Azim Premji University

A publication of Azim Premji University together with Community Mathematics Centre, Rishi Valley

At Right Angles

A RESOURCE FOR SCHOOL MATHEMATICS

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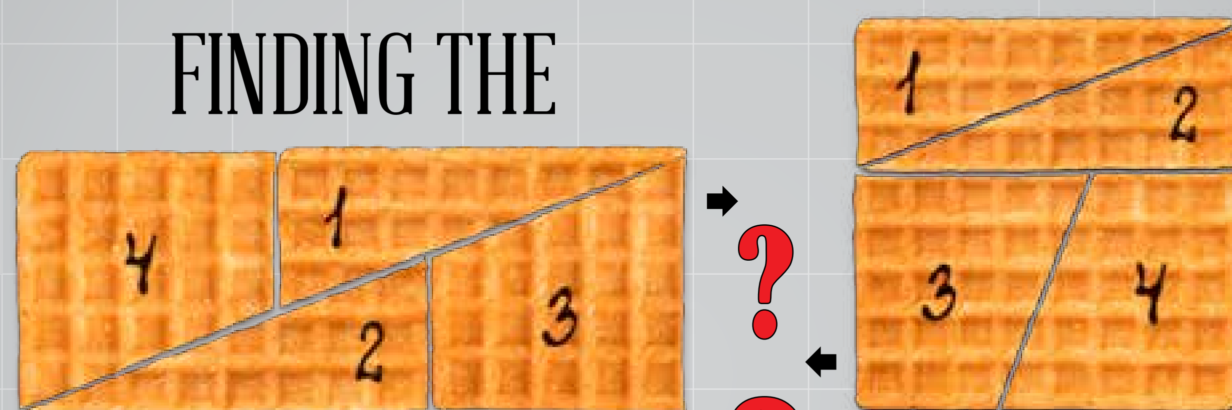
November 2016

CHILDREN DOING MATH

FINDING THE



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PULLOUT
Data Handling

CHILDREN DOING MATH

'Children doing math' is a cause for celebration at At Right Angles, and we are delighted to carry not one or two but three articles by students in this appropriately timed November issue which coincides with the celebration of Children's Day in India. What does it take to engage students with this subject? Why are there reams written about math phobia and millions spent on out-of-class engagement with the subject – unfortunately, most of it to do with drill and dull repeated practice? For our three contributors, mathematics is clearly more than that – it is a subject to play with, to think about and to build on.

Featured here are Muralidharan Somasundaram of Mahila Samithi School, Thane; Rohini Lakshmi of Pushpalata Vidyamandir, Tirunelveli; and Ishan Magon, Maya Reddy and Rishabh Suresh of Centre for Learning, Bangalore. Kids who engage, who think out of the box, who perceive, pursue and pin down patterns, who use their skills of visualization, representation and generalization to make sense of their experiences – may we find the missing link, that piece of the puzzle that increases this tribe of math warriors!

- Sneha Titus

From the Editor's Desk . . .

November 14th is celebrated as Children's Day in India and fortuitously, At Right Angles had a sudden influx of articles by students (*see facing page*). We took a journey down memory lane to 2012 when our Chief Editor, Dr. Shailesh Shirali penned the following rationale for the magazine.

In recent months, two news items have come to light which I believe to be significant. One was about how some pupils from a primary school in Devon, U.K., investigated how bumblebees see colours and patterns and published their findings in Biology Letters, a journal published by the Royal Society: http://www.physorg.com/news/2010-12-british-eight-year-olds-publish-science-journal.html

The other item was about how three middle school children who collaborated with their mathematics teacher, found something fresh and new in the world of mathematics (related to Pascal's triangle), and published a paper called "The Rascal Triangle" in the College Mathematics Journal of the Math Association of America (Nov 2010): http://www.maa.org/pubs/cmj_nov10-Rascal.html

I believe such phenomena are highly unlikely to ever take place in India – not because our children are not capable of original discovery, but because the prevailing educational climate in the country and the lack of opportunity and availability of platforms make it nearly impossible.

I believe there is a serious need to create viable platforms – in print and on the internet – where teachers and children from schools across the country can read about matters not in the regular school curriculum, where they can solve problems, where they can interact with one another, and where they can contribute; where individuals can share their original observations and discoveries; where beautiful results in school level mathematics can be written about and discussed.

Four years down the line and with this, our thirteenth issue, we have a joyous sense of satisfaction – in the March 2016 issue, we carried the article about the Rascal Triangle and lo and behold, three students of a school in Bangalore have built on it and presented their findings in this issue. We are both delighted to provide the platform to present this and challenged to extend our reach to more and more students of mathematics.

In this issue, Shailesh Shirali presents the beautiful Napoleon's theorem and its proof, and Marcus Bizony discovers even more treasures in the 3-4-5 triangle. CoMaC explains a shorter test for Divisibility by 8 and picks up on one of the posts in AtRiUM to explain Approximate Angle Constructions. And we carry an article on Dyscalculia by Pooja Singh – a first for us, well in keeping with understanding children doing math. We have hands on activities in both the Low Floor High Ceiling article on Circular Solids and in Tech Space; where Jonaki Ghosh works magic with Sierpinski triangles. A Ramachandran has some useful observations and practical suggestions for Middle School Mathematics, we are also happy to announce a rework of our Middle School Problem Corner section – we have a handy reference sheet for those middle schoolers who would like to engage with problems based on parity and divisibility. For some time now, this section had been becoming a closed space which only 'insiders' could access; we are making a deliberate attempt to make this a more welcoming and comfortable corner.

Problem Corner has plenty of challenges to offer, some of which have been taken up by our readers.

From the
Editor's Desk . . .(Contd.,)

Two contributions have been featured here. Review continues to give us glimpses of mathematicians worlds, this time we feature Ramesh Sreekantan's review of Ken Ono's book, 'My Search for Ramanujan: how I learned to count'.

And we close with Padmapriya Shirali's PullOut on Statistics for Primary School. Readers will be happy to know that we now have compilations of Pullouts 1-7 in Hindi, Kannada and English. You may write in with your complete postal address to AtRightAngles@apu.edu.in if you would like a copy of the PullOut compilation.

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At Right Angles is a publication of Azim Premji University together with Community Mathematics Centre, Rishi Valley School and Sahyadri School (KFI). It aims to reach out to teachers, teacher educators, students & those who are passionate about mathematics. It provides a platform for the expression of varied opinions & perspectives and encourages new and informed positions, thought-provoking points of view and stories of innovation. The approach is a balance between being an 'academic' and 'practitioner' oriented magazine.

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Contents

Features

Our leading section has articles which are focused on mathematical content in both pure and applied mathematics. The themes vary: from little known proofs of well-known theorems to proofs without words; from the mathematics concealed in paper folding to the significance of mathematics in the world we live in; from historical perspectives to current developments in the field of mathematics. Written by practising mathematicians, the common thread is the joy of sharing discoveries and the investigative approaches leading to them.

ClassRoom

This section gives you a 'fly on the wall' classroom experience. With articles that deal with issues of pedagogy, teaching methodology and classroom teaching, it takes you to the hot seat of mathematics education. ClassRoom is meant for practising teachers and teacher educators. Articles are sometimes anecdotal; or about how to teach a topic or concept in a different way. They often take a new look at assessment or at projects; discuss how to anchor a math club or math expo; offer insights into remedial teaching etc.

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TechSpace

'This section includes articles which emphasise the use of technology for exploring and visualizing a wide range of mathematical ideas and concepts. The thrust is on presenting materials and activities which will empower the teacher to enhance instruction through technology as well enable the student to use the possibilities offered by technology to develop mathematical thinking. The content of the section is generally based on mathematical software such dynamic geometry software (DGS), computer algebra systems (CAS), spreadsheets, calculators as well as open source online resources. Written by practising mathematicians and teachers, the focus is on technology enabled explorations which can be easily integrated in the classroom.

Problem Corner

Review

We are fortunate that there are excellent books available that attempt to convey the power and beauty of mathematics to a lay audience. We hope in this section to review a variety of books: classic texts in school mathematics, biographies, historical accounts of mathematics, popular expositions. We will also review books on mathematics education, how best to teach mathematics, material on recreational mathematics, interesting websites and educational software. The idea is for reviewers to open up the multidimensional world of mathematics for students and teachers, while at the same time bringing their own knowledge and understanding to bear on the theme.

PullOut

The PullOut is the part of the magazine that is aimed at the primary school teacher. It takes a hands-on, activity-based approach to the teaching of the basic concepts in mathematics. This section deals with common misconceptions and how to address them, manipulatives and how to use them to maximize student understanding and mathematical skill development; and, best of all, how to incorporate writing and documentation skills into activity-based learning. The PullOut is theme-based and, as its name suggests, can be used separately from the main magazine in a different section of the school.

Features

Complex Napoleon's Theorem Part 1 Made Simple

In this article we discuss a gem from Euclidean geometry that was discovered in post-revolution France, a result known as Napoleon's theorem.

Shailesh Shirali

It is rare to come across a result belonging to Euclidean geometry which does not date all the way back to some ancient Greek mathematician (Euclid; Pythagoras; Thales; Archimedes; Apollonius), or some ancient Indian mathematician (Brahmagupta). One such result—Morley's theorem—has been the subject of a three-part series of articles in earlier issues of this magazine. In this note, which will also be in three parts, we discuss another such gem whose discovery goes back to nineteenth century France: a result known as *Napoleon's theorem*. The feature it shares with Morley's theorem is the unexpected occurrence of an equilateral triangle within a given triangle. However, it is far easier to prove than Morley's theorem. That makes it particularly attractive for us; it means that students of classes 11-12 would be able to understand the proof without much difficulty.

You may be puzzled by the name of this theorem: *Napoleon's theorem*. Which Napoleon is this, you may wonder. Well, the reference is indeed to Napoleon Bonaparté, who was known to be a patron of both the sciences and mathematics; he generally moved around with an entourage of scientists and mathematicians, including scholars as established as Fourier,

Keywords: *Napoleon, equilateral triangle, complex number, rotation, cosine rule, area of triangle, cube roots of unity*

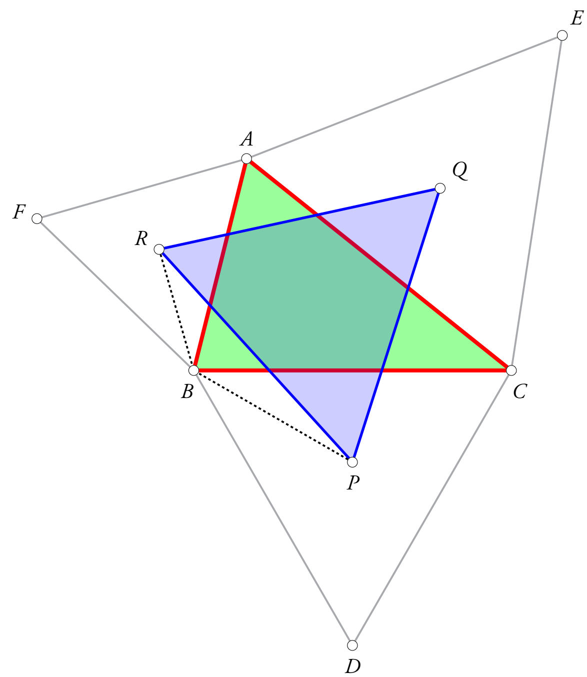


Figure 1

Laplace and Lagrange. The attribution of the theorem to Napoleon may derive from this fact. Other than this, there is no evidence that Napoleon knew of the result that would one day be attributed to him. (It is of course possible that he himself stumbled upon the discovery. Let us not be unfair to him ....)

