Lurking within any triangle …

Morley's Miracle – Part III

…is an equilateral triangle

This article concludes the three-part series begun in the July 2014 issue, wherein we study one of the most celebrated theorems of Euclidean geometry: Morley's Miracle. In this segment we examine an unusual proof due to Professor John H Conway.

feature

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In Part I of this article we narrated the history of this theorem and discussed a pure geometry proof (M. T. Naraniengar's). We remarked that the proof *starts* with an equilateral triangle and then proceeds to construct a configuration similar to the original one; thus it reaches the desired conclusion.

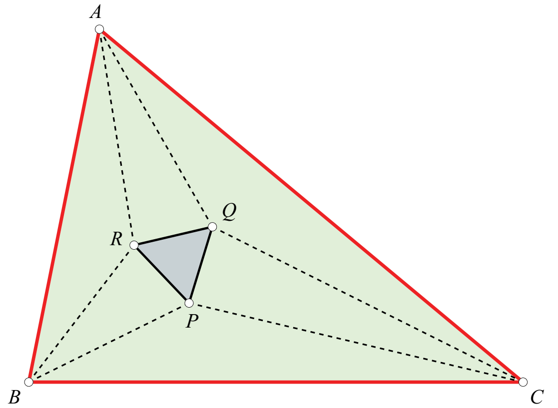


Figure 1. Morley's theorem: The angle trisectors closest to each side intersect at points which are the vertices of an equilateral triangle

Keywords: *Angle trisector, equilateral triangle, congruent, sine rule, backward proof*

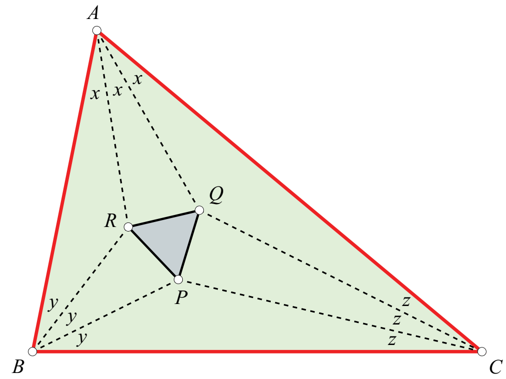


Figure 2.

We added that many of the pure geometry proofs known today proceed in just this way.

Now, in the concluding piece of this three-part series, we give another such proof; this one has sprung from the fertile mind of Professor John Conway [2]. (See https://en.wikipedia.org/wiki/John_Horton_Conway for information on this remarkable individual.) It may well be the most unusual of all the proofs of Morley's theorem. (Actually, our proof is a slight adaptation of Conway's proof.)

Given $\triangle ABC$, let angles x, y, z be defined by $A = 3x$, $B = 3y$ and $C = 3z$ (see Figure 2). We shall assume henceforth that all angles are measured in degrees, so that $x + y + z = 60$.

Conway starts by introducing the following operation for angles. Let θ be any angle (measured in degrees, of course). Then he defines θ^+ to be the angle $\theta + 60$ and θ^{++} to be the angle $\theta^+ + 60 = \theta + 120$. So a triangle exists with angles of $0^+, 0^+, 0^+$: it is an equilateral triangle. In the same way we can assert that:

- A triangle exists with angles x, y^+, z^+; for, $x + y^+ + z^+ = 180$.

  Similarly, a triangle exists with angles x^+, y, z^+, and a triangle exists with angles x^+, y^+, z.
- A triangle exists with angles x^{++}, y, z; for, $x^{++} + y + z = 180$.

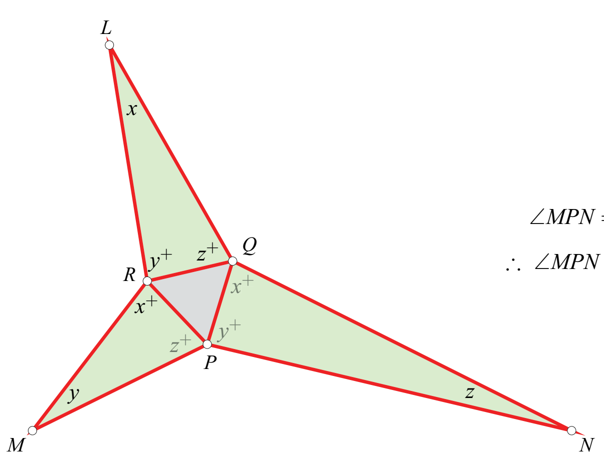
  Similarly, a triangle exists with angles x, y^{++}, z, and a triangle exists with angles x, y, z^{++}.

