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Functional Equations

Part II

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In Part 1, I explained the different techniques to solve functional equations. In most cases, the steps to solve them are similar to those in solving algebraic equations. However there is one scenario where using the algebraic method to solve a functional equation may lead to an incomplete solution; i.e., only a subset of functions that solve the FE may be identified and not the complete list. This error is known as *pointwise trap*. In Part II of the two-part article on functional equations, we discuss the notion of pointwise trap and learn different ways to overcome it.

Pointwise Trap

What is a 'pointwise trap'? Consider the following example:

Find all functions $f: \mathbb{R} \to \mathbb{R}$ such that $f(x)^2 = xf(x)$.

Factorising the equation $f(x)^2 - xf(x) = 0$ gives $f(x)(f(x) - x) = 0$. Solving it like a normal equation would yield $f(x) = 0$ for all x **or** $f(x) = x$ for all x.

Is this the complete list of functions that solve the FE?

Consider a function $f(x)$ defined as follows: $f(x) = x$ for all x except $f(5) = 0$. If $x \neq 5$, then $f(x) - x = 0 \implies f(x)^2 = xf(x)$. If $x = 5$, then $f(x) = 0 \implies f(x)^2 = xf(x)$. So clearly this function also satisfies the FE. This proves that our earlier list of functions that satisfy the FE was incomplete.

It is not too difficult to deduce that there are infinitely many solutions to this FE. Refer Figure 1 as another example of such a function that satisfies the FE.

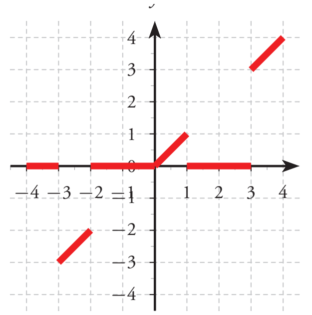

Figure 1. Example of a function solving the equation $f(x)^2 = xf(x)$

Keywords: Functional equation, function, domain, range, injective, surjective, even, odd, continuous, pointwise trap, solution

This mistake is known as a *pointwise trap*. In an equation, if $y(y-5)=0$, then $y=0$ or $y=5$ are the solutions and that is correct. However, if we blindly apply this approach to solve the FE, it leads to *pointwise trap*.

Properties of functions used to deal with the trap. In cases when an additional condition is imposed on the function, solving the problem becomes easier. We consider what happens when the following conditions are imposed separately: continuity, monotonicity, injectivity and surjectivity.

Continuity

Example 1. Find all continuous functions $f:\mathbb{R}\to\mathbb{R}$ satisfying the following condition: $f(x)^2=xf(x)$ for all $x\in\mathbb{R}$.

Solution 1. After factorisation, we get $f(x)(f(x)-x)=0\Rightarrow f(x)=0$ or $f(x)=x$ for all x. This means that for every real number x, either $f(x)=0$ or $f(x)=x$. In particular, we have $f(0)=0$. The trivial solution is $f(x)=0$ for all x.

In order to see if there is a non-trivial solution, assume that there is some real $r\neq 0$ such that $f(r)=r$. Suppose firstly that $r>0$. Then, continuity coupled with the condition $f(x)(f(x)-x)=0$ implies that $f(x)=x$ for all $x>0$. Next, suppose that $r<0$. Then, arguing exactly as earlier, we see that $f(x)=x$ for all $x<0$. This line of reasoning leads to four possible solutions:

- $f(x)=0$ for all x;
- $f(x)=x$ for all x;
- $f(x)=x$ for all $x\geq 0$ and $f(x)=0$ for all $x<0$;
- $f(x)=x$ for all $x\leq 0$ and $f(x)=0$ for all $x>0$.

So the continuity condition has brought down the number of solutions to just four (refer Figure 2).