The statement of the theorem is given below (Box 1).

Napoleon's theorem

Let ABC be an arbitrary triangle. With the three sides of the triangle as bases, construct three equilateral triangles, each one outside $\triangle ABC$. Next, mark the centres P, Q, R of these three equilateral triangles. Napoleon's theorem asserts that $\triangle PQR$ is equilateral, irrespective of the shape of $\triangle ABC$. (See Figure 1.)

Box 1

A trigonometric proof. Perhaps the most straightforward proof of the result is computational, through the use of trigonometry. We consider $\triangle BPR$ and compute the length of side PR, using the cosine rule. In the derivation below, we use the standard short forms for the elements of $\triangle ABC$ (a, b, c for the lengths of the sides, $s = (a+b+c)/2$ for the semi-perimeter, Δ for the area of the triangle, and so on). Here are the steps. Using the cosine rule in $\triangle BPR$ we get:

$$PR^2 = BP^2 + BR^2 - 2BP \cdot BR \cdot \cos \measuredangle RBP.$$

Since $\measuredangle RBP = B + 60^\circ$, we have:

$$\cos \measuredangle RBP = \cos(B + 60^\circ) = \frac{\cos B}{2} - \frac{\sqrt{3}\,\sin B}{2}.$$

We also have:

$$BP = \frac{2}{3} \times \text{altitude of triangle } DCB = \frac{2}{3} \times \frac{\sqrt{3}}{2}a = \frac{a}{\sqrt{3}},$$

$$BR = \frac{2}{3} \times \text{altitude of triangle } FBA = \frac{2}{3} \times \frac{\sqrt{3}}{2}c = \frac{c}{\sqrt{3}}.$$

Hence:

$$\begin{aligned} PR^2 &= \frac{a^2}{3} + \frac{c^2}{3} - \frac{2ac}{3}\left(\frac{\cos B}{2} - \frac{\sqrt{3}\sin B}{2}\right) \\ &= \frac{a^2}{3} + \frac{c^2}{3} - \frac{ac\cos B}{3} + \frac{ac\sin B}{\sqrt{3}}. \end{aligned}$$

Next we have, using the cosine rule:

$$2ac\cos B = c^2 + a^2 - b^2.$$

Also, one of the formulas for the area of a triangle ("area of triangle equals half the product of any two sides and the sine of the included angle") yields:

$$\Delta = \frac{1}{2}ac\sin B.$$

Hence:

$$PR^2 = \frac{a^2}{3} + \frac{c^2}{3} - \frac{c^2 + a^2 - b^2}{6} + \frac{2\Delta}{\sqrt{3}},$$

and this simplifies to:

$$PR^2 = \frac{a^2 + b^2 + c^2}{6} + \frac{2\Delta}{\sqrt{3}}.$$

The crucial aspect of the above result is that the expression for PR^2 is ***symmetric*** in a, b, c. This tells us that we will get exactly the same expression for QR^2 as well as PQ^2. It follows that $PQ = QR = RP$, i.e., $\triangle PQR$ is equilateral. □

This proof is purely computational. Such proofs are not to the liking of all readers, but they certainly accomplish whatever is desired; we cannot fault them in any way. So while we are at it, we give another such proof!

A proof using complex numbers. We use the following elegant result which comes from the geometry of complex numbers. Let A, B, C be three distinct points such that in $\triangle ABC$, the direction $[A, B, C, A]$ is counterclockwise (see Figure 2). Let a, b, c denote the complex numbers which represent the points A, B, C respectively. Then if $\triangle ABC$ is equilateral, we have:

$$a + bw + cw^2 = 0, \tag{1}$$

where $w = \cos 120° + i\sin 120°$ is that complex cube root of unity which has argument 120°. Since $w^3 = 1$, relation (1) can be written in the following equivalent forms:

$$c + aw + bw^2 = 0, \quad b + cw + aw^2 = 0.$$

To see why these relations are true, recall the geometrical meaning of multiplication by a complex number. Multiplication by $\cos\theta + i\sin\theta$ accomplishes rotation around the origin through an angle θ, in a counterclockwise direction; so in particular:

- multiplication by w accomplishes rotation by $120°$ in a counterclockwise direction;

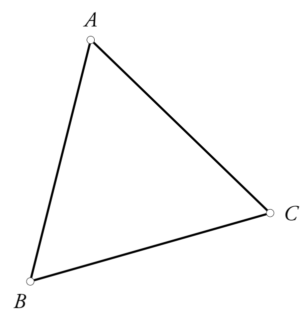


Figure 2

- multiplication by w^2 accomplishes rotation by $240°$ in a counterclockwise direction, which is the same as rotation by $120°$ in a clockwise direction;
- multiplication by $-w^2$ accomplishes rotation by $60°$ in a counterclockwise direction (because $-w^2 = \cos 60° + i\sin 60°$);
- multiplication by $-w$ accomplishes rotation by $60°$ in a clockwise direction.

In each case, the rotation is about the origin.

Referring to Figure 2, since rotation about B through a $60°$ angle in the counterclockwise direction takes C to A, it follows that

$$a - b = -w^2(c - b).$$

This relation may be written as:

$$a - \left(1 + w^2\right) b + w^2 c = 0.$$

Since $1 + w^2 = -w$, this yields $a + wb + w^2c = 0$, as claimed. $\square$

The proof as presented can be reversed at every step; please check that this is so. This means that the following converse statement is true as well: if a, b, c are distinct complex numbers such that $a + wb + w^2c = 0$, then $\triangle ABC$ with vertices A, B, C corresponding to a, b, c (respectively) is equilateral. (More can be said: the orientation of the cycle $[A, B, C, A]$ will be *counterclockwise*; but generally we are not concerned by this part of the result.) It is this converse statement which comes of use in proving Napoleon's theorem.

The reader will no doubt notice a lack of symmetry in the relations,

$$a + wb + w^2c = 0, \quad c + aw + bw^2 = 0, \quad b + cw + aw^2 = 0; \tag{2}$$

namely, they do not treat a, b, c 'equally.' But it is easy to see the reason for the lack of symmetry: it stems from the assumption that when we traverse the cycle $[A, B, C, A]$, we travel in a *counterclockwise* direction. Once we realise this, we see immediately that the following result is true as well: If $\triangle ABC$ is such that the direction $[A, B, C, A]$ is clockwise, and $\triangle ABC$ is equilateral, then the following equalities must hold:

$$a + bw^2 + cw = 0, \qquad b + cw^2 + aw = 0, \qquad c + aw^2 + bw = 0. \tag{3}$$

As earlier, the converse proposition is true as well. Note again the lack of symmetry in these relations: they do not treat a, b, c equally.

If we bring together these two non-symmetric results, we obtain a result which is fully symmetric in a, b, c. We have seen that if $\triangle ABC$ is equilateral and the direction $[A, B, C, A]$ is counterclockwise, then $a + bw + cw^2 = 0$; and if the direction $[A, B, C, A]$ is clockwise, then $a + bw^2 + cw = 0$. These two statements taken together imply the following: if $\triangle ABC$ is equilateral, then

$$\left(a + bw + cw^2\right) \cdot \left(a + bw^2 + cw\right) = 0.$$

Moreover, the converse statement must be true as well; if the above product is 0, then one of the two bracketed terms must be 0, hence $\triangle ABC$ must be equilateral. If we multiply out the above product, the result comes as a surprise. We obtain:

$$a^2 + b^2 + c^2 - ab - bc - ca = 0. \tag{4}$$

Note that we have obtained a relation which is fully symmetric in a, b, c! So we obtain the following result as a bonus: if a, b, c are such that $a^2 + b^2 + c^2 = ab + bc + ca$, then $\triangle ABC$ with vertices A, B, C at the points represented by a, b, c is equilateral. Box 2 summarises all these results.

Conditions that make a triangle equilateral

Let A, B, C be three distinct points in the coordinate plane, and let the complex numbers representing these points be a, b, c. The following claims may now be made:

- If the direction $[A, B, C, A]$ is counterclockwise, then $\triangle ABC$ is equilateral if and only if $a + bw + cw^2 = 0$; equivalently, if and only if each of the following equalities holds:

$$a + bw + cw^2 = 0, \quad c + aw + bw^2 = 0, \quad b + cw + aw^2 = 0.$$

- If the direction $[A, B, C, A]$ is clockwise, then $\triangle ABC$ is equilateral if and only if $a + bw^2 + cw = 0$; equivalently, if and only if each of the following equalities holds:

$$a + bw^2 + cw = 0, \quad c + aw^2 + bw = 0, \quad b + cw^2 + aw = 0.$$

- $\triangle ABC$ is equilateral if and only if $\left(a + bw + cw^2\right) \cdot \left(a + bw^2 + cw\right) = 0$, i.e., if and only if

$$a^2 + b^2 + c^2 - ab - bc - ca = 0.$$

- The above may also be written as:

$$\triangle ABC \text{ equilateral} \iff (a - b)^2 + (b - c)^2 + (c - a)^2 = 0.$$

Box 2

Before moving on, we note that equality (4) can be written in the following still more elegant form:

$$(a - b)^2 + (b - c)^2 + (c - a)^2 = 0. \tag{5}$$

If a, b, c are ***real*** numbers, then equality (5) holds if and only if $a = b = c$. It is striking that the shift from the real domain to the complex domain can result in so dramatic a change in conclusion.

Let us use these findings to prove Napoleon's theorem. We refer to Figure 1 and use lower case letters to denote the complex numbers representing the respective points (a for A, b for B, ...). Noting carefully the orientations of the various triangles, we obtain the following:

$$d + wc + w^2 b = 0,$$

$$c + we + w^2 a = 0,$$

$$b + wa + w^2 f = 0.$$

We also have (since P, Q, R are the centroids of the respective triangles):

$$3p = b + d + c, \quad 3q = c + e + a, \quad 3r = a + f + b.$$

To prove that $\triangle PQR$ is equilateral, we must prove that $p + wq + w^2 r = 0$. Hence we must prove that

$$(b + d + c) + w(c + e + a) + w^2(a + f + b) = 0.$$

This is equivalent to proving that:

$$\left(d + wc + w^2 b\right) + \left(c + we + w^2 a\right) + \left(b + wa + w^2 f\right) = 0.$$

But this is immediate, since each of the bracketed terms is itself equal to 0. Hence the conclusion follows, that $\triangle PQR$ is equilateral.

