Some Down-to-Earth Trigonometry

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This article aims to present some applications of trigonometry to earth sciences. We assume that the earth is a perfect sphere.

Latitudes and longitudes are imaginary circles that run East-West and North-South respectively, on the earth's surface. Unlike longitudes (or meridians), latitudes (or parallels) vary in length. The Equator is the longest. The others decrease in size till the poles, which are just points. The length of the other latitudes can be expressed as fractions of the length of the Equator.

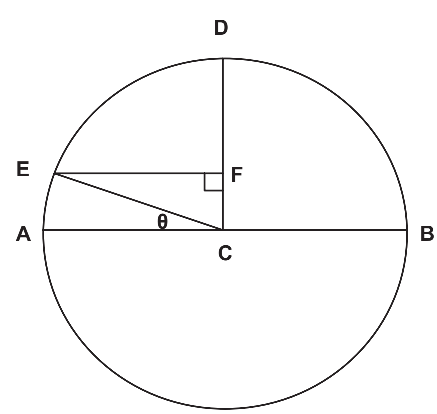

Figure 1

In Figure 1, which shows a section of the earth through the poles, AB is an equatorial diameter. C is the centre of the earth and D the North Pole. E is a point on latitude θ°. The radius of the latitude

$$EF = EC\cos\theta = R\cos\theta,$$

R being the radius of the earth. So we see that the radii of the latitudes and thereby the latitudes themselves, vary as the cosine function as we move from 0° (Equator) to 90° (the poles). The latitudes that are half the length of the Equator are the 60° latitudes.

Keywords: Trigonometry, latitude, longitude, sphere, curved surface area, horizon.

Now we consider the earth's surface area enclosed by a given latitude and the Equator. We invoke the celebrated theorem of Archimedes which states that the surface area of a sphere enclosed between two parallel planes equals a similar part of the (curved) surface area of an enveloping cylinder with axis perpendicular to the planes (Refer Figure 2).

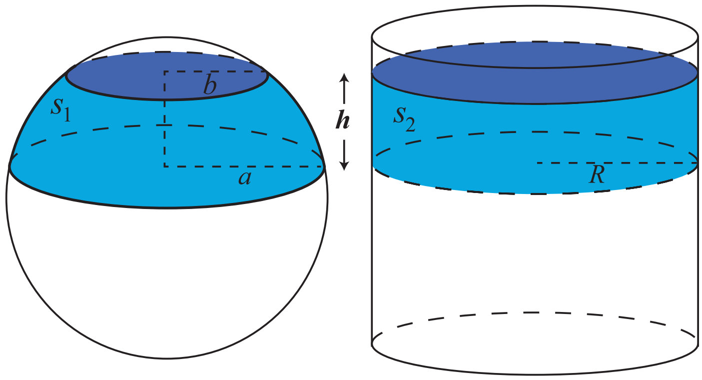

Figure 2. Source: http://mathworld.wolfram.com/

So we need to obtain the perpendicular distance between the equatorial plane and the plane of the given latitude. This is represented in Figure 1 by the distance

$$FC = R \sin \theta.$$

So the extent of the earth's surface enclosed between the equator and a given latitude θ is

$$2\pi R\ (R \sin \theta) = 2\pi R^2 \sin \theta.$$

This reaches the hemispherical curved surface area of $2\pi R^2$ at the poles. Half the earth's surface area lies between the 30° N and S latitudes. It may be of geographical interest to note that about 40% of the earth's surface lies in the tropical zone, about 52% in the temperate zone and about 8% in the frigid zone.

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The shortest route between two places on earth over the earth's surface lies along the great circle passing through them. Great circles are the largest circles that can be drawn on the surface of the globe. There are an infinite number of them. The equator is one. Two opposite longitudes together make one. There is a unique great circle passing through a pair of points, unless they happen to be antipodal (diametrically opposite) points, in which case there are an infinite number of them.