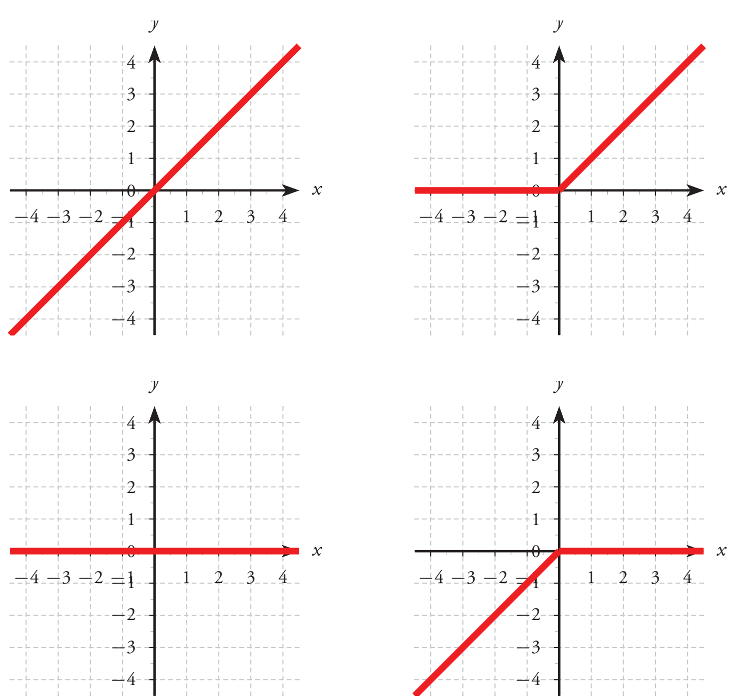

Figure 2. Solutions to example 1

Remark. A casual look at the equation may suggest that a possible solution is $f(x) = |x|$. Note that it satisfies the continuity requirement. However, it is easy to check that it does not satisfy the given condition for $x < 0$. (The quantity on the left side simplifies to x^2, whereas the quantity on the right side simplifies to $-x^2$.)

Monotonicity

Example 2. Find all strictly increasing functions $f\colon \mathbb{R} \to \mathbb{R}$ satisfying the condition $f(x)^2 = xf(x)$ for all $x \in \mathbb{R}$.

Solution 2. After factorisation, we get $f(x)(f(x) - x) = 0 \Rightarrow f(x) = 0$ or $f(x) = x$ for all x. This means that for every real number x, either $f(x) = 0$ or $f(x) = x$. In particular, we have $f(0) = 0$. From the given condition, we have $f(x) > f(y)$ for all real $x > y$. Plugging $y = 0$, we have $f(x) > 0$ for $x > 0$ and plugging $x = 0$ gives $f(y) < 0$ for $y < 0$. Hence we have:

$$f(x) = x \text{ for all } x \in \mathbb{R}.$$

So the strict monotonicity condition has brought down the number of solutions to just one (refer Figure 3).

Injectivity

Example 3. Find all injective functions $f\colon \mathbb{R} \to \mathbb{R}$ satisfying the condition $f(x)^2 = xf(x)$ for all $x \in \mathbb{R}$.

Solution 3. After factorisation, we get $f(x)(f(x) - x) = 0 \Rightarrow f(x) = 0$ or $f(x) = x$ for all x. This means that for every real number x, either $f(x) = 0$ or $f(x) = x$. In particular, we have $f(0) = 0$. Injectivity implies that there cannot be $a \neq 0$ such that $f(a) = 0$. Hence we have:

$$f(x) = x \text{ for all } x \in \mathbb{R}.$$

So the injectivity condition has brought down the number of solutions to just one (refer Figure 3).