Box 3 describes the basic strategy followed in this proof. □

Strategy for proving Napoleon's theorem (outline)

- We use relation (1) which connects the complex numbers representing the vertices of an equilateral triangle and apply it to the three constructed equilateral triangles.
- Then we find expressions for the centroids of the three equilateral triangles in terms of the complex numbers representing the vertices of the triangle.
- Finally, we use the converse of relation (1) to arrive at the desired result.

Box 3

Yet another computational proof. Another approach, involving more manipulations than the one above, is to obtain explicit expressions for p, q, r. We first obtain an expression for d. Since rotation about C through 60° (counterclockwise) takes B to D,

$$d - c = \left(-w^2\right)(b - c),$$

which yields $d = -w^2 b + \left(1 + w^2\right) c$, i.e.,

$$d = -w^2 b - wc,$$

since $w^2 = -1 - w$. We similarly get expressions for e and f. Since P is the centroid of $\triangle BCD$, we have: $3p = b + c + d = \left(1 - w^2\right) b + (1 - w)c$, and similarly for q and r. Thus:

$$\begin{aligned} 3p &= \left(1 - w^2\right) b + (1 - w)c, \\ 3q &= \left(1 - w^2\right) c + (1 - w)a, \\ 3r &= \left(1 - w^2\right) a + (1 - w)b. \end{aligned}$$

We need to verify that $p + wq + w^2 r = 0$. The coefficient of a in $3\left(p + wq + w^2 r\right)$, obtained by adding suitable multiples of the above three equations, is:

$$\left(w - w^2\right) + \left(w^2 - w^4\right) = w - w^2 + w^2 - w = 0,$$

and similarly for the coefficients of b and c. Hence $p + wq + w^2 r = 0$, and it follows that $\triangle PQR$ is equilateral. □

Closing remark 1. It is worth drawing attention to the use of the word 'symmetry' and words such as 'similarly' which also indicate a kind of symmetry. For most younger students, the word *symmetry* has a strongly geometrical connotation. Here, though we are proving a geometric theorem, our methods have been heavily algebraic; yet we have made use of symmetry at various points: not geometrical symmetry, but algebraic symmetry, the symmetry of symbols, in which we implicitly make use of the fact that nature does not have preferences between the sides of a triangle. This lack of preference naturally carries over to the symbols denoting the sides of the triangle.

Closing remark 2. There are yet other proofs of Napoleon's theorem. In particular, there are proofs that use no computations whatever; rather, they use transformations. There are also striking generalisations of Napoleon's theorem. There is even a result which is a kind of converse to Napoleon's theorem! We shall study all these and more in subsequent parts of this article.

The cube roots of unity: a short tutorial

By the term *cube roots of unity*, we mean the solutions of the cubic equation $x^3 = 1$ over the complex numbers. As the equation is of degree 3, it has 3 roots. We can get them explicitly by noting that: (i) $x - 1$ is a factor of $x^3 - 1$, (ii) the remaining factor is quadratic:

$$x^3 - 1 = (x-1)\left(x^2 + x + 1\right).$$

Hence the solutions are: $x = 1$, and

$$x = \frac{-1 \pm \sqrt{1-4}}{2}, \quad \text{i.e.,} \quad x = \frac{-1 + i\sqrt{3}}{2}, \quad x = \frac{-1 - i\sqrt{3}}{2},$$

where $i = \sqrt{-1}$. The latter two roots are non-real complex numbers. Note that they are conjugates of each other. It is traditional to denote the first one (with positive imaginary part) by w; then the second one is its conjugate $\overline{w}$. We list below a number of properties of these complex numbers. They frequently come of use in geometric applications.

(1) $|w| = 1$; $\left|w^2\right| = 1$; $\arg w = 120^\circ$; $\arg w^2 = -120^\circ$.

(2) $\overline{w} = w^2$ and $w = (\overline{w})^2$; that is, each non-real cube root of unity is the square of the other one. In the same way, each non-real cube root of unity is the reciprocal of the other one. So the three cube roots of unity may be expressed as $1, w, w^2$ or as $1, w, 1/w$.

(3) $1 = w^3 = w^6 = w^9 = w^{12} = w^{15} = \cdots$.

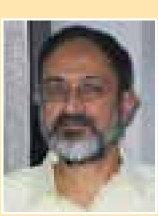
(4) $w = w^4 = w^7 = w^{10} = w^{13} = w^{16} = \cdots$.

(5) $w^2 = w^5 = w^8 = w^{11} = w^{14} = w^{17} = \cdots$.

(6) $1 + w + \overline{w} = 0$; otherwise put, $1 + w + w^2 = 0$.

(7) The three cube roots of -1 are: -1, $-w$, $-w^2$.

(8) The six sixth roots of unity are: 1, w, w^2, -1, $-w$, $-w^2$.



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Features

The 3-4-5 Triangle: Some Observations

Marcus Bizony

In Figure 1 we see a right-angled 3-4-5 triangle ABC in which $AB = 3$, $BC = 4$ and $AC = 5$. The incircle (centre I) has been drawn; also the angle bisector AIP through vertex A, and a fourth tangent PQ to the incircle.

Since $\tan A = 4/3$, we get:

$$\tan\frac{A}{2} = \frac{\sqrt{(4/3)^2+1}-1}{4/3} = \frac{5/3-1}{4/3} = \frac{2/3}{4/3} = \frac{1}{2}.$$

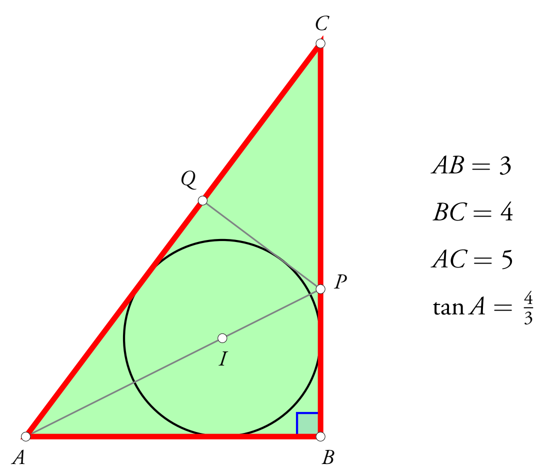

Figure 1

Keywords: *3-4-5 triangle, incircle, golden ratio, golden point*

Therefore $PB = 3/2$ and $CP = 5/2$. (Note: We could also have found the length of PB using the angle bisector theorem, which tells us that $CP : PB = 5 : 3$.)

Since $\triangle APQ \cong \triangle APB$ (angle-side-angle or ASA congruence; for: $\measuredangle PAQ = \measuredangle PAB$; $\measuredangle PQA = \measuredangle PBA$, both being right angles; and AP is a shared side), we have $PQ = 3/2$.

Note that $\triangle CPQ \sim \triangle CAB$, the similarity ratio being $PQ/AB = 1/2$.

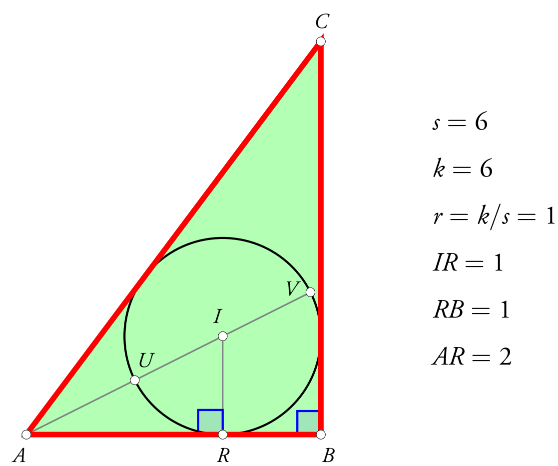

Figure 2

For any triangle, the radius r of its incircle is given by the formula $r = k/s$ where k is the area and s is the semi-perimeter of the triangle. In the case of the 3-4-5 triangle this gives an in-radius of $r = 6/6 = 1$ unit.

Let line AI intersect the incircle at points U and V as shown, and let R be the point of tangency of AB; then $IR = 1$, $RB = 1$, so $AR = 2$. But $AI = \sqrt{5}$ (by Pythagoras), so $AV = \sqrt{5} + 1$, which means that the ratio $AV : UV$ is

$$\frac{AV}{UV} = \frac{\sqrt{5}+1}{2} = \text{the Golden Ratio } \varphi.$$

We may therefore describe the point U as a *Golden Point* of AV.

Now we consider triangle PAB, where PA is the bisector of angle A. We already know that $PB = 3/2$. Draw the incircle of $\triangle PAB$; let its centre be J, and let its radius be x. Let T be the point of tangency of the circle and PB. (See Figure 3.)

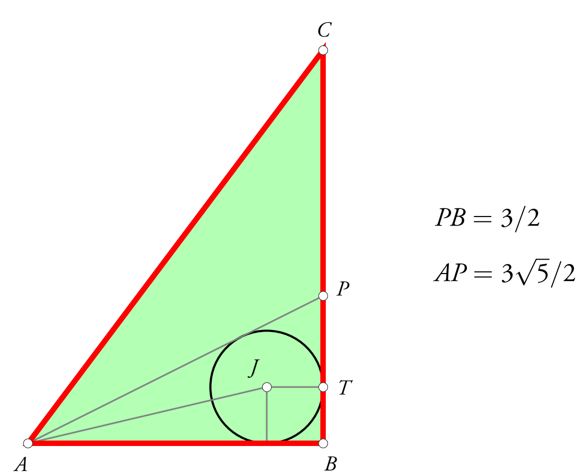

Figure 3

Since $AB = 3$ and $PB = 3/2$, we have (Pythagoras) $AP = 3\sqrt{5}/2$. Hence the semi-perimeter of $\triangle PAB$ is

$$\frac{1}{2}\left(3 + \frac{3}{2} + \frac{3\sqrt{5}}{2}\right) = \frac{3\left(3+\sqrt{5}\right)}{4}.$$

The area of $\triangle PAB$ is $1/2 \times 3 \times 3/2 = 9/4$. Hence the radius x of its incircle is

$$x = \frac{9/4}{3(3+\sqrt{5})/4} = \frac{3}{3+\sqrt{5}} = \frac{3\left(3-\sqrt{5}\right)}{4}.$$

Hence the ratio PB/PT is

$$\frac{PB}{PT} = \frac{3/2}{3/2 - 3(3-\sqrt{5})/4} = \frac{2}{2-(3-\sqrt{5})} = \frac{2}{\sqrt{5}-1} = \frac{\sqrt{5}+1}{2}.$$

In other words, T is a Golden Point of PB.

