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A Property of Primitive PYTHAGOREAN TRIPLES

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A *primitive Pythagorean triple*, or *PPT* for short, is a triple (a, b, c) of coprime positive integers satisfying the relation $a^2 + b^2 = c^2$. Some well-known PPTs are: $(3, 4, 5)$, $(5, 12, 13)$ and $(8, 15, 17)$. See Box 1 for some basic facts about PPTs.

A note on PPTs

There have been several articles in past issues of *At Right Angles* exploring PPTs and ways of generating them. Here are some features about PPTs which you need in this article (we invite you to provide proofs): If (a, b, c) is a PPT, then:

(i) c is odd;

(ii) one out of a, b is odd and the other one is even;

(iii) the even number in $\{a, b\}$ is a multiple of 4.

We agree to list the numbers in the PPT so that a is the odd number and b is the even number.

The following property is worth noting: *b is a multiple of 4.* To see why, write $b^2 = c^2 - a^2$. Note that a and c are odd, and recall that any odd square is of the form $1 \pmod 8$. This implies that b^2 is a multiple of 8 and hence that b is a multiple of 4. (If b were even but not a multiple of 4, then b^2 would be a multiple of 4 but not a multiple of 8.)

This article focuses on one particular family of PPTs, those having $b = c - 1$. For this family we have:

$$a^2 + (c-1)^2 = c^2, \quad \therefore \quad a^2 = 2c - 1,$$

so:

$$c = \frac{a^2+1}{2}, \quad b = \frac{a^2-1}{2}.$$

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This note describes a feature of PPTs (a, b, c) in which $b = c - 1$. Here are some PPTs with this feature:

$$(3,4,5), \qquad (5,12,13),$$
$$(7,24,25), \qquad (9,40,41),$$
$$(11,60,61), \qquad (13,84,85),$$
$$(15,112,113), \qquad (21,220,221),$$
$$(33,544,545), \qquad (35,612,613),$$
$$(39,760,761), \qquad \ldots .$$

Here is the property I discovered:

> If (a, b, c) is a PPT with $b = c - 1$, then $a^b + b^a$ is divisible by c.

For example:

- For the PPT $(3, 4, 5)$: $3^4 + 4^3 = 145 = 5 \times 29$;
- For the PPT $(5, 12, 13)$: $5^{12} + 12^5 = 244389457 = 13 \times 18799189$.

But in the other PPTs such as $(15, 8, 17)$, $(21, 20, 29)$, $(33, 56, 65)$, $(35, 12, 37)$, $(39, 80, 89)$, etc., where $b \neq c - 1$, this property is not to be seen. Why should the property belong to just this type of PPT?

I will prove the following: if (a, b, c) is a PPT with $b = c - 1$, then $a^b + b^a$ is divisible by c.

Proof. Since $b = c - 1$ we have (see Box 1):

$$c = \frac{a^2+1}{2}, \qquad b = \frac{a^2-1}{2}.$$

From $b = c - 1$ we get $b \equiv -1 \pmod{c}$, therefore

$$b^a \equiv (-1)^a \pmod{c} \equiv -1 \pmod{c},$$

since a is odd. Next, from $a^2 = 2c - 1$ we get $a^2 \equiv -1 \pmod{c}$. Raising both sides to power $b/2$ (remember that b is an even number), we get

$$a^b \equiv (-1)^{b/2} \pmod{c} \equiv 1 \pmod{c},$$

since, as per Box 1, b is a multiple of 4 (which implies that $b/2$ is an even number). Hence

$$a^b + b^a \equiv 1 - 1 \equiv 0 \pmod{c}.$$

In other words, $a^b + b^a$ is divisible by c. $\square$

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