A TRIANGLE PROBLEM and THREE VARIANTS

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In this short note we start with a simple problem concerning a triangle and then analyze new problems derived from it by changing the hypothesis. Our starting point is the following:

Problem. *In triangle ABC, the largest angle is twice the smallest angle and the lengths of the sides are consecutive positive integers. Determine the lengths of the sides.*

The simplest way to tackle this problem is to use the sine rule. If θ is the smallest angle and if the side-lengths are $x-1$, x and $x+1$, then by the sine rule,

$$\frac{x-1}{\sin\theta} = \frac{x}{\sin(180^\circ - 3\theta)} = \frac{x+1}{\sin 2\theta}. \tag{1}$$

From this it follows, by using standard identities for $\sin 2\theta$ and $\sin 3\theta$, that

$$\begin{aligned}\frac{x+1}{x-1} &= 2\cos\theta;\\ \frac{x}{x-1} &= 3 - 4\sin^2\theta = (2\cos\theta)^2 - 1.\end{aligned} \tag{2}$$

Therefore:

$$\frac{x}{x-1} = \left(\frac{x+1}{x-1}\right)^2 - 1, \tag{3}$$

which yields the equation $x(x-5) = 0$, from which follows $x = 5$, since x is a positive integer. Thus the side-lengths are 4, 5 and 6. We note in passing that $\theta \approx 41.41^\circ$ (computed using the fact that $\cos\theta = 3/4$).

Keywords: Triangle, angle, AP, GP, HP, similarity, golden ratio, intermediate value theorem

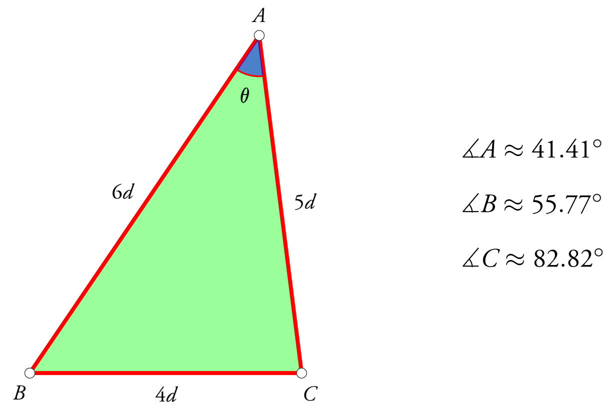


Figure 1. Sides in AP

Variation 1. We now modify the hypothesis of this problem. Instead of requiring that the side-lengths be consecutive positive integers, suppose we demand that they be in arithmetic progression (AP). We keep the angle condition unchanged. What are the side-lengths? Let them be $x - d$, x and $x + d$ where d is the common difference and $0 < d < x$. A triangle exists with these side-lengths if and only if the sum of the two smaller lengths exceeds the largest length, i.e.,

$$x - d + x > x + d. \tag{4}$$

The inequality is equivalent to $x > 2d$. By an argument similar to the one used to analyze the given problem, we have

$$\frac{x}{x-d} = \left(\frac{x+d}{x-d}\right)^2 - 1 \tag{5}$$

which gives $x = 5d$. Thus the side-lengths are $4d$, $5d$ and $6d$. Therefore the effect of enlarging the difference between the side-lengths is that of scaling the side-lengths in the original problem by the common difference. The resulting triangle is similar to the one obtained earlier, with side-lengths 4, 5 and 6. The smallest angle naturally remains unchanged ($\theta \approx 41.41°$). The relevant triangle is depicted in Figure 1.

Variation 2. What if the side-lengths are in geometric progression (GP) and the largest angle is (as earlier) twice the smallest angle? Let the side-lengths be x, xr and xr^2 where r is the common ratio (we may assume that $r > 1$ by agreeing to denote the smallest side-length by x). A triangle exists with these side-lengths if and only if the sum of the two smaller lengths exceeds the largest length, i.e.,

$$x(1 + r) > xr^2, \quad \text{i.e.,} \quad r^2 - r - 1 < 0. \tag{6}$$

This inequality yields $1 < r < \phi$ where $\phi = \frac{1}{2}(1 + \sqrt{5})$, the Golden Ratio. (Recall that the Golden Ratio ϕ is the positive solution of the quadratic equation $x^2 = x + 1$. Its value is approximately 1.618034.) Invoking the sine rule, we get:

$$\frac{x}{\sin\theta} = \frac{xr}{\sin(180° - 3\theta)} = \frac{xr^2}{\sin 2\theta}, \tag{7}$$