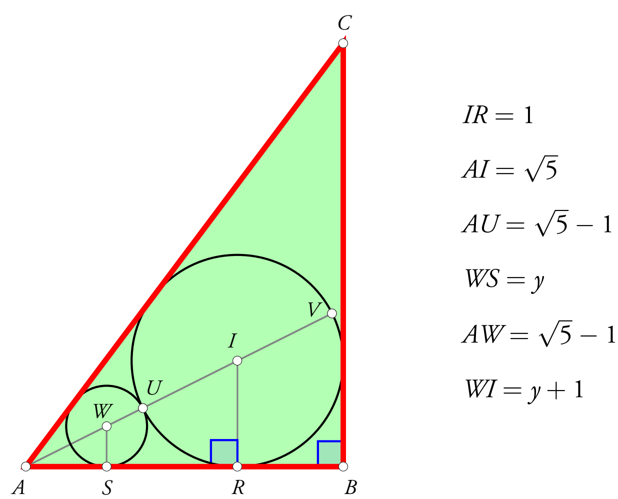

Figure 4

Let a circle be fitted into the region between A and the incircle of $\triangle ABC$, with its centre at W (see Figure 4). Let y be the radius of this small circle.

By using the properties of similar triangles, we get: $y/AW = IR/AI$, i.e.,

$$\frac{y}{\sqrt{5}-1-y} = \frac{1}{\sqrt{5}},$$

which yields:

$$y = \frac{\sqrt{5}-1}{\sqrt{5}+1} = \frac{3-\sqrt{5}}{2}.$$

Hence

$$WI = y + 1 = \frac{5-\sqrt{5}}{2},$$

and therefore

$$\frac{AI}{WI} = \frac{\sqrt{5}}{(5-\sqrt{5})/2} = \frac{2}{\sqrt{5}-1} = \frac{\sqrt{5}+1}{2}.$$

In other words, W is a Golden Point of AI.

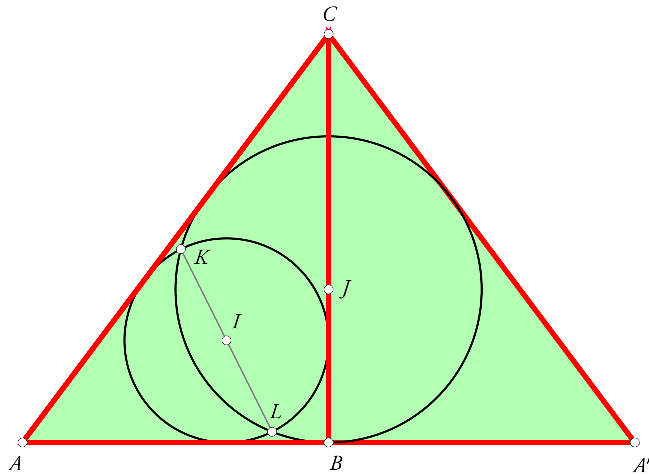

Figure 5

Consideration of similar triangles shows us that if a third circle were fitted in between A and the circle centred at W, we would get a new Golden Section, and so on.

If we "double" the 3-4-5 triangle to make an isosceles triangle (with equal sides 5 and base 6), and consider the incircles of the two triangles (see Figure 5), their common chord is a diameter of the smaller circle; i.e., the common chord KL passes through I.

To see why, let J denote the centre of the larger incircle; both I and J lie on the internal bisector of $\measuredangle BAC$. The radius of this larger incircle is

$$\frac{\text{Area of } \triangle AA'C}{\text{Semi-perimeter of } \triangle AA'C} = \frac{(6 \times 4)/2}{(5+6+5)/2} = \frac{3}{2}.$$

The radius of this larger incircle is $3/2$, while the radius of the smaller circle was earlier established as 1. That means that the radius of the large incircle is $3/2$ times that of the smaller one. Now consideration of the perpendicular from I to AB together with JB shows us that AJ and AI are in the same ratio as the radii of the larger and the smaller incircle; that is:

$$AJ = \frac{3}{2} AI, \quad \therefore \quad IJ = \frac{1}{2} AI = \frac{\sqrt{5}}{2}.$$

Now focus attention on the triangle whose vertices are K, I, J. We have:

$$KI = 1, \quad KJ = \frac{3}{2}.$$

We observe that $KJ^2 = KI^2 + IJ^2$, which indicates that $\measuredangle KIJ$ is a right angle. By symmetry, so is $\measuredangle LIJ$. Hence KL is a diameter of the smaller incircle.

This implies that points K, I, L lie in a straight line.

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The Rascal Triangle

A Rascal full of Surprises!

Ishaan Magon, Maya Reddy, Rishabh Suresh and Shashidhar Jagadeeshan

As a teacher, one is always looking for so-called 'non-standard' problems. These should be based on material that has been taught, and yet be neither trivial nor too hard. This article illustrates an example of such a non-standard problem. Reading the article on the Rascal Triangle [1], I felt it would fit the bill, given that it was discovered by students in the first place. As an introduction to the problem, I put down the following six rows (Figure 1), and titled it

'The Rascal Triangle'

1
1 1
1 2 1
1 3 3 1
1 4 5 4 1
1 5 7 7 5 1

Figure 1

Keywords: *Pascal triangle, investigation, pattern, prediction, successive differences*

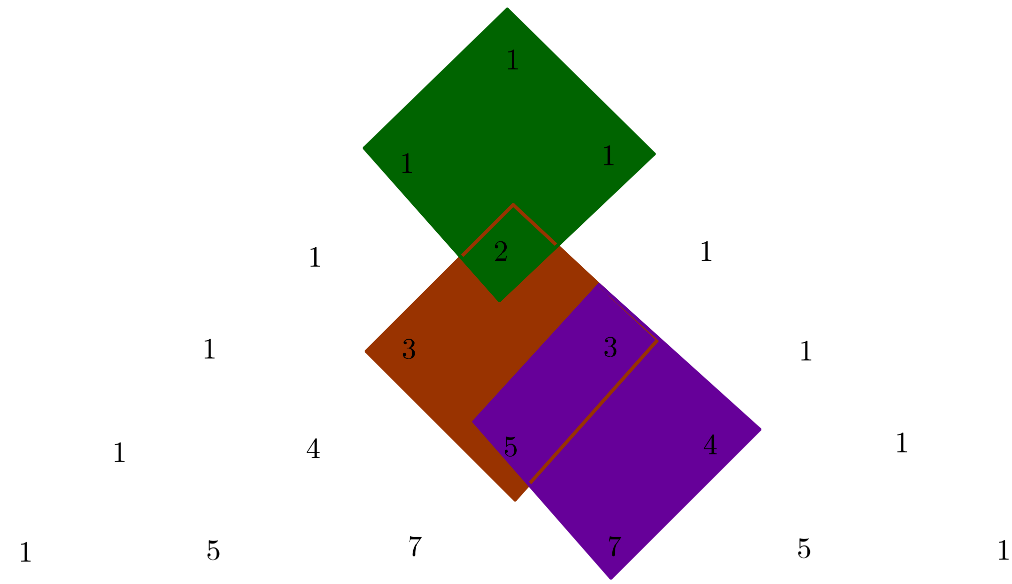

Figure 2

Since my students were already familiar with the Pascal triangle, I told them the story of the Rascal triangle and how it was discovered, and asked my students to find the pattern for the next few rows and come up with a rule to generate rows, somewhat like the rule for Pascal's triangle.

As a hint, I drew Figure 2 and explained (following the lead from [1]) that just as we can think of the rule generating elements in the Pascal triangle to be based on a triangle, we can think of the rule generating the elements of the Rascal triangle to be based on a diamond.

My hope was that my students would rediscover the formula that the American students had found. Knowing their familiarity with finding formulae for sequences with constant differences, I also asked them to find the formula for the r^{th} entry in the m^{th} row. I was not sure how much progress they would make in this regard, and thought I would have to explain the method the article uses to derive the formula.

I did not realize that there would be many surprises awaiting me! The first was that three of my students discovered an entirely new 'diamond rule'. Here is a description of their discovery in Ishaan and Maya's own words:

"Given some time to stare at it, we suggested that the number x in the diamond (Figure 3) could be obtained by the formula $x = b + c + 1 - a$.

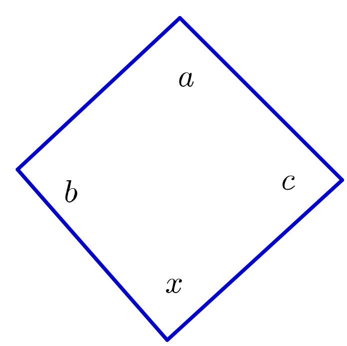

Figure 3

While arriving at the answer seemed to be an almost instantaneous process, we had very different approaches to finding the formula. One approach was to try various arithmetic processes (addition, subtraction, multiplication and division) indiscriminately! The other approach was to try addition first driven by the belief that it is the only guaranteed method by which one would always get an integer value for x using integers a,b,c.

Our formula appeared to work for all the values in the entries of Rascal triangle given to us and it surprised our math teacher, because it did not resemble the one that the students who originally created the Rascal triangle had come up with."

The second surprise was that Rishabh came up with a different approach to derive the formula for the r^{th} entry in the m^{th} row. To begin with almost all students recognized the pattern among the diagonal rows [1], so they were able to generate rows of the Rascal triangle at will.

Here is Rishabh in his own words.

