Interpretation of ERRORS IN ARITHMETIC

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Errors should not be viewed as a setback but as an opportunity to learn more about the student's thought processes.

Correcting errors to get the right product instead of analysing the learning trajectory is a quick fix that does not nurture deep learning

Very often, errors are caused by over-generalisation of rules which are transmitted to the student as short-cuts to getting the required answer.

Certain repetitive drill and practice tasks set by teachers can contribute to students developing these quick fixes which in fact, can divert them from the concept that they are meant to understand and practise.

Teachers must plan tasks that help children understand the basic concept being taught. They must also observe the child doing the task and, together with the child, examine the child's procedural thinking.

Errors are often referred to as windows into the minds of learners and also as part of the ladder of learning. The effort to look at the work of the learner and attempting to construct the conceptual and procedural underpinning of the response-expressions of the child provides many insights. This analysis can also be useful in mathematics as it points out the care the teacher may take in conversing with learners as it may offer many learnings and takeaways for the teacher. It has been spelt out sufficiently well that the work of children has to be considered far more carefully than merely sorting into the categories of right and wrong. The manner in which the learner has reached the answer provides insights about the way she thinks and approaches the question. Many times, the answer being correct does not imply that the problem has been tackled appropriately. The correct answer could well be a consequence of fortuitous mistakes and coincidences. In an article 'Errors or ladders of learning' many years ago, Agnihotri[1] suggested that errors could be the steps in the ladders of learning and indicate the path through which the journey of learning could take place. This has indeed been stated and argued before and after by many persons. In the learning of language there are many examples of this and some of them are relatively better understood. For example, there is the phenomenon of over generalisation in responding to exceptions to the rules. The simplest example is of 'go' and 'goed' or the example in Hindi of saying टूटाना for तोडना. The formulation presented to the child, 'इस गिलास को मत तोडना' became 'इस गिलास को मत टूटाना. The implied

[1] Agnihotri, Rama Kant. 1988. 'Errors as learning strategies'. In Indian Journal of Applied Linguistics 14.1.: 1-14. Bahri Publications Delhi

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meaning in both is 'breaking' as a part of the imperative, 'Do not break the glass'. Many more examples can be added to this from language and from other areas as well.

While this happens in the natural learning process of the child, as she tries to engage with situations and attempts to create expressions that communicate her ideas, the patterns and rules that she has over-generalised start entering her conversations and then, based on the responses received, she herself slowly corrects them. Yet, there is a tendency of teachers to forcibly correct the rules children need to use and not give the children the opportunity to recognise them and correct them. By setting up a formal process to correct the error, the teacher loses the chance to help the student understand the underlying pattern and the reason for the error.

If we ask ourselves the question as to how this, or something else that is akin to an over-generalisation of rules happens in the context of mathematics, then we can try to analyse the work of children and connect it to the kind of processes they have gone through and the possible generalisations that they can make. Given the nature of mathematical objects and the manner in which relationships between them are constructed, there are no or very few exceptions to the definitions and the rules. Where there are exceptions, they are indicated in the beginning itself. For example, in the defining relationships for a rational number, where p/q, where $q \neq 0$ and p, q are within the set of integers or in calculus finding limits with the denominator tending to zero. In doing operations and in solving problems, there are however, created algorithms. And these may be inappropriately spelt out and wrongly used. These algorithms and short-cuts are created to find quick answers and are sometimes provided by teachers and sometimes shared through exchanges. The effort is focused towards finding a response to the task without fully understanding the expectation or the task, leading to inappropriate use of the technique. It is these phenomena that we would explore in this article. We do this through the piecing together of response patterns to three distinct kinds of mathematical situations. In all these, some form of shortcut strategy to reach the expected answer is seen. We will then try to link it to some phenomena that may be fairly widely used by children to respond to the tasks they have been given so as to complete them with minimum effort. All the examples presented in this paper have at their base a widely used task that is common across schools. This task continues to influence the development of mathematical understanding and impacts the subsequent mathematical ideas developed by the learner. The strategy or the method adopted for the task, produces the desired answer and is efficient in providing the response the teacher wants but the manner in which it constructs understanding has wider implications for the conceptual structures of the child. The purpose of the analysis of these examples is to illustrate the often stated point that short-cuts and special algorithms to arrive at the answer can lead to such conceptual confusions and generalisations that fail to give the user the ability to appreciate the procedures being used and use them in appropriate situations in an appropriate manner only. The tasks that are given to children must be well thought out and must expect her to use her conceptual, procedural and cognitive knowledge instead of just reproducing information. It also suggests that examining how children do the task is important for the teacher to observe and she must also talk to the child about the way she is doing the task and the logic she has behind it.

