Tonic AMBIGUITY

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Musical tones

A musical tone is the musical perception of a physical sound frequency. It has been empirically found out that there are seven whole tones and five semi-tones. Most of the musical cultures of the world work with these or a sub-set of these. They arrange them into scales, which are hierarchically placed musical tones. Solfege is the notational system for the musical tones. The Indian solfege system denotes the whole tones as Sa, Ri, Ga, Ma, Pa, Dha, Ni and the western counter-part is Do, Re, Mi, Fa, So, La, Ti. A musical note however, is a symbol in the sign system that refers to musical tones.

Tonic pitch

Pitch is the quality of sound that is directly related to the number of vibrations per second of a sound wave. The unit of measurement of pitch is Hertz (Hz). The Tonic Pitch is the pitch against which all other pitches of musical tones are hierarchically referenced. Any pitch can be the Tonic Pitch. Hence it is variable. The musical note referring to this variable tonic-pitch is called 'Do' in the Western solfege system and 'Sa' (short for *Shadjam*) in the Indian solfege system.

Frequency ratios of musical tones

It has been found (right from Pythagorean times) that the frequency of the tonic and the frequencies of the rest of the tones and semi-tones form a simple ratio. A particular musical tone always has the same frequency ratio relationship with the tonic. The western solfege syllables corresponding to Sa, Ri, Ga, Ma, Pa, Dha, Ni are Do, Re, Mi, Fa, So, La, Ti respectively.

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These frequency ratios are given in Table 1.

Sa	Ri	Ga	Ma	Pa	Dha	Ni
1	9/8	5/4	4/3	3/2	5/3	15/8

Table 1

For example if the variable Sa has a frequency value of 440 Hertz, then Ga will have a frequency value of 440 * (5/4) = 550 Hz.

If the variable Sa has a frequency value of 330 Hz, then Ma will have a frequency value of 330 * (4/3) = 440 Hz.

Why these particular ratios?

This has been a speculation right from Pythagorean times. Pythagoras considered that musical tones are good to hear because of their simple ratios. Later Euler formulated a whole theory of consonance and dissonance based on 'degree of agreeableness'. He classified all possible ratios into classes/degrees. But his theory does not seem to predict certain agreeable harmonies.

Ambiguity in perception of tones

There is a possibility that a particular pitch be perceived as different musical tones by different hearers even though the frequency is the same. This arises when the hearers assume different tonic pitches, i.e., in each of their minds the variable Sa has different frequency values.

Let P_1 and P_2 be two frequencies. Let x and y be frequency ratios of two different tones to the tonic pitch.

There can be cases of ambiguity when the same frequency is interpreted as different musical tones. For example,

The pitch 550 Hertz can be with different frequencies of tonic pitches (say 440 Hz and 330 Hz) different musical tones (say Ga and Dha).

550 Hz = 440 Hz * 5/4 = 330 Hz * 5/3

Hence in a case of ambiguity, the same frequency is interpreted as different musical tones (frequency ratios x and y) with frequencies of tonic pitches as P_1 and P_2 respectively.

$$P_1 * x = P_2 * y$$

Let there be another set of frequencies which are also interpreted as different musical tones (with ratios a and b) by two different hearers assuming frequencies of tonic pitches as P_1 and P_2 respectively.

$$P_1 * a = P_2 * b$$

From the above two equations,

$$\frac{x}{y} = \frac{a}{b} \quad \text{or} \quad x * \frac{1}{a} = y * \frac{1}{b}$$

The product grid

To compute the product of x and $1/a$ or y and $1/b$, we make a table with row headings as ratios of frequencies of a tone to its tonic pitch and the reciprocals of the ratios as column headings.

We now make the 'product grid' where value of each cell is the product of the ratio of a particular tone to the tonic and its reciprocal.

For example, the product of Ga and Ri' is same as Dha and Pa'. This means that the tones Ga and Ri of Hearer 1 correspond to Dha and Pa of Hearer 2.