"I started by looking (Figure 4) at the first difference of all the rows:

I saw that for any given row of first differences, successive elements differ by –2. Also the first element in each of the blue rows can be found by subtracting 2 from the number of the row, that is, for the m^{th} row, the first element of the corresponding blue row of successive differences will be m–2. Using the same notation as [1] to denote the r^{th} element in the m^{th} row to be Entry (m, r). Writing $1 = m - (m - 1)$, I got the following list:

Entry $(m, 1) = m - (m - 1)$

Entry $(m, 2) = m - (m - 1) + (m - 2)$

Entry $(m, 3) = m - (m - 1) + (m - 2) + (m - 4)$

Entry $(m, 4) = m - (m - 1) + (m - 2) + (m - 4) + (m - 6)$

Entry $(m, 5) = m - (m - 1) + (m - 2) + (m - 4) + (m - 6) + (m - 8)$

Simplifying:

Entry $(m, 1) = m - (m - 1) = 0m + 1$

Entry $(m, 2) = m - (m - 1) + (m - 2) = 1m - 1$

Entry $(m, 3) = m - (m - 1) + (m - 2) + (m - 4) = 2m - 5$

Entry $(m, 4) = m - (m - 1) + (m - 2) + (m - 4) + (m - 6) = 3m - 11$

Entry $(m, 5) = m - (m - 1) + (m - 2) + (m - 4) + (m - 6) + (m - 8) = 4m - 19$

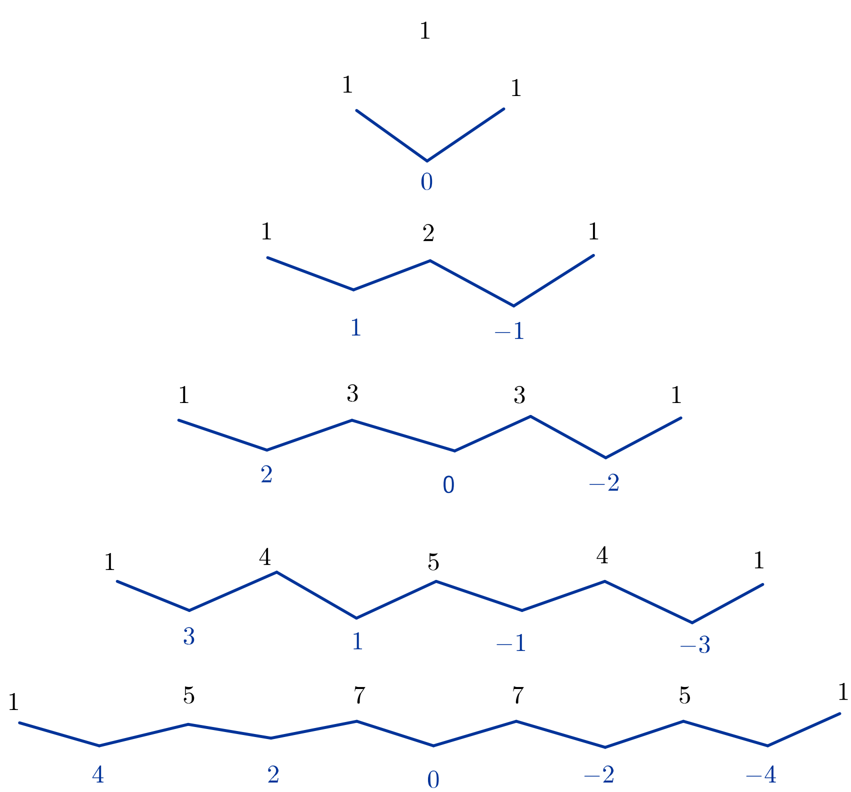

Figure 4

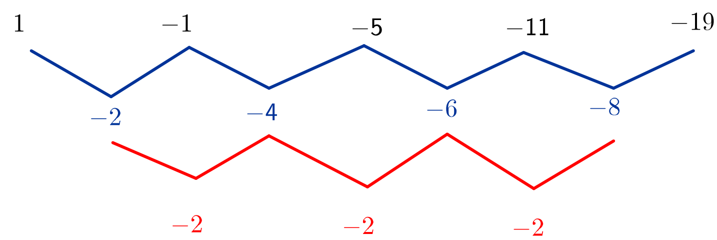

Figure 5

It is easy to see that the coefficient of m in each entry is simply $r - 1$. I tried to find a formula for the terms $1, -1, -5, -11, -19, \ldots$.

When I examined the differences amongst the terms (Figure 5), I found the second difference to be a constant. Using the standard technique for finding the formula for sequences whose second difference is a constant, the formula for this sequence is given by $-r^2 + r + 1$. Therefore:

$$\text{Entry } (m, r) = m(r-1) - r^2 + r + 1 = (r-1)(m-r) + 1.$$

Here $m = 1, 2, 3 \ldots$ and $r = 1, 2, \ldots m$."

Finishing up. To finish up, we need to show two things:

1. The formula we have found for the r^{th} entry of the m^{th} row is the same as that derived in the original article
2. Our new diamond rule is equivalent to the diamond rule discovered by the students from America.

Recall that the formula for the k^{th} element of the n^{th} row given in the original is Entry $(n, k) = k(n - k) + 1$, where $n = 0, 1, 2, \ldots$ and $k = 0, 1, 2 \ldots n - 1$. It is obvious that all we need to do is set $m = n + 1$ and $r = k + 1$ to show the two are equivalent.

To establish the second we prove the following theorem (replace with similar to [1]). Here we would like to thank Dr Shirali for pointing us in this direction. We will use Rishabh's formula for the r^{th} element of the m^{th} row to do so!

Theorem. *For the sub-array:*

$$\boxed{\begin{array}{ccc} & a & \\ b & & c \\ & x & \end{array}}$$

the new entry x is given by:

$$x = b + c + 1 - a.$$

Proof. Examining the framed box above, all we need to verify is the following:

Entry $(m + 1, r + 1)$ = Entry (m, r) + Entry $(m, r + 1) + 1$ − Entry $(m - 1, r)$,

i.e.,

$$r(m-r) + 1 = \big((r-1)(m-r) + 1 + \big(r(m-r-1) + 1 + 1 - \big((r-1)(m-r-1) + 1\big).$$

The verification takes but a moment!

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1. $\mathscr{C}\otimes\mathscr{M}\alpha\mathscr{C}$, "The Rascal Triangle". *At Right Angles*, Volume 5, No.1 , March 2016

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SUMS OF **CONSECUTIVE** NUMBERS THAT YIELD POWERS OF 9

$$1 = 9^0$$

$$2+3+4 = 9^1$$

$$5+6+7+8+9+10+11+12+13 = 9^2$$

$$14+15+16+17+\cdots+37+38+39+40 = 9^3$$

$$41+42+43+44+\cdots+118+119+120+121 = 9^4$$

And so on.

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[1] www.webalice.it/francesco.daddi [2] www.facebook.com/matematicadaddi

Rascal Triangle Synopsis

In the March 2016 issue of At Right Angles, we carried an article describing the discovery of the whimsically named Rascal Triangle. You can find the article at http://teachersofindia.org/en/ebook/rascal-triangle. We give below a brief synopsis of this story.

The story unfolds in a classroom in America where a teacher displayed the first four lines of the well-known Pascal triangle (namely, rows 0, 1, 2 and 3), and asked them to guess or to deduce what could be the next few lines.

					1					
				1		1				
			1		2		1			
		1		3		3		1		
	?		?		?		?		?	
?		?		?		?		?		?

Table 1 : The familiar Pascal aray

What he had displayed was the array shown in Table 1

The teacher's intention was for them to discover that, in the Pascal triangle, each new row is generated additively, using the numbers in the row above it (namely, by adding the two numbers closest to the entry to be filled). Thus if we have:

a		b
	x	

then the new entry x is given by: $x = a+b$.

Instead, the students surprised him by proposing that the row after 1,3,3,1 should be 1,4,5,4,1; and the one after that should be 1,5,7,7,5,1.

In the rule used by the students, the numbers in each new row are computed using the two rows preceding it. Thus if we have:

	a	
b		c
	x	

then the new entry x is given by: $x = \dfrac{bc+1}{a}$

If the generating rule for the Pascal array could be called a "triangular rule" (based on the underlying shape), then the one used by the students could be called a "diamond rule." The students who put forward this new rule and explored this new array whimsically named it the Rascal triangle.

1									
1	1								
1	2	1							
1	3	3	1						
1	4	5	4	1					
1	5	7	7	5	1				
1	6	9	10	9	6	1			
1	7	11	13	13	11	7	1		
1	8	13	16	17	16	13	8	1	
1	9	15	19	21	21	19	15	9	1

Table 2 : The first ten rows of the Rascal array

The article discussed the validity of this rule and proves the following:

1. Despite the division, all the entries do turn out to be positive integers.

2. Entry(k) in Row(n) of the Rascal array is $k(n-k)+1 = kn-k^2+1$.

References

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   http://www.maa.org/sites/default/files/pdf/pubs/cmj393-395.pdf

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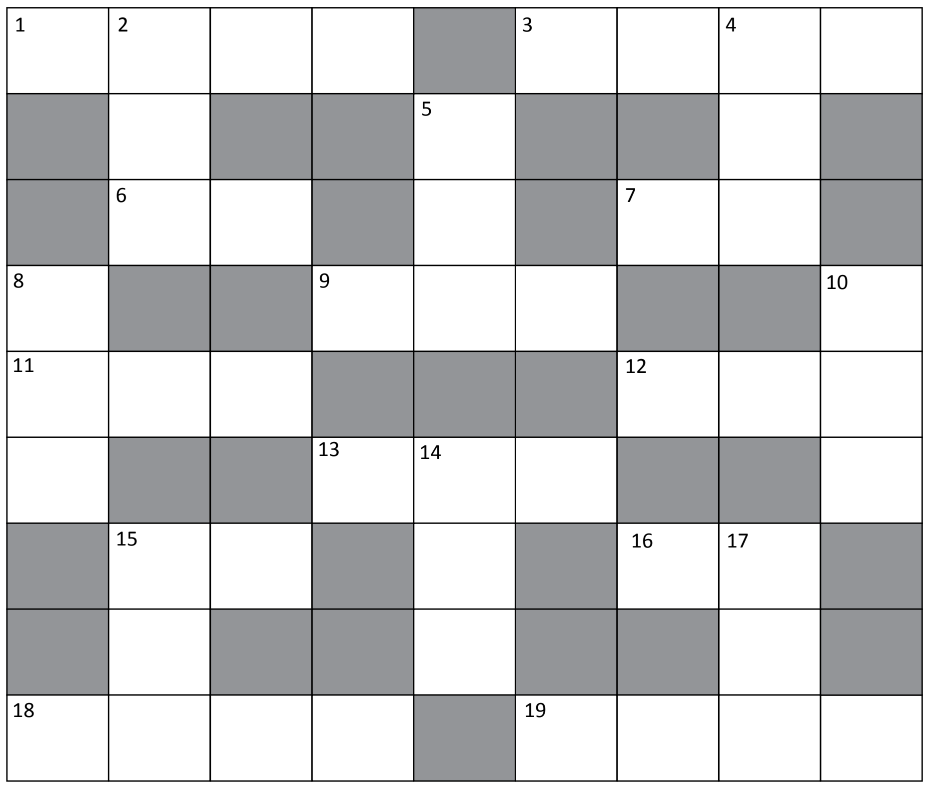
NUMBER CROSSWORD

Solutions on Page 76

D.D. Karopady



CLUES ACROSS

1. Sum of internal angles of a penta-decagon (a 15 sided polygon)
3. Reverse of 19A
6. Number of days in February 2020
7. Smallest two-digit square that can be written as a sum of two squares
9. All three digits are the same
11. 15D - 93
12. Fifty dozens minus half a dozen
13. Tenth term in the sequence 1, 2, 4, 8, ...
15. Smallest number that becomes a perfect square if its reverse is either added to it or subtracted from it
16. Square root of 10D
18. Sum of answers for all the Down clues
19. Largest four digit perfect square

CLUES DOWN

2. Middle digit is the difference of the other two digits
4. 15A x 13
5. Area of a triangle with sides 21, 28, 35
8. Sum of squares of 1 through 12
10. Each interior angle of a regular decagon (10 sided polygon)
14. Palindrome with 0 and 1
15. 8D - 6A
17. Sum of integers from 1 through 20

ClassRoom

Counting Triangles

Sundarraman Madhusudanan

In this article, I study the problem of counting the number of triangles formed in a triangle if n segments are drawn from one vertex to its opposite side, and h segments are drawn from another vertex to its opposite side. This kind of counting problem is often seen in puzzle collections; e.g.: "Count the number of triangles visible in Figure 1". Making a manual count for such a problem is tedious; also, it is easy to make an error in the count. We need a more analytic and systematic procedure.