To find the distance along a great circle route we need to know the angular separation of the points, i.e., the angle between the radii connecting the given points to the centre of the earth. This can be obtained from the latitude and longitude of these points. We take the following observations from 3-D coordinate geometry:

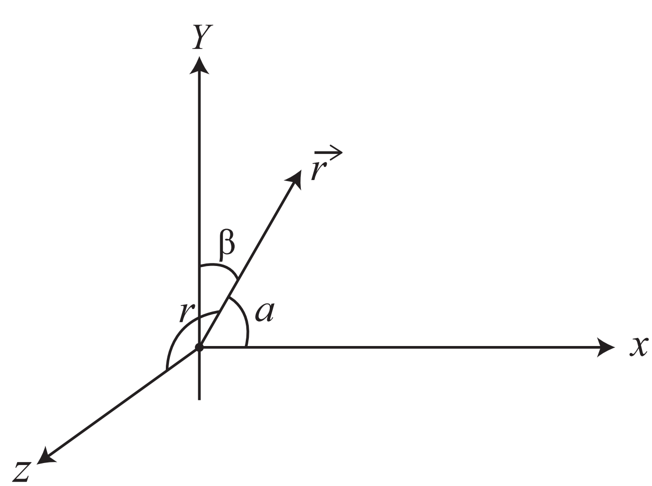

Figure 3

Any line in 3-space makes angles α, β, γ with the X, Y, Z axes respectively (Figure 3). If α_1, β_1, γ_1 and α_2, β_2, γ_2 are the angles made by two lines with the coordinate axes respectively, then the angle θ between the lines can be obtained from the equation

$$\cos\theta = \cos\alpha_1 \cos\alpha_2 + \cos\beta_1 \cos\beta_2 + \cos\gamma_1 \cos\gamma_2.$$

In the context of the earth, we can choose the lines from the centre of the earth to the North pole, the point (0°, 0°) and the point (0°, 90° E) respectively as the coordinate axes. Then, if the latitudes of the two places are ϕ_1 and ϕ_2 and longitudes λ_1, λ_2 we can say that the cosine of the central angle θ

$\cos\theta = \sin\phi_1 \sin\phi_2 + \cos\phi_1 \cos\lambda_1 \cos\phi_2 \cos\lambda_2 + \cos\phi_1 \sin\lambda_1 \cos\phi_2 \sin\lambda_2$, which simplifies to

$\cos\theta = \sin\phi_1 \sin\phi_2 + \cos\phi_1 \cos\phi_2 (\cos\lambda_1 \cos\lambda_2 + \sin\lambda_1 \sin\lambda_2)$ or

$\cos\theta = \sin\phi_1 \sin\phi_2 + \cos\phi_1 \cos\phi_2 \cos\Delta\lambda.$

We follow the convention that North, East are positive, while South, West are negative. See Boxed item at the end of the article for a fuller explanation.

The product of the central angle (in radians) and the radius of the earth gives the distance between the given places 'as the crow flies.'

The following could be checked out by the reader:

If both points lie on the Equator the central angle is the difference in longitude between the places; if the points lie on the same longitude the central angle is the difference in latitude; if the points are antipodal {in such a case the points can be taken to be [x° N, y° E] and [x° S, ($180^\circ - y^\circ$)W]} the central angle is 180°.

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There is a limit to the extent of the earth's surface visible from any point above ground level. The limit of vision is the horizon. How far is the horizon? Obviously it is a function of the height of the observer/detector above ground level. (Again we assume a perfectly spherical earth lacking atmosphere.)

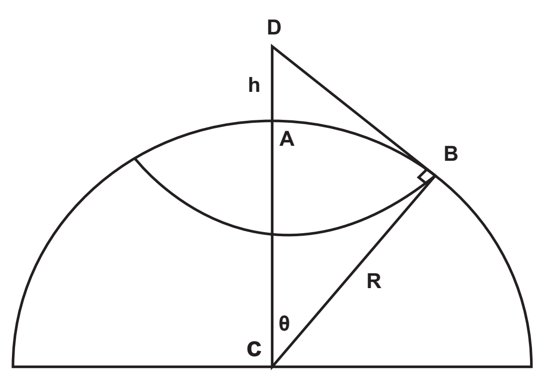

Figure 4

As can be seen from Figure 4, the horizon is the locus of the points of contact of tangents drawn from the observer to the earth's surface. The required distance d is then the arc length $AB = R\theta$ (θ in radians).

$$\text{But } \cos\theta = \frac{R}{R+h}.$$

From the above we get

$$d = R\cos^{-1}\frac{R}{R+h}.$$

This equation can be rearranged to give

$$h = R[\sec(d/R) - 1],$$

enabling one to calculate the height required to see to a given distance.