	Ri' (8/9)	Ga' (4/5)	Ma' (3/4)	Pa' (2/3)	Dha' (3/5)	Ni' (8/15)
Ri(9/8)	1	9/10	27/32	3/4	27/40	3/5
Ga(5/4)	10/9	1	15/16	5/6	3/4	2/3
Ma(4/3)	32/27	16/15	1	8/9	4/5	32/45
Pa(3/2)	4/3	2/15	9/8	1	9/10	4/5
Dha(5/3)	40/27	4/3	5/4	10/9	1	8/9
Ni(15/8)	5/3	3/2	45/32	5/4	9/8	1

Table 2: The cells having the same value are colored the same. The column headings are the reciprocals (denoted by an apostrophe)

Let P_1 = 440 Hz, x = Ri (9/8), y = Pa (3/2)

$9/8 * 440 = 3/2 * P_2$

$9*55*2/3 = 330$ Hz $= P_2$

So there exists a frequency of tonic pitch 330 Hz, which along with 440Hz produces ambiguity when 495 Hz (Ri or Pa) and 550 Hz (Ga or Dha) are played.

The cells having the same value are colored the same. These pairs of cells correspond to the tone pairs (x, a) and (y, b) which are interpretations of Hearers 1 and 2 respectively with frequencies of tonic pitches P_1 and P_2.

Permutations

The equation

$$x * \frac{1}{a} = y * \frac{1}{b}$$

Can also be written as

$$x * \frac{1}{y} = a * \frac{1}{b}$$

This means that (x, a) of Hearer1 can be interpreted as (y, b) by Hearer2. So the frequency ratio pair (x, y) can be interpreted as (a, b).

For example, if Ri and Ga of Hearer1 are Pa and Dha of Hearer2 then Ri and Pa of Hearer1 are Ga and Dha of Hearer2.

Semitones

The frequency ratios of the semitones are:

ri	ga	ma	dha	ni
16/15	6/5	7/5	8/5	7/4

Table 3

The product grid including the semitones with the whole tones is:

	ri' 15/16	Ri' 8/9	ga' 5/6	Ga' 4/5	Ma' 3/4	ma' 5/7	Pa' 2/3	dha' 5/8	Dha' 3/5	ni' 4/7	Ni' 8/15
ri 16/15	1	128/135	8/9	64/75	4/5	16/21	32/45	2/3	16/25	64/105	128/225
Ri 9/8	135/128	1	15/16	9/10	27/32	45/56	3/4	45/64	27/40	9/14	3/5
ga 6/5	9/8	16/15	1	24/25	9/10	6/7	4/5	3/4	18/25	24/35	48/75
Ga 5/4	75/64	10/9	25/24	1	15/16	25/28	5/6	25/32	3/4	5/7	2/3
Ma 4/3	5/4	32/27	10/9	16/15	1	20/21	8/9	5/6	4/5	16/21	32/45
ma 7/5	21/16	56/45	7/6	28/25	21/20	1	14/15	7/8	21/25	4/5	56/75
Pa 3/2	45/32	4/3	5/4	6/5	9/8	15/14	1	15/16	9/10	6/7	4/5
dha 8/5	3/2	64/45	4/3	32/25	6/5	8/7	16/15	1	24/25	32/35	64/75
Dha 5/3	75/48	40/27	25/18	4/3	5/4	25/21	10/9	25/24	1	20/21	8/9
ni 7/4	105/64	14/9	35/24	7/5	21/16	5/4	7/6	35/32	21/20	1	14/15
Ni 15/8	225/128	5/3	25/16	3/2	45/32	75/56	5/4	75/64	9/8	15/14	1

Table 4

In this product grid, it is surprising that we find more than 2 cells with the same value. This correspondingly means that there are more interpretations for the two pitch frequencies.

- There are 2, 3 and 5 cells with the same value.
- There are no 4 cells with the same value.

Glossary

- *Pitch*: Distinctive quality of a sound, dependent primarily on the frequency of the sound waves produced by its source.
- *Note*: A symbol for a tone, indicating pitch.
- *Swara*: It is a Sanskrit word that denotes a note in the successive steps of the octave. Swaras are the selected pitches from which the musician constructs the scales, melodies. The seven notes of the musical scale in Indian classical music are shadjam (Sa), rishabham (Ri), gandharam (Ga), madhyamam (Ma), panchamam (Pa), dhaivatham (Dha) and nishadham (Ni).

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