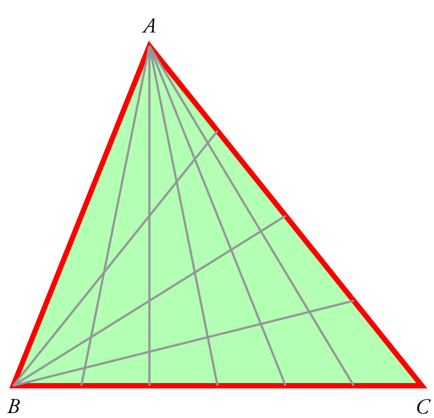


Figure 1

Keywords: *triangles, counting, combinations*

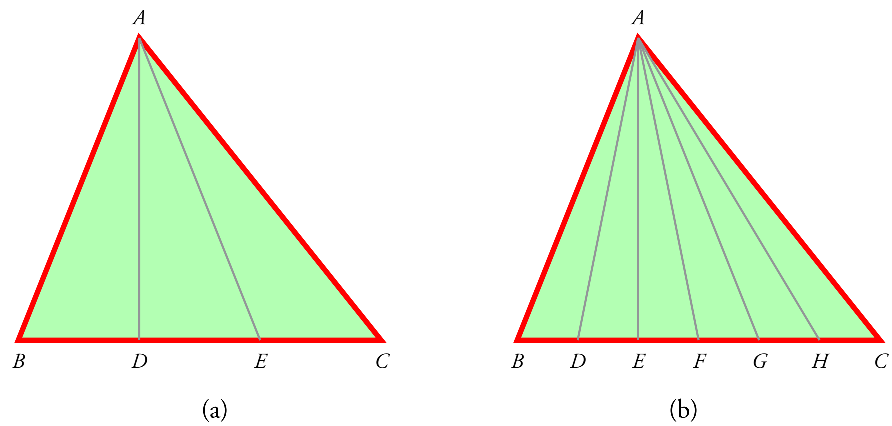


Figure 2

Step 0: Segments drawn from just one vertex

We first solve the sub-problem in which segments are drawn from just one vertex. In Figure 2 (a), lines AD, AE have been drawn from vertex A to points D, E on BC. The number of triangles thus formed can be manually counted; it comes to be 6. Now draw n segments from A to n points on BC, as in Figure 2 (b); here $n = 5$. If we take any two segments from the set of $n + 2$ segments which emanate from vertex A (i.e., the n segments along with AB and AC), we get precisely one triangle (the base lying on BC). So the number of triangles will be equal to the number of different ways of choosing 2 segments from $n + 2$ segments; this is $\binom{n+2}{2}$. Hence, for n segments drawn from a single vertex, the number of triangles is $(n + 2)(n + 1)/2$.

Step 1: Back to the original problem

Now I return to the main problem. I have divided the problem into two cases, depending on whether the number of segments emanating from the two vertices are equal or unequal.

Case 1: Same number of segments drawn from the two vertices, n $=$ h. Consider for example $\triangle ABC$ (Figure 3), in which two segments each have been drawn from vertices B and C to the opposite sides. We first count all triangles which have B as a vertex. Triangle CBE contains $\binom{4}{2} = 6$ such triangles; likewise for triangles CBD and CBA. Therefore there are $6 + 6 + 6 = 18$ triangles which have B as a vertex. The computation may also be written as $\binom{4}{2} \times \binom{3}{1} = 6 \times 3 = 18$.

The triangles which do not have B as a vertex all lie inside $\triangle ACG$, and they all have C as a vertex. To count these, note that they can be generated by choosing any two segments out of the segments CA, CF, CG and choosing one segment from the segments BA, BD, BE. So we get $\binom{3}{2} \times \binom{3}{1} = 3 \times 3 = 9$ triangles.

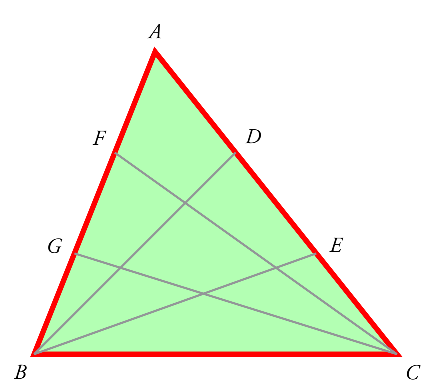


Figure 3

So, the total number of triangles is $18+9=27$.

Some of you may observe that $27=3^3$ which may be written as $(2+1)^3$, and guess that this is not pure coincidence!

Now we will try to prove that if n segments are drawn from both vertices, there will be $(n+1)^3$ triangles.

The proof of this general claim runs on exactly the same lines as above. Thus, in place of the quantity $\binom{4}{2}\times\binom{3}{1}$, we have the term

$$\binom{n+2}{2}\times\binom{n+1}{1};$$

and in place of the quantity $\binom{3}{2}\times\binom{3}{1}$, we have the term

$$\binom{n+1}{2}\times\binom{n+1}{1}.$$

Hence the total number of triangles in the configuration is

$$\begin{aligned}\binom{n+2}{2}\times\binom{n+1}{1}+\binom{n+1}{2}\times\binom{n+1}{1} &= \frac{(n+2)(n+1)^2}{2}+\frac{(n+1)^2 n}{2}\\ &= \frac{(n+1)^2}{2}(2n+2)\\ &= (n+1)^3.\end{aligned}$$

Case 2: Unequal number of segments drawn from the two vertices, n $\neq$ h. Now we consider the situation (Figure 4) when there are n line segments drawn from vertex B to side AC, and h line segments drawn from vertex C to side AB. Here it is assumed that $n\neq h$.

Very conveniently for us, the analysis for the general configuration runs on exactly the same lines as earlier.

Thus, in place of the term $\binom{n+2}{2}\times(n+1)$ we have the term

$$\binom{n+2}{2}\times\binom{h+1}{1};$$

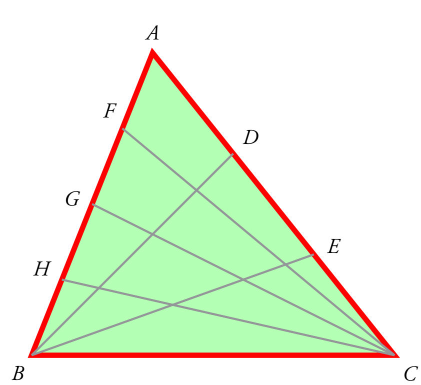


Figure 4

and in place of the term $\binom{n+1}{2} \times \binom{n+1}{1}$ we have the term

$$\binom{h+1}{2} \times \binom{n+1}{1}.$$

Hence the total number of triangles in the configuration is

$$\begin{aligned}
&\binom{n+2}{2} \times \binom{h+1}{1} + \binom{h+1}{2} \times \binom{n+1}{1} \\
&= \frac{(n+2)(n+1)(h+1)}{2} + \frac{(h+1)h(n+1)}{2} \\
&= \frac{(n+1)(h+1)}{2}(n+h+2) \\
&= \frac{(n+1)(h+1)(n+h+2)}{2}.
\end{aligned}$$

Remarks.

- If we interchange values of n and h, we will get the same answers by using this formula. This seems logical, as the two configurations are mirror images of each other.
- If we put $n = h$, we get the formula derived earlier, i.e., $(n+1)^3$.

Open Question

Can you find the number of triangles when n, h and k line segments are drawn from the three vertices respectively to the opposite sides? We may assume for simplicity that no three of these $n + h + k$ line segments concur.

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ClassRoom

An Iteration on the Prime Factors of a Number

R. Rohini Lakshmi

In this short note I study the behaviour of a function f defined in the positive integers exceeding 1 (namely, the set $\{2, 3, 4, 5, \ldots\}$), when it is applied over and over again on itself. Here is its definition. Given a positive integer $n > 1$, we compute $f(n)$ as follows. First, we check whether n is prime or composite. If n is prime, then $f(n) = n + 1$. If n is composite, then we set $f(n)$ to be equal to the sum of all the prime numbers which divide n, each prime number being added as many times as it divides n. I illustrate how the definition works in Table 1.

n	Prime/composite	Prime factorisation	Computation	$f(n)$
5	prime	5	$5+1$	6
6	composite	2×3	$2+3$	5
7	prime	7	$7+1$	8
8	composite	2^3	$2+2+2$	6
9	composite	3^2	$3+3$	6
10	composite	2×5	$2+5$	7
12	composite	$2^2 \times 3$	$2+2+3$	7
20	composite	$2^2 \times 5$	$2+2+5$	9
100	composite	$2^2 \times 5^2$	$2+2+5+5$	14

Table 1

Keywords: *function, prime number, prime factorisation, composite, iteration*

Next, we iterate the function definition; that is, we start with some n, compute $f(n)$, then compute $f(f(n))$, then $f(f(f(n)))$, and so on, and we list the outputs in sequence. The results certainly come as a surprise; please see Table 2, where we have listed the outputs for various inputs. In every single case, the sequence ultimately settles down to . . . , 5, 6, 5, 6, 5, 6, . . .!

Starting number n	Sequence of outputs: $n, f(n), f(f(n)), \ldots$
5	5, 6, 5, 6, 5, 6, 5, 6, 5, 6, 5, 6, 5, 6, 5, 6, . . .
6	6, 5, 6, 5, 6, 5, 6, 5, 6, 5, 6, 5, 6, 5, 6, 5, . . .
7	7, 8, 6, 5, 6, 5, 6, 5, 6, 5, 6, 5, 6, 5, 6, 5, . . .
8	8, 6, 5, 6, 5, 6, 5, 6, 5, 6, 5, 6, 5, 6, 5, 6, . . .
9	9, 6, 5, 6, 5, 6, 5, 6, 5, 6, 5, 6, 5, 6, 5, 6, . . .
10	10, 7, 8, 6, 5, 6, 5, 6, 5, 6, 5, 6, 5, 6, 5, 6, . . .
11	11, 12, 7, 8, 6, 5, 6, 5, 6, 5, 6, 5, 6, 5, 6, 5, . . .
12	12, 7, 8, 6, 5, 6, 5, 6, 5, 6, 5, 6, 5, 6, 5, 6, . . .
20	20, 9, 6, 5, 6, 5, 6, 5, 6, 5, 6, 5, 6, 5, 6, 5, . . .
30	30, 10, 7, 8, 6, 5, 6, 5, 6, 5, 6, 5, 6, 5, 6, 5, . . .
50	50, 12, 7, 8, 6, 5, 6, 5, 6, 5, 6, 5, 6, 5, 6, 5, . . .
100	100, 14, 9, 6, 5, 6, 5, 6, 5, 6, 5, 6, 5, 6, 5, 6, . . .
1000	1000, 21, 10, 7, 8, 6, 5, 6, 5, 6, 5, 6, 5, 6, 5, 6, . . .
123456	123456, 658, 56, 13, 14, 9, 6, 5, 6, 5, 6, 5, 6, 5, 6, 5, . . .

