AQB > \measuredangle ABQ$ (proof: $\measuredangle AQB > \measuredangle ACQ$, which equals $\measuredangle ABQ$). Invoking the fact that the larger angle in a triangle has the larger side opposite it, we deduce that $AB > AQ$ and so $AB/AQ > 1$.

Therefore $\sin \measuredangle PAQ / \sin \measuredangle BAP > 1$, and it follows that $\measuredangle BAP < \measuredangle PAQ$. (Once again, we implicitly make use of a fact from trigonometry: that over the domain of acute angles, sine is an increasing function of the angle.)

The reader is invited to find other non-computational proofs showing that $\measuredangle BAP < \measuredangle PAQ$.

The **COMMUNITY MATHEMATICS CENTRE** (CoMaC) is an outreach arm of Rishi Valley Education Centre (AP) and Sahyadri School (KFI). It holds workshops in the teaching of mathematics and undertakes preparation of teaching materials for State Governments and NGOs. CoMaC may be contacted at shailesh.shirali@gmail.com.