Conway starts by constructing an equilateral triangle PQR with side 1 unit (Figure 3). Then he constructs:

- On side PQ as base: $\triangle PQN$ with angles y^+, x^+, z at vertices P, Q, N respectively;
- On side QR as base: $\triangle QRL$ with angles z^+, y^+, x at vertices Q, R, L respectively;
- On side RP as base: $\triangle RPM$ with angles x^+, z^+, y at vertices R, P, M respectively.

Each of these is a legitimate triangle, in the sense that the prescribed angles add up to 180. Each one is uniquely fixed both in shape and size.

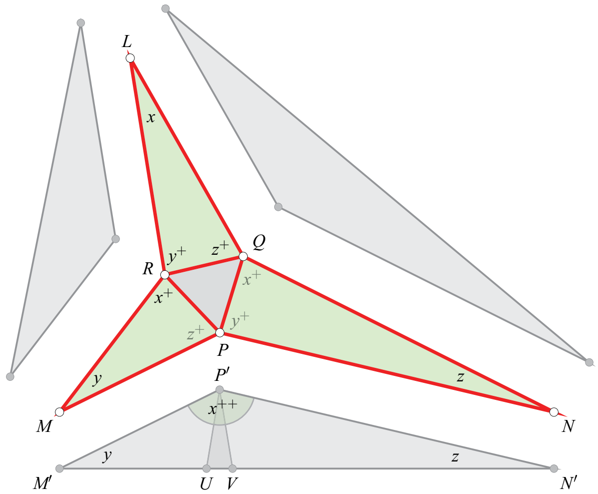
The computation in Figure 3 shows that $\angle MPN = x^{++}$. This fact allows us to insert into angle MPN a triangle $P'M'N'$ with angles x^{++}, y and z. (We have noted earlier that there does exist a triangle with these angles, as the angles do add up to 180.) But we need to fix the size of the triangle first. We do this as follows.



$\angle MPN = 360 - 60 - y^+ - z^+$

$\therefore\ \angle MPN = x^{++}$

Figure 3.



- $\angle M'P'N' = x^{++}$
- $\angle P'M'N' = y$
- $\angle P'N'M' = z$
- $\angle P'VM' = x^+$
- $\angle P'UN' = x^+$

Figure 4.

Let the candidate triangle $P'M'N'$ be drawn as shown, with its prescribed angles (see Figure 4). Next, let rays $P'U$ and $P'V$ be drawn from P' such that $\angle M'P'V = z^+$ and $\angle N'P'U = y^+$. Let these rays intersect the side $M'N'$ at U and V respectively. Then $\angle P'VM' = 180 - y - z^+ = x^+$ and $\angle P'UN' = 180 - z - y^+ = x^+$. Note that this makes $\triangle P'UV$ isosceles, with $P'U = P'V$. Now we fix the scale of the triangle so that $P'U$ and $P'V$ have the same length as the side of the equilateral triangle PQR. This is clearly possible.

With this in place, we consider $\triangle MPR$ and $\triangle M'P'V$. They are clearly congruent to each other ('ASA congruence'), as they have the same sets of angles, and the sides opposite angle y have equal length; hence $MP = M'P'$. In just the same way we have $NP = N'P'$ (consider $\triangle NPQ$ and $\triangle N'P'U$). Hence when we insert $\triangle M'P'N'$ into angle MPN, the fit is exact: $M'P'$ lines up with MP; $N'P'$ lines up with NP; and $M'N'$ lines up with MN.

The same kinds of actions can be repeated on the other two sides of the triangle: we insert into angle NQL and angle LRM triangles of suitable size, which then match up exactly with the spaces occupied by the angles. (See Figure 4. We have not named the triangles to avoid a visual clutter.) With these three triangles thus in place, the seven triangles together make up triangle LMN, whose angles at L, M and N are $3x$, $3y$ and $3z$. This means that $\triangle LMN$ is similar to the given $\triangle ABC$ (they have the same sets of angles). Moreover, the lines LQ and LR trisect $\angle MLN$; the lines MR and MP trisect $\angle LMN$; and the lines NP and NQ trisect $\angle LNM$. So the trisectors of the angles of $\triangle LMN$ give rise to an equilateral triangle, and it follows that the same must be true of $\triangle ABC$, just as Morley's theorem asserts. This proves the theorem.