Considering a few response patterns

It is generally noticed that: i) children have difficulties considering a fractional representation as one number; ii) they also have difficulties in dealing with two-digit numbers and on operations with them. The nature of responses has been reported on at many places and if a teacher considers the responses of her students to such problems beyond marking them as right or wrong, she will discover many patterns in the responses. One such pattern likely to be present in the responses even till class 8 and beyond is typified by the following example

$$\frac{1}{3}+\frac{1}{2}=43 \quad \text{and} \quad \frac{1}{3}+\frac{1}{2}=\frac{2}{5}$$

and similar responses in subtraction and multiplication sums.

The second example is of addition of two or more numbers with carry over and also to some extent addition of numbers with an unequal number of digits.

The examples of these are

$$\begin{array}{rrr} & 2 & 7 \\ + & 3 & 8 \\ \hline 5 & 1 & 5 \end{array}$$

and other such equivalent examples.

When the number of digits is unequal or a problem of algorithmic addition with numbers presented in the same manner as above with digits displaced by a small amount from a strict column format, interesting responses may be seen. For example

128 + 64 can become

$$\begin{array}{rrr} 1 & 2 & 8 \\ +\ 6 & 4 & \\ \hline 7 & 6 & 8 \end{array}$$

Or

$$\begin{array}{rrrr} & 1 & 7 & 9 \\ +\ 2 & 6 & 1 & \\ \hline 2 & 7 & 8 & 9 \end{array}$$

Similar responses are seen in subtraction where the larger digit in the numerals is the one from which the other smaller digit is subtracted. For example, 64 – 38 gives the answer 34

$$\begin{array}{rrr} & 6 & 4 \\ - & 3 & 8 \\ \hline & 3 & 4 \end{array}$$

Another task that illustrates this rather well is the belief that once the child knows the rule of carry over, then she can be given numbers of any size to add. So generally, when we think children in class 4 or 5 (if not earlier) know addition of 2- and 3-digit numbers we start giving numbers like this-

$$\begin{array}{r} 74345212 \\ 52136128 \\ +\ 214321 \\ \hline \end{array} \quad \text{or} \quad \begin{array}{r} 28750 \\ 13250 \\ +\ 8950 \\ \hline \end{array}$$

Clearly students are unable to read the numbers and know what they are and what the sum should be and so they are doing column additions without reading the numbers as one number. When given subtraction problems, they are at a loss when they have to borrow. So even if they get the correct answer in these, it is not as if they are learning any mathematical concepts or developing any ability in mathematics. It would seem that the rules of addition have been over-generalised without comprehension.

Then there is division as well. Consider these for example:

$$\begin{array}{r} 11 \\ 3\overline{)44} \\ 3 \\ \hline 14 \\ 3 \\ \hline 11 \end{array} \quad \text{or} \quad \begin{array}{r} 11 \\ 3\overline{)44} \\ 44 \\ \hline 00 \end{array}$$

$$\begin{array}{r} 11 \\ 6\overline{)85} \\ 66 \\ \hline 21 \end{array} \quad \text{or} \quad \begin{array}{r} 11 \\ 5\overline{)\ 75} \\ -55 \\ \hline 20 \end{array}$$

All these children do problems like these correctly:

$$\begin{array}{r} 21 \\ 4\overline{)\ 84} \\ -8 \\ \hline 4 \\ 4 \\ \hline 0 \end{array} \quad \text{and} \quad \begin{array}{r} 7 \\ 8\overline{)58} \\ 56 \\ \hline 02 \end{array} \quad \begin{array}{r} 20 \\ 2\overline{)\ 40} \\ -40 \\ \hline 00 \end{array}$$

It is when they have to deal with numbers that have to be seen as 2-digit numbers and a clear understanding of place value is required that they slip into errors/short cuts. These are all examples from Class 3 of a school.

All teachers of mathematics have seen these examples. They become a path over which many travel and slowly overcome but many others get stuck with it and as they are faced with more and more mathematics they become more entrapped in them.

The responses below are from adults who have enrolled in an open course on teaching-learning of mathematics. In response to what is wrong with the solution to $\frac{8}{15} + \frac{3}{15} = \frac{11}{30}$ the reasons were interesting as we can decipher a pattern underlying them.

One response was that she has added the numerators and the denominators. What would have been the correct thing to do is to add the numbers above and below each other and then take the sum of the two numbers.

$$\text{So, } 23 + 18 = 41$$

This is a simple response emerging from the habit of column addition to which we would return.

In response to the solution to $\frac{3}{4} + \frac{3}{5}$, the answer given was even more complex; it goes

$$\frac{3}{4} + \frac{3}{5} = \frac{4}{3} + \frac{5}{3} = \frac{9}{3}$$

The numbers are reversed to give a common denominator and then the numbers on top are added.