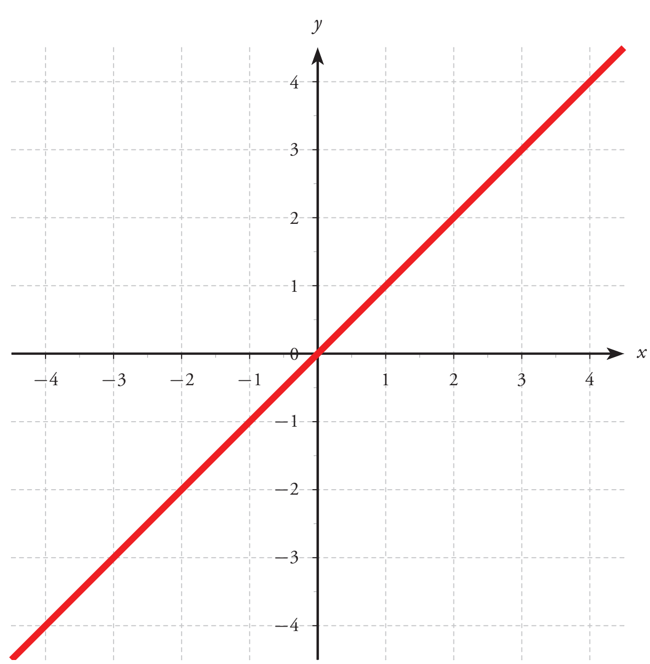

Figure 3. Solution to examples 2, 3, 4

Surjectivity

Example 4. Find all surjective functions $f\colon \mathbb{R} \to \mathbb{R}$ satisfying the condition $f(x)^2 = xf(x)$ for all $x \in \mathbb{R}$.

Solution 4. After factorisation, we get $f(x)(f(x) - x) = 0 \Rightarrow f(x) = 0$ or $f(x) = x$ for all x. This means that for every real number x, either $f(x) = 0$ or $f(x) = x$. In particular, we have $f(0) = 0$.

Next, from surjectivity we realize that for every real b there exists a real a such that $f(a) = b$. Suppose $b \neq 0$; then, using the condition that $f(x) = 0$ or $f(x) = x$ for each x, we deduce that $a = b$ and hence that $f(x) = x$ is the only solution. Hence we have:

$$f(x) = x \text{ for all } x \in \mathbb{R}.$$

So the surjectivity condition has brought down the number of solutions to just one (refer Figure 3).

Use of contradiction in solving the pointwise trap. Another powerful technique to resolve pointwise trap is contradiction, particularly in multi-variable functional equations. Suppose in a multivariate FE, we end up getting $f(x) = g(x)$ or $f(x) = h(x)$ for each x (in the above examples, $g(x) = 0$ and $h(x) = x$). In such a case, we can use contradiction to prove that $f(x) = g(x)$ for all x or $f(x) = h(x)$ for all x are the only solutions, in the following manner. We assume for the sake of contradiction that there are distinct elements a, b in the domain such that $f(a) = g(a)$ and $f(b) = h(b)$. After that, we cleverly plug a, b either in the original equation or its variant to prove that $a = b$. This contradicts our assumption, implying that the only solutions are $f(x) = g(x)$ for all x and $f(x) = h(x)$ for all x.

Example Problems

Example 5. Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that

$$f(x^2) + f(xy) = f(x)f(y) + yf(x) + xf(x+y)$$

for all $x, y \in \mathbb{R}$.

Solution 5. Let $P(x, y)$ be the assertion of the problem statement.

$P(0, 0)$ gives $2f(0) = f(0)^2 \Rightarrow f(0) = 0$ or $f(0) = 2$.

Case 1: $f(0) = 2$. $P(0, y)$ gives $2 + 2 = 2f(y) + 2y \Rightarrow f(y) = 2 - y$. On checking, we find that $f(y) = 2 - y$ does indeed solve the problem.

Case 2: $f(0) = 0$. $P(x, 0)$ gives $f(x^2) = xf(x)$. $P(-x, 0)$ gives $f(x^2) = -xf(-x)$. Thus

$$xf(x) = f(x^2) = -xf(-x) \Rightarrow x(f(x) + f(-x)) = 0.$$

For $x \neq 0$, $f(x) = -f(-x)$, therefore f is odd.

Now that we know that f is odd, it might be worth putting $y = -x$. We then get:

$$0 = f(x^2) + f(-x^2) = f(x)f(-x) - xf(x) = -f(x)^2 - xf(x)$$

$$\therefore \quad 0 = f(x)^2 + xf(x) = f(x)(f(x) + x).$$

Hence $f(x) = 0$ or $f(x) = -x$ for each value of x.

This is the pointwise trap. We must avoid the trap using contradiction.

Clearly $f(x) = 0$ for all x is a trivial solution. Let $f(a) = 0$ and $f(b) = -b$ where $a, b \neq 0$. We plug these values in the original equation and try to work towards a contradiction.