Table 2

You will notice that in Table 2, we skipped the numbers below 5; i.e., we did not explore the output when the starting numbers are 2, 3 or 4. Table 3, below, lists the outcomes in these cases.

Starting number n	Sequence of outputs: $n, f(n), f(f(n)), \ldots$
2	2, 3, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, . . .
3	3, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, . . .
4	4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, . . .

Table 3

What these two tables show (or suggest) is that no matter what the starting number is, the outputs ultimately settle down to either the unending sequence . . . , 4, 4, 4, 4, . . ., or the unending sequence . . . , 5, 6, 5, 6, ...

How may this be explained?

ROHINI LAKSHMI is a 9th grade student from Pushpalata Vidyamandir, Tirunelveli, Tamil Nadu. She is a Math enthusiast and a student of Bharatanatyam and the Violin. She also plays Tennis. During a session conducted by Shri Vinay Nair in her school on math explorations, she chanced upon some explorations done on Happy Numbers. This motivated her to explore iterations on her own, thus leading to this observation concerning Factor Iteration.

ClassRoom

Low Floor High Ceiling Tasks

Circle Time

Solid Learning!

Swati Sircar & Sneha Titus

In the November 2014 issue, we began a new series which was a compilation of 'Low Floor High Ceiling' activities. A brief recap: an activity is chosen which starts by assigning simple age-appropriate tasks which can be attempted by all the students in the classroom. The complexity of the tasks builds up as the activity proceeds so that each student is pushed to his or her maximum as they attempt their work. There is enough work for all but as the level gets higher, fewer students are able to complete the tasks. The point, however, is that all students are engaged and all of them are able to accomplish at least a part of the whole task.

This time we focus on regular polygons inscribed in circles. As with all such hands-on activities, there is ample scope for skill development. But careful facilitation is key to this aspect of activities. Students should be given time for reflection, for discussion and, most importantly, they should not be afraid to make mistakes. While emphasis is given to rigour, this discipline is best imposed with gentle questioning – students must be encouraged to ask each other the question 'Why?' as often as possible.

Keywords: *angle, equilateral triangle, square, regular pentagon, regular hexagon, regular octagon, inscribed polygon, platonic solid, circle, diameter, intersection, collaborative, pattern, dimension*

The tasks require that each student has a large supply of paper, a compass, ruler, sharp pencil and scissors. Each student must use circles of the same radius throughout the activity; however, the radius can vary from student to student.

In the spirit of eco-friendliness, we recommend that you use paper which is used only on one side for this task. Folds to create the regular polygons should be made in toward the blank side.

These activities can be attempted by students in classes from 8 to 10 though some of the questions may need to be toned down for the younger students.

TASK 1: Construction to create a Square in a Circle

1.1 Draw a circle of reasonably large radius and any one diameter of it. Mark its end points B and C.

Construct the perpendicular bisector of this diameter and extend it to meet the circle at D and E. Mark the centre A of the circle and cut it out.

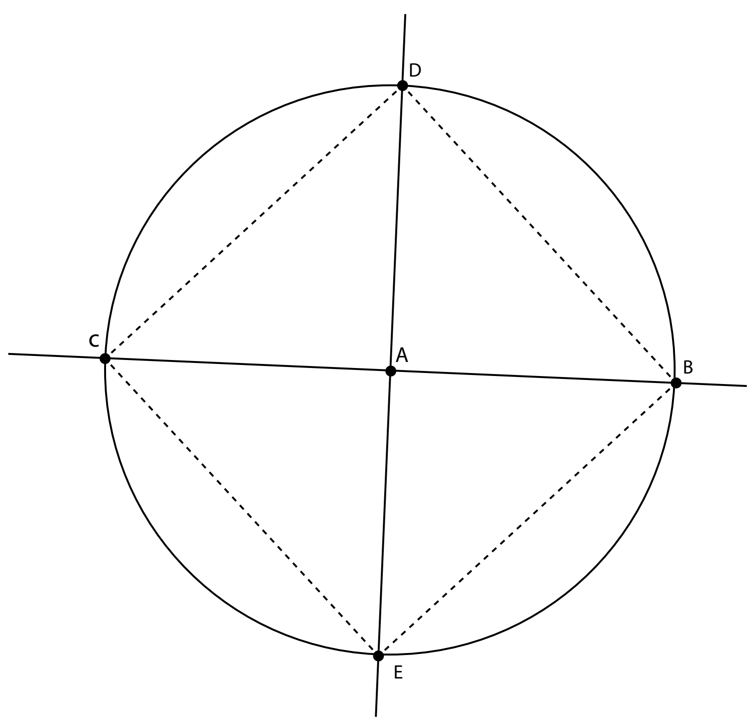


Figure 1: Fold along DC, CE, EB and BD, and verify that BDCE is a square. Explain how you verified this.

1.2 Prove that BDCE is a square.

1.3 If the side of the square is 1 unit, find the radius of the circle.

1.4 If the radius of the circle is 1 unit, find the length of the sides of the square.

1.5 How can you fold a regular octagon inscribed in another circle of the same radius?

1.6 Find the area of each sector of the circle which has been created in the figure, taking the radius to be one unit.

TASK 2: Construction to create a Triangle in a Circle

2.1 Use another circle with the same radius and draw any diameter CB. Mark the centre A of the circle.

Construct the perpendicular bisector of AC and extend it to cut the circle at E and F. Mark the intersection D of AC and EF. Cut out the circle.

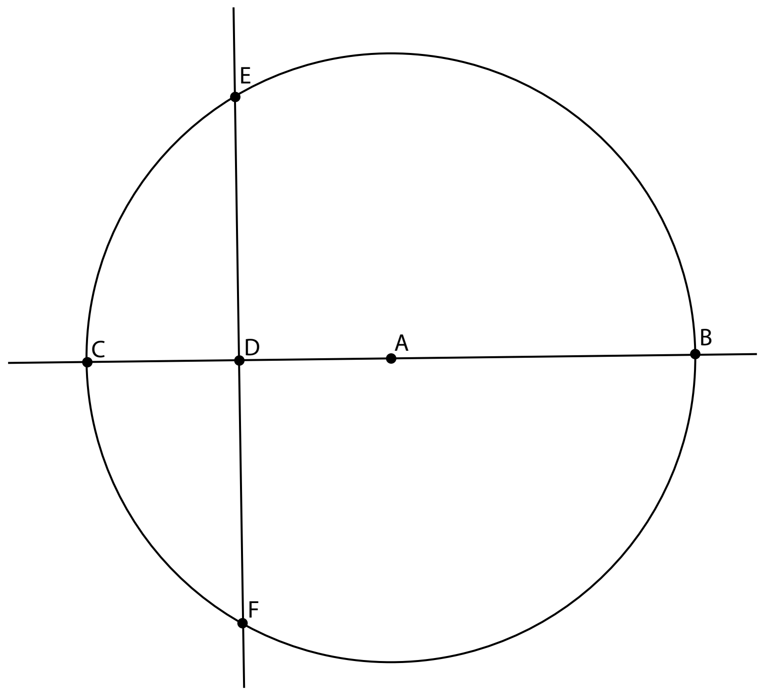


Figure 2

2.2 Verify that the points E, F and B trisect the circle. Explain how you verified this.

2.3 Fold the lines joining E, B & F to get a triangle. What kind of triangle is this? Justify.

2.4 If the side of the triangle is 1 unit, find the radius of the circle.

2.5 If the radius of the circle is 1 unit, find the lengths of the sides of the triangle.

2.6 How can you fold the above paper to get a regular hexagon inscribed in the given circle?

2.7 Join BE and BF and find the area of each sector of the circle in the figure.

Teacher's Note:

Tasks 1 and 2 are simple activities which are a good opening exercise that helps students practise the skills they have learnt in their geometry classes. It would be particularly interesting for them to attempt to do the same activity with paper folding instead of constructions and see the similarities between both the activities.

Notice the Low Floor activity 'Verify' which students can easily do with a protractor and ruler, and which helps them to revise their understanding of regular polygons.

In the subsequent step up to the High Ceiling, students are asked to 'Prove'- a task that they could do by using the concepts of angles in a semi-circle, chord properties, congruent triangles and by applying Pythagoras' theorem or simple trigonometric ratios. The last question in each task helps them to use mensuration formulae and symmetry, and their findings will be collated in a subsequent task. Some amount of familiarity with operations on irrational numbers is required.

TASK 3: Construction to create a Pentagon in a Circle.

3.1 Draw a circle of the same radius used previously. Mark the centre A and any point B on the circle and draw the radius AB. Construct the diameter perpendicular to AB and mark any one of its ends. Call this point C and locate the mid-point D of AC. Calculate BD in terms of 'r', the radius of the circle.

3.2 With D as centre and radius BD, draw an arc. Mark the point of intersection E of this arc with the diameter AC. (E is inside the first circle). Calculate BE in terms of 'r'.

3.3 With B as centre and radius BE, draw another arc. Mark the point of intersection F of this arc and the radius AC of the first circle. Also mark the points of intersection G and H of this arc with the first circle.

3.4 Now using the same radius BE, draw 2 arcs centred at G and H. Mark the points of intersection J and L of these arcs with the first circle. Draw the polygon BHLJG. What are the sides of this polygon in terms of 'r'?

3.5 What kind of polygon is BHLJG? What are the angles subtended by each side of this polygon at the centre A of the circle? Cut out the circle and fold along the sides of the polygon.