An alternative approach is to use the Pythagoras theorem to get

$$DB^2 = DC^2 - CB^2 = (h+R)^2 - R^2 = h^2 + 2hR$$

$$\text{So} \quad DB = \sqrt{(h^2 + 2hR)}$$

Now, if h << R, we can say

$$DB = \sqrt{2hR}.$$

And we could then say $d = \sqrt{2hR}$, approximating the arc length AB to the line of sight distance DB.

Substituting 6370 km for R, and converting to metres we have $d = 3570\sqrt{h}$ metres.

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Now we shall derive an expression for the extent of the earth's surface visible from a point.

Again taking a cue from Archimedes we need to find the vertical extension of the part of the sphere enclosed by the horizon. If BE is perpendicular to CD (Figure 5), then the required distance is AE, which we shall denote by 'k.' Now

$$BC^2 = CE.CD, \text{ or}$$

$$R^2 = (R - k)(R + h), \text{ which gives}$$

$$k = Rh/(R + h).$$

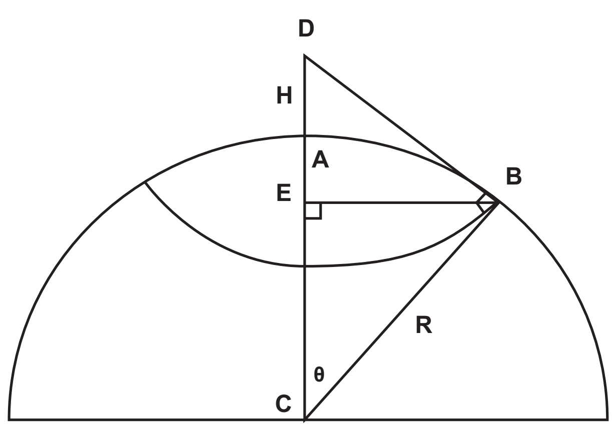

Figure 5

So the extent of the earth's surface visible from point D is

$$2\pi R[Rh/(R + h)] = 2\pi R^2[h/(R + h)] = 2\pi R^2 (1- \cos \theta).$$

As a fraction of the curved surface area of the hemisphere, this becomes $h/(R + h) = 1 - \cos \theta$.

It can be seen that when h = 0, the visible area = 0. When h is very large the area approaches $2\pi R^2$. When h = R, visible area $=\pi R^2$, half the curved surface of the hemisphere.

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The above expression could also be considered as an expression for the extent of the earth's surface from any point of which an object located at the given height above the earth would be in line of sight. This view of the matter has more practical implications. The Global Positioning System, the American satellite-based navigation system, initially depended on a set of 24 satellites placed in orbits around the earth at an approximate height of 20200 km above the earth's surface, which is more than 3 times the earth's radius. So, substituting 3R for h in the equation derived above we find that each satellite has 3/4 of the earth's hemispherical surface area or 3/8 of earth's total surface in line of sight. So 24 satellites judiciously located would cover the earth's surface $\frac{3}{8} \times 24 = 9$ times over. This fully bears out the claim made that any place on earth would at any instant be in line of sight of at least 6 satellites. The system was later augmented to 32 satellites which by similar arguments can be said to cover earth's surface 12 times over, justifying the claim that at least 9 satellites would always be in line of sight from any point on earth.

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In the course of the earth's revolution around the sun, different latitudes in the tropical zone get vertical rays of the sun at midday on different days of the year. The Equator experiences it on the equinoxes (21, March and 23, September), the Tropic of Cancer on the summer solstice (21, June) and the Tropic of Capricorn on the winter solstice (21, December). The movement of the 'overhead midday sun' between the Tropics follows a sinusoidal path. We can calculate the date(s) of direct midday sun at a particular latitude or conversely the latitude that experiences it (declination, δ) on a given date.