The emphasis and importance given to column writing and dealing with numbers (or rather digits) in the same column independent of the place in the whole number seems to suggest there is either a cognitive inability or there is something that is done in early classes that forms this pattern. The fact that all children display more elaborate cognitive and conceptual abilities in their standard or routine tasks leaves us to examine the manner in which they are expected to engage with numbers and the opportunities that are created for them. One example of a strategy observed by a person closely observing children while they are attempting to quickly do the tasks being given by their teachers in rural schools of central India illustrates this and offers an interesting insight. It is usual practice in beginning classes to have children write numbers from 1 to 100 in a column or a row format.

1	2	3	4	5	.	.	.	.	10
11	12	13	14	15	.	.	.	.	20
21	.	.	.	.	.	.	.	.	30

Or in the form

1	11	21	.	.	.	.	.	.	91
2	12	22	.	.	.	.	.	.	92
9	19	.	.	.	.	.	.	.	99

Children are asked to do this and it is supposed that they would write these as is required, through a systematic following of the order. What was observed was that children follow a strategy that makes the task easier for them; they write one row or column as required and then repeat the same up to 9. The matrix you get is

1	2	3	4	5	.	.	.	.	10
11	12	13	14	15	.	.	.	.	20
21	_2	_3	_4	.	.	.	.	.	30
31	_2	_3	_4	.	.	.	.	.	40

Then two small steps lead to listing all the numbers from 1 to 100.

The step is to write 2, 3, 4, 5, 6, 7, 8, 9 in front of each number of the respective column giving this on the slate:

1	2	3	4	5	.	.	.	.	10
11	12	13	14	15	.	.	.	.	20
21	22	23	24	.	.	.	.	.	30
31	32	33	34	.	.	.	.	.	40

and then you can write 10, 20, ..., etc. in the last column or write 1, 2 ....., 9 and then add a zero at the end of each. They repeat this pattern when numbers above 100 are to be written or numbers are to be written in horizontal rows.

Very often, children using a slate have these part columns already written. And these can be quickly filled in the way required. This gives them time to do other things that they want to do. It is fascinating to see how they manage to do this when asked to write counting numbers in reverse sequence. They only need to write digits 9 to 1 and manage the second column appropriately in the same 9 to 1 pattern. In other tasks also, similar patterns are evoked to produce the expected answer. There is no need to understand why in the 2 digit numbers, the digits at different locations are different in the sense they represent different quantities, or in the order of the numbers, what is the logic of the number names, etc. What is enough is to know 1 to 9 and the strategy to write the answer.

So what is the learning from all this for teachers of mathematics?

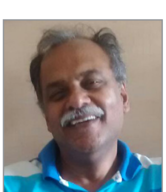
a. Giving tasks that are mechanical, particularly in the absence of conversation about the task, what they did and why, children can use strategies that change the task to one that only requires the production of the answers without engaging with any concept that should have been required to do the task with understanding. It does not also make them think about what, how or even the procedures involved. The tasks become amenable to alternative short-cuts and do not even help with procedural knowledge or memorisation of facts. Attempts to make tasks complex and challenging has to go along with ensuring that children understand the question and its relevance, be able to read out the questions and estimate the answer.

b. Avoid giving children short-cuts in the form of alternative simple strategies or routes to reach the answer. The strategy of column addition, just looking at the digits and without reading and understanding the numbers is another such example. The problem of short-cuts does not end here and carries on to later classes as well. Some of the examples are short-cuts in word problems, namely hints like all together means addition, spent and left mean subtraction, sharing means division or following BODMAS without thinking. All these lead to confusions that complicate. Similarly, in trying to simplify the meaning of letter numbers, terms like 4a, calling it (say) 4 apples, and so 4a cannot be added to 3b as apples and bananas cannot be added. This suggests, however, that 4a + 3b can be added to give 7f (fruits). And also the confusion that the other letters in the book represent other objects and the consequent confusions and road blocks in learning. These are just a few types of confusions but each of these confusions manifest themselves in multiple ways.

c. Practice is useful but only when it is not mechanical and repetitive and requires the need to develop an understanding of the problem and about how to proceed to the solution.

d. In situations where the number of children is large and the time available for the teacher is at a premium, resorting to techniques of occupying children with tasks like writing numerals can lead to the false sense that an understanding has developed in children. In addition, it gives children generalisations and rules that are grossly inappropriate and wrong

In considering the work of children and while constructing tasks for them, it is important to ensure that they are able to form and articulate their understanding and get reasonable feedback on that. The fact that the teacher may not be able to look at all of them necessitates that they work with each other not in an 'expert' and novice relationship but in a peer relationship. It is of course true that these small group interactions would yet have hierarchies of knowledge but those are what they have constructed as peers. The groups and the relationships cannot be designed and constructed by the teacher such that the 'smart' student guides the weaker one, learning should be independent and supported by a discussion of errors.



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