whence

$$r^4 - r - 1 = 0. \tag{8}$$

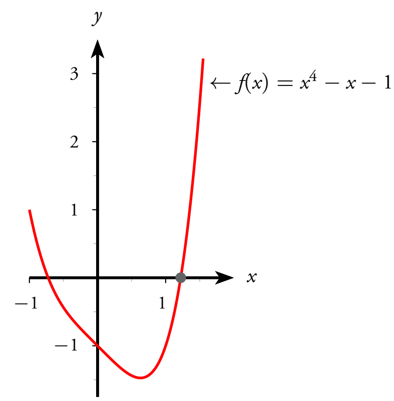


Figure 2. Graph of the function $f(r) = r^4 - r - 1$

Let $f(r) = r^4 - r - 1$. We need to estimate the positive roots of this polynomial. The relevant portion of the graph of this function is shown in Figure 2.

We observe that $f(r)$ has a positive root r_0 between 1 and 2. We can see why without appealing to the graph. First, note that $f(r)$ is strictly increasing for $r > 1$. For, the slope function is $f'(r) = 4r^3 - 1$, which is positive for $r > 1$. Next, note that $f(1) = -1 < 0$ and $f(2) = 13 > 0$. Indeed, we also have $f(1.5) = 2.5625 > 0$. Hence, by the intermediate value theorem, there exists a root between 1 and 1.5, and it is the sole such real root. (Editor's note: The *intermediate value theorem* states that if f is a continuous function defined on an interval I, and its values at the endpoints of I have opposite sign, then it must take a value of 0 somewhere in I.) The use of a computer algebra system such as *Mathematica* or *Derive* reveals the root r_0 to be approximately equal to 1.2207. The corresponding value of θ is given by:

$$2\cos\theta = r_0^2, \quad \therefore \quad \theta \approx \cos^{-1}\frac{1.2207^2}{2} \approx 41.83^\circ. \tag{9}$$

The relevant triangle is depicted in Figure 3.

Variation 3. The third case is a triangle with side-lengths in harmonic progression (HP) and the largest angle is (as earlier) twice the smallest angle. (Recall that a harmonic progression is obtained by taking the reciprocals of the terms of an arithmetic progression whose terms are all of one sign.) Let the side-lengths be

$$\frac{1}{x-d}, \qquad \frac{1}{x}, \qquad \frac{1}{x+d}, \tag{10}$$

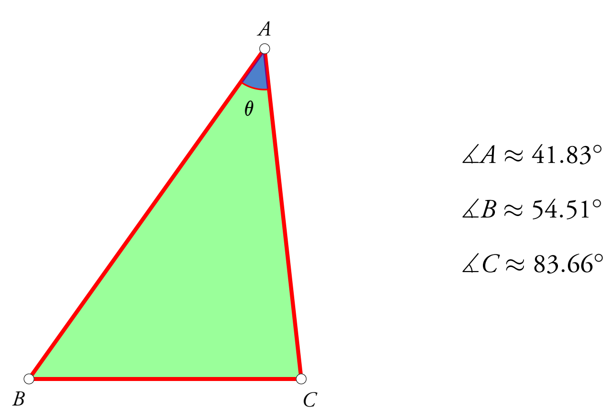


Figure 3. Sides in GP

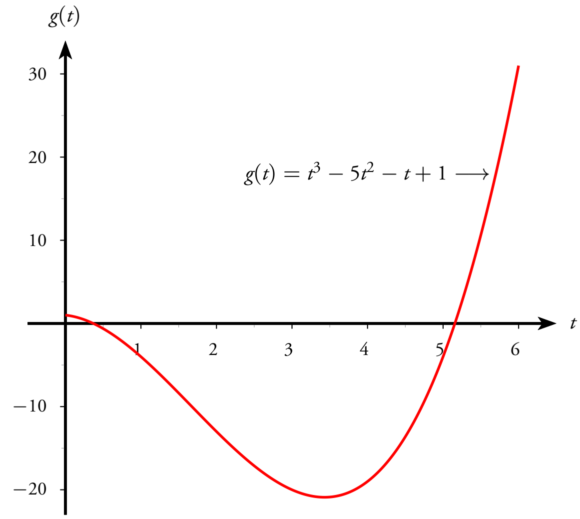


Figure 4. Graph of the function $g(t) = t^3 - 5t^2 - t + 1$

where $0 < d < x$. A triangle exists with these side-lengths if and only if the sum of the two smaller lengths exceeds the largest length, i.e.,