We notice that there are a lot of $f(x)$ terms which may need to be eliminated to simplify the problem. We plug in $x = a$. Note that $f(a^2) = af(a) = 0$. Hence we have:

$$P(a, y): \quad f(ay) = af(a+y).$$

There are two f-terms here: $f(ay)$ and $f(a+y)$. We use the fact that $f(b) = -b$ and select a value of y such that the latter term becomes $f(b)$:

$$P(a, b-a): \quad f(a(b-a)) = af(a+(b-a)) = af(b) = -ab,$$

so

$$f\left(ab - a^2\right) = -ab.$$

But we know that $f(ab - a^2) = 0$ or $a^2 - ab$. This implies that either $0 = -ab$ or

$$a^2 - ab = -ab, \quad \therefore \ ab = 0.$$

So either $ab = 0$ or $a = 0$. But we had said at the start that both $a, b \neq 0$, so we have a contradiction. Hence it cannot be that $f(a) = 0$ and $f(b) = -b$ where $a, b \neq 0$. Therefore, the possible solutions are:

- $f(x) = 0$ for all x;
- $f(x) = -x$ for all x;
- $f(x) = 2 - x$ for all x.

It is easy to check by substitution that these are indeed solutions to the given problem.

Example 6. Find all functions $f\colon \mathbb{Z} \to \mathbb{Z}$ such that

$$f(n^2) = f(n+m)f(n-m) + m^2$$

for all $m, n \in \mathbb{Z}$.

Solution 6. Let $P(m, n)$ be the assertion of the problem statement. The problem looks like the identity $a^2 - b^2 = (a-b)(a+b)$. The solution to the problem seems to be $f(n) = n$ for all n. We proceed in a similar way.

$P(0, 0)$ gives $f(0) = f(0)^2 \Rightarrow f(0) = 0$ or 1.

Case 1: $f(0) = 1$. We need to eliminate the product term in the RHS. $P(n, n)$ gives $f(n^2) = f(2n) + n^2$. Putting $n = 2, f(4) = f(4) + 4$ which is not possible and we have no solution in this case. So we eliminate this possibility.

Case 2: $f(0) = 0$. $P(n, n)$ gives $f(n^2) = n^2$, hence

$$(n-m)(n+m) = n^2 - m^2 = f(n+m)f(n-m).$$

We cannot conclude from this that $pq = f(p)f(q)$ for all p, q as we are dealing with integers and $n + m$ and $n - m$ are not independent. Let $a = n - m$ and $b = n + m$. This means that a, b are of the same parity, hence $f(a)f(b) = ab$ for $a \equiv b \pmod 2$. We'll try using the original condition a little more. $P(n, 0)$ gives $f(n^2) = f(n)^2$. We also have $f(n^2) = n^2$. Hence $f(n)^2 = n^2$, so:

$$f(n) = \pm n.$$

This is the pointwise trap in this problem. For the sake of contradiction, assume that there exists an integer $k \neq 0$ (why can we assume this?) such that $f(k) = -k$. We know that $f(a)f(b) = ab$ for $a \equiv b \pmod 2$. Suppose m is any integer with the same parity as k. We have
$km = f(k)f(m) = -kf(m) \implies f(m) = -m$ for all m with the same parity as k, but we had $f(n^2) = n^2$.

Consider any perfect square $r^2 \neq 0$ with the same parity as k. For example, if k is odd, then we can choose $r^2 = 9$; if k is even, then we can choose $r^2 = 16$. Therefore

$$r^2 = f(r^2) = -r^2,$$

hence $r = 0$, which gives us a contradiction.

This means that there doesn't exist any k such that $f(k) = -k$.

Hence the answer is: $f(n) = n$ for all n which is indeed a solution to the FE.

Practice Problems

(1) (Japan MO Final 2004) Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that for all $x, y \in \mathbb{R}$,

$$f(xf(x) + f(y)) = f(x)^2 + y.$$

(2) (Iran 1999) Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that for all $x, y \in \mathbb{R}$,

$$f(x^2 + y) = f(f(x) - y) + 4f(x)y.$$

(3) Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that for all $x, y \in \mathbb{R}$,

$$xf(y) + yf(x) = 2f(x)f(y).$$

(4) Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that for all $x, y \in \mathbb{R}$,

$$f(x)f(x - y) + f(y)f(x + y) = x^2 + f(y)^2.$$

Note. Solutions to these problems will be given in the next issue (March 2019). Solutions to the practice problems posed in Part 1 of the article are given elsewhere in this issue.

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