Another presentation of Conway's proof

Conway's proof can be presented in a different way. See Figure 5. Consider $\triangle PMN$. Since $\angle MPN = x^{++}$, it follows that $\angle PMN + \angle PNM = y + z$.

Now we consider the ratio $PM : PN$ in $\triangle PMN$. We compute the ratio via $\triangle MPR$ and $\triangle NQP$:

$$\frac{PM}{PN} = \frac{PM/PR}{PN/PQ} = \frac{\sin x^+/\sin y}{\sin x^+/\sin z} == \frac{\sin z}{\sin y}.$$

It follows that

$$\frac{\sin\angle PNM}{\sin\angle PMN} = \frac{\sin z}{\sin y}.$$

We also know that $\angle PNM + \angle PMN = z + y$. From these relations we may conclude that

$$\angle PNM = z, \qquad \angle PMN = y.$$

(This may seem intuitively clear but it needs justification. Let $\angle PMN = u$, $\angle PNM = v$, and let $w = y + z$. Then $w = u + v$ too, and $\sin u : \sin v = \sin y : \sin z$. We now have:

$$\frac{\sin u}{\sin v} = \frac{\sin y}{\sin z},$$

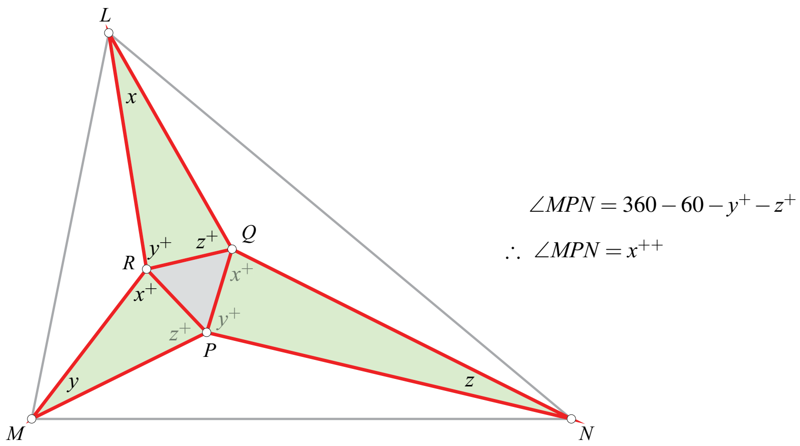


Figure 5.

$$\therefore\ \frac{\sin(w-v)}{\sin v} = \frac{\sin(w-z)}{\sin z},$$

$$\therefore\ \sin(w-v)\sin z = \sin(w-z)\sin v,$$

$$\therefore\ (\sin w\cos v - \sin v\cos w)\sin z = (\sin w\cos z - \sin z\cos w)\sin v,$$

$$\therefore\ \cos v\sin z = \cos z\sin v \quad (\text{since } \sin w \neq 0),$$

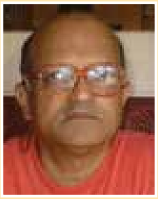
$$\therefore\ \sin(z-v) = 0,$$

which yields $v = z$ and hence $u = y$ as well. The reader could look for different ways of arguing this out.)

In just the same way we find that $\angle QNL = z$, $\angle QLN = x$, $\angle RLM = x$, $\angle RML = y$. We conclude, as earlier, that $\triangle LMN$ is similar to $\triangle ABC$; LQ and LR trisect $\angle MLN$; MR and MP trisect $\angle LMN$; and NP and NQ trisect $\angle LNM$. So the trisectors of the angles of $\triangle LMN$ give rise to an equilateral triangle, and the same must be true of $\triangle ABC$. This proves Morley's theorem.

References

[1] Bankoff, L. "A simple proof of the Morley theorem", Math. Mag., **35** (1962) 223–224
[2] Conway, J. H. "Proof of Morley's Theorem", http://www.cambridge2000.com/memos/pdf/conway2.pdf
[3] Coxeter, H. S. M. "Introduction to Geometry", 2nd ed., Wiley, New York, 1969
[4] Coxeter, H. S. M. & Greitzer, S. L. "Geometry Revisited", Random House/Singer, New York, 1967



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