For the earth-sun system we could designate the following coordinate axes:

- The line of centres of earth and sun at spring equinox as the X axis
- The line perpendicular to the above and lying in the earth's orbital plane (ecliptic) as the Y axis
- The line perpendicular to the above lines (perpendicular to ecliptic) as the Z axis.

Now assuming the earth's orbit to be circular with the sun at the centre, and a uniform orbital speed for the earth, it can be shown that

$$\sin \delta = \sin 23.45^\circ \sin (360^\circ N/365.25),$$

where N is the number of days elapsed after the spring equinox. A positive value indicates a latitude in the northern hemisphere while a negative value indicates a latitude in the southern hemisphere.

The above formula can be approximated to

$$\delta = 23.5^\circ \sin (360^\circ N/365).$$

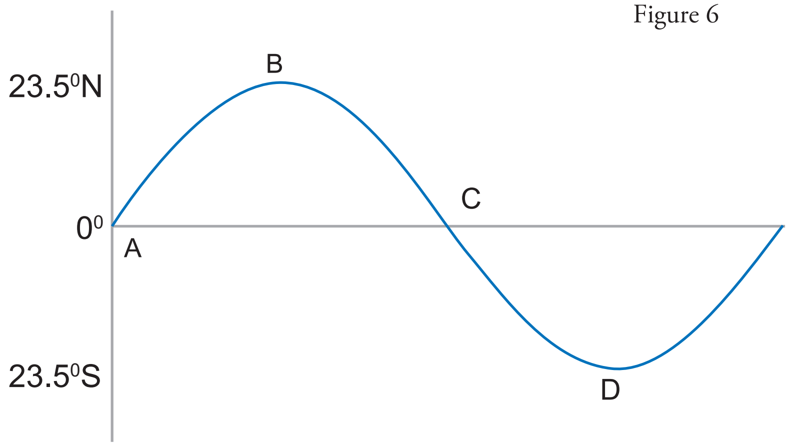
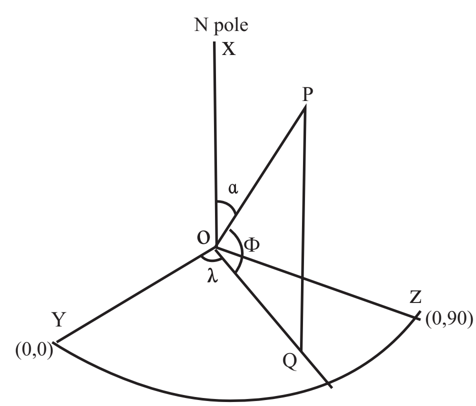


Figure 6

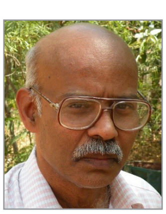
As can be seen (Figure 6) the graph of this relation is steep around the equinoxes and has low values of slope around the solstices, where the sun seems to linger, justifying the term 'solstice' (= stationary sun).

A, B, C, D stand for spring equinox, summer solstice, autumn equinox and winter solstice respectively.



The cosines of the angles made by a straight line in 3-space with the X, Y and Z axes are generally denoted by α, β and γ. In the context of the earth, we can take the centre of the earth to be the origin and the ray towards the North pole as the X axis. Then the ray towards the point on the earth's surface with latitude 0° and longitude 0° could be the Y axis, while the ray towards the point with latitude 0° and longitude 90° E could be the Z axis. Now if P is a point on the earth's surface its α value is the complement of Φ, the latitude of P. So cos α = sin Φ. To obtain cos β we multiply cos Φ with cos λ, the longitude of point P. That is, we project P on the equatorial plane to Q, and then project Q onto the Y axis. To obtain cos γ we multiply cos Φ with sin λ as the angle between OQ and the Z axis is the complement of the longitude λ. If α_1, β_1, γ_1 and α_2, β_2, γ_2 are the angles made by two radii of earth with the coordinate axes respectively, and θ is the angle between the radii, we have

$\cos\theta = \sin\Phi_1 \sin\Phi_2 + \cos\Phi_1 \cos\lambda_1 \cos\Phi_2 \cos\lambda_2 + \cos\Phi_1 \sin\lambda_1 \cos\Phi_2 \sin\lambda_2$, as mentioned earlier in this article.



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