$$\frac{1}{x} + \frac{1}{x+d} > \frac{1}{x-d}. \tag{11}$$

This yields:

$$\frac{1}{x} > \frac{2d}{x^2 - d^2}, \quad \text{i.e.,} \ \ x^2 - d^2 > 2dx, \tag{12}$$

which may be expressed as:

$$t^2 - 2t - 1 > 0, \tag{13}$$

where $t = x/d > 1$. This inequality may be expressed as $(t-1)^2 > 2$; hence it is valid if and only if $t > 1 + \sqrt{2}$. After going through the usual route of sine rule followed by algebraic manipulations, we are left with:

$$\frac{x+d}{x-d} = 2\cos\theta, \quad \frac{x+d}{x} = 4\cos^2\theta - 1. \tag{14}$$

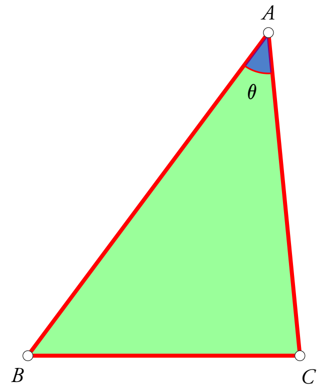
Eliminating θ and setting $t = x/d$, we obtain:

$$t^3 - 5t^2 - t + 1 = 0. \tag{15}$$

Let $g(t) = t^3 - 5t^2 - t + 1$. Figure 4 shows a graph of $g(t)$ for $0 < t < 6$. We see that g has a root between 0 and 1, and another root between 5 and 6. (Alternative proof: $g(0) = 1 > 0$, $g(1) = -4 < 0$, $g(5) = -4 < 0$, $g(6) = 31 > 0$. Now invoke the intermediate value theorem, as earlier.) The former root is not of interest to us, as we know that t exceeds 1; but the latter is, and use of a computer algebra software reveals the root to be approximately equal to 5.1563.

The corresponding approximate magnitude of θ is

$$\theta = \cos^{-1}\frac{t+1}{2(t-1)} \approx 42.22^\circ. \tag{16}$$

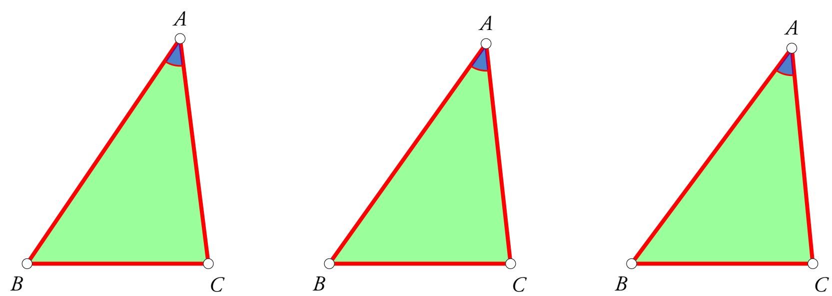


$\measuredangle A \approx 42.22°$

$\measuredangle B \approx 53.35°$

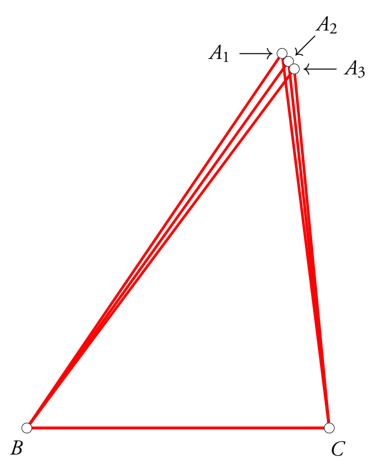
$\measuredangle C \approx 84.44°$

Figure 5. Sides in HP



Sides in AP | Sides in GP | Sides in HP

Figure 6



$\triangle A_1BC$: sides in AP

$\triangle A_2BC$: sides in GP

$\triangle A_3BC$: sides in HP

Figure 7

Since the side-lengths are in the ratio $1/(t+1) : 1/t : 1/(t-1)$, by substituting the value of t, we see that the side-lengths of the required triangle are approximately in the ratio $0.1624 : 0.1939 : 0.2405$. The relevant triangle is depicted in Figure 5.

Figure 6 displays the three triangles next to one another.

Figure 7 shows the three triangles superimposed upon each other, with a common segment as base. Observe how close are the three vertices to one another: A_1, A_2, A_3.

We conclude by saying that in the three different cases, we obtain three triangles whose angles do not differ much from each other.

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