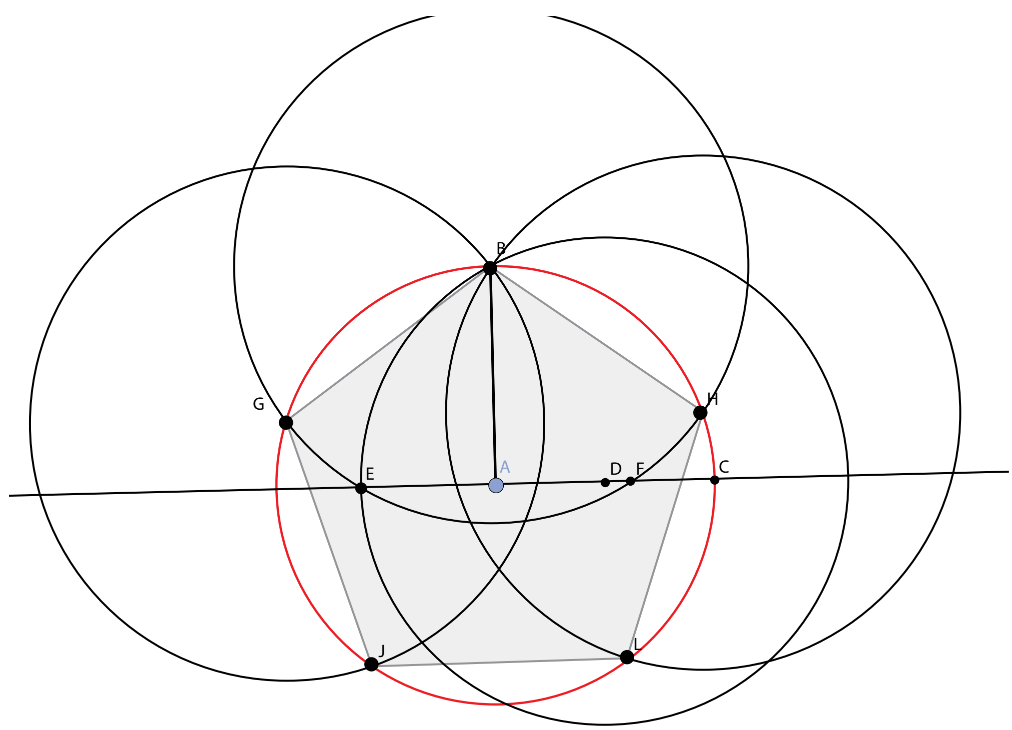


Figure 3

Teacher's Note:

This is a slightly more complex construction and students may get confused with the number of circles that have to be drawn. It would be a good idea to draw the original circle in a different colour. The calculation of the side of the pentagon in terms of 'r' may be difficult for students who are not comfortable with symbolic manipulation; in this case, we suggest that the student is asked to assume that the radius of the circle is 1 unit.

Refer to the Low Floor High Ceiling article "The Midas Touch" in the November 2015 issue of At Right Angles. Explain how the construction is related to the Golden Ratio.

An alternative way of getting a pentagon by a ruler and compass construction is found at http://www.mathopenref.com/constinpentagon.html and it would be useful for students to find similarities between the two methods. Using paper folding to get an inscribed pentagon is slightly complex; you could find clear instructions at https://www.youtube.com/watch?v=1X-UAjSGCpU. It's rich with mathematical connections and a great 'pause and question why' exercise for a teacher to do with a class.

TASK 4: From 2D to 3D

4.1 Fill in the following table

Regular Polygon	Interior angle α	Angle subtended by a side at the centre of the circle	Length of side in terms of 'r'	Multiples of α which are less than 360
Quadrilateral				
Triangle				
Pentagon				
Hexagon				
Octagon				

In this part of the activity, you will be making platonic solids. A platonic solid is a polyhedron in which all the polygonal faces are regular and congruent, and which has the same number of polygons at each vertex with all angles at the vertices equal. Make multiple cutouts of circles of the same radius which have equilateral triangles, squares, hexagons and pentagons inscribed in them. Take several congruent circles with the same regular polygon inscribed in each and join the segments of different circles to form solids.

4.2 Can you make a solid which has 3 equilateral triangles at each vertex? If so, how many faces does this solid have?

4.3 Can you make a solid which has square faces on all sides? How many squares meet at each vertex?

4.4 Can there be more than one solid which has square faces on all sides? If so, how many squares meet at each vertex in that case? How many faces does this solid have? How many such solids exist?

4.5 Can there be more than one solid which has equilateral triangles on all sides? If so, how many triangles meet at each vertex? How many faces does this solid have? How many such solids exist?

4.6 Which other regular polygon can be used to make a platonic solid?

4.7 Which polygons cannot be used?

4.8 How many platonic solids are there?

Teacher's Note:

This part of the activity is a shot at the ceiling but with concrete models to work with, students can definitely arrive at the conclusion that there are only 5 platonic solids. The table in the first part of the activity will help students direct their reasoning and it is good practice for them to draw conclusions by extrapolating from available data. Of course, no teacher can resist bringing up Euler's formula of $F + V - E = 2$ and this is a solid opportunity for students to verify this formula. Do encourage your students to use this formula to arrive at other proofs of the fact that there are only 5 platonic solids. If the teacher is so inclined, (s)he can take this a step further into Archimedean solids.

Here is a question which could trigger off such an investigation:

If a solid had the same combination of 3 faces (triangle, square, pentagon, hexagon or octagon- all of them regular) at each vertex, what are the possible solids that could be formed?

Teacher's Note:

You may be surprised at what students will come up with as they venture into trying all possible combinations. Do encourage them to write down their discoveries and begin to reason what can work and what won't. You can even bring in a parity argument and help them see that it must be odd-even-even or even-even-even (and therefore 3-3-even, 5-5-even, 3-5-even and 3-3-5 and 3-5-5 are ruled out.)

These two would give them several of the Archimedean solids, though not all…

Conclusion

Going round in circles has never been so much fun! These Platonic Solids make for great decorations as I am sure our photos reveal. Students work so much with pen and paper that 3 dimensions often fazes them – ironic, since it has been prescribed that the study of mathematics should connect with their experience of the real world. Do share photos of your students' creations on AtRiUM – our FaceBook page.

The Making of the Tetrahedron

Figure. 4: Equilateral Triangles inscribed in congruent circles.
Notice the one-sided paper and the folds toward the blank side.

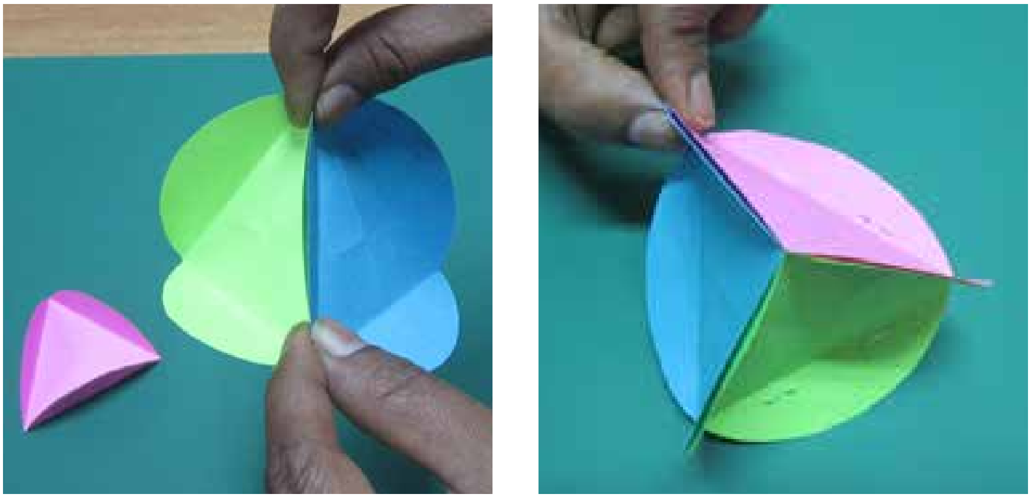


Figure. 5: Joining the circles along the folds to get the tetrahedron

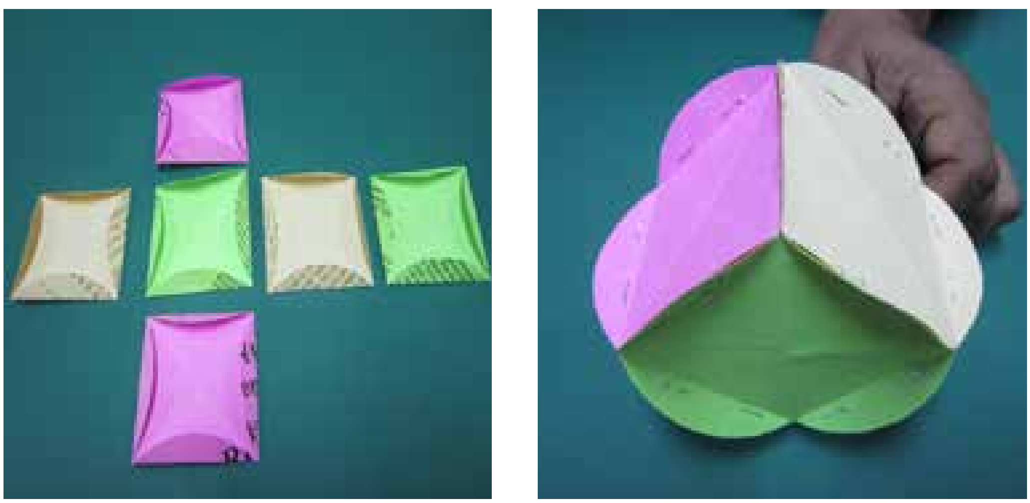


Figure. 6: The inscribed squares (notice the arrangement of the circles in the net of the solid) and a vertex of the cube formed when the six faces are joined.

Figure. 7: Preparations for the octahedron – and the finished product

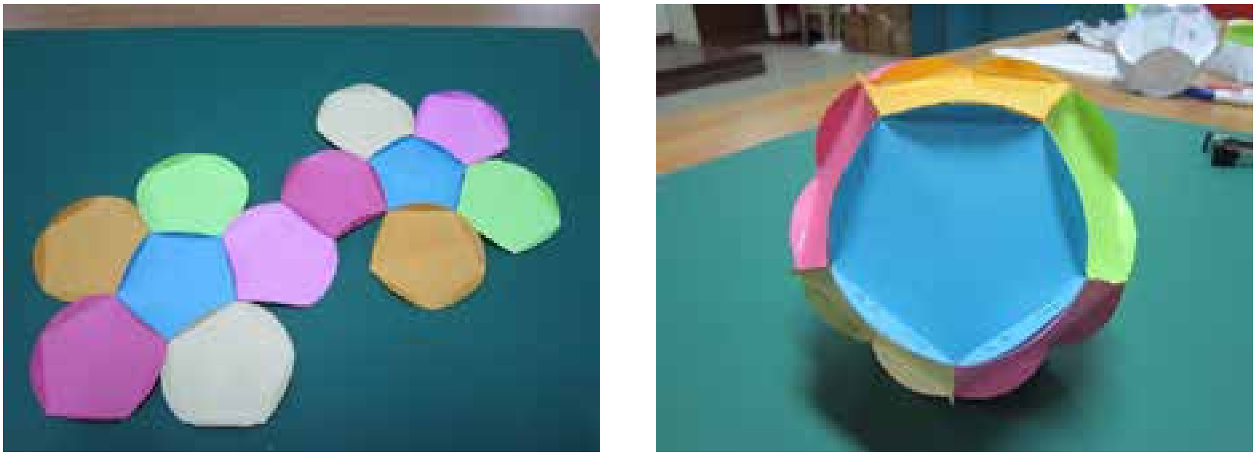


Figure. 8: The inscribed pentagons join together to form the 12 sided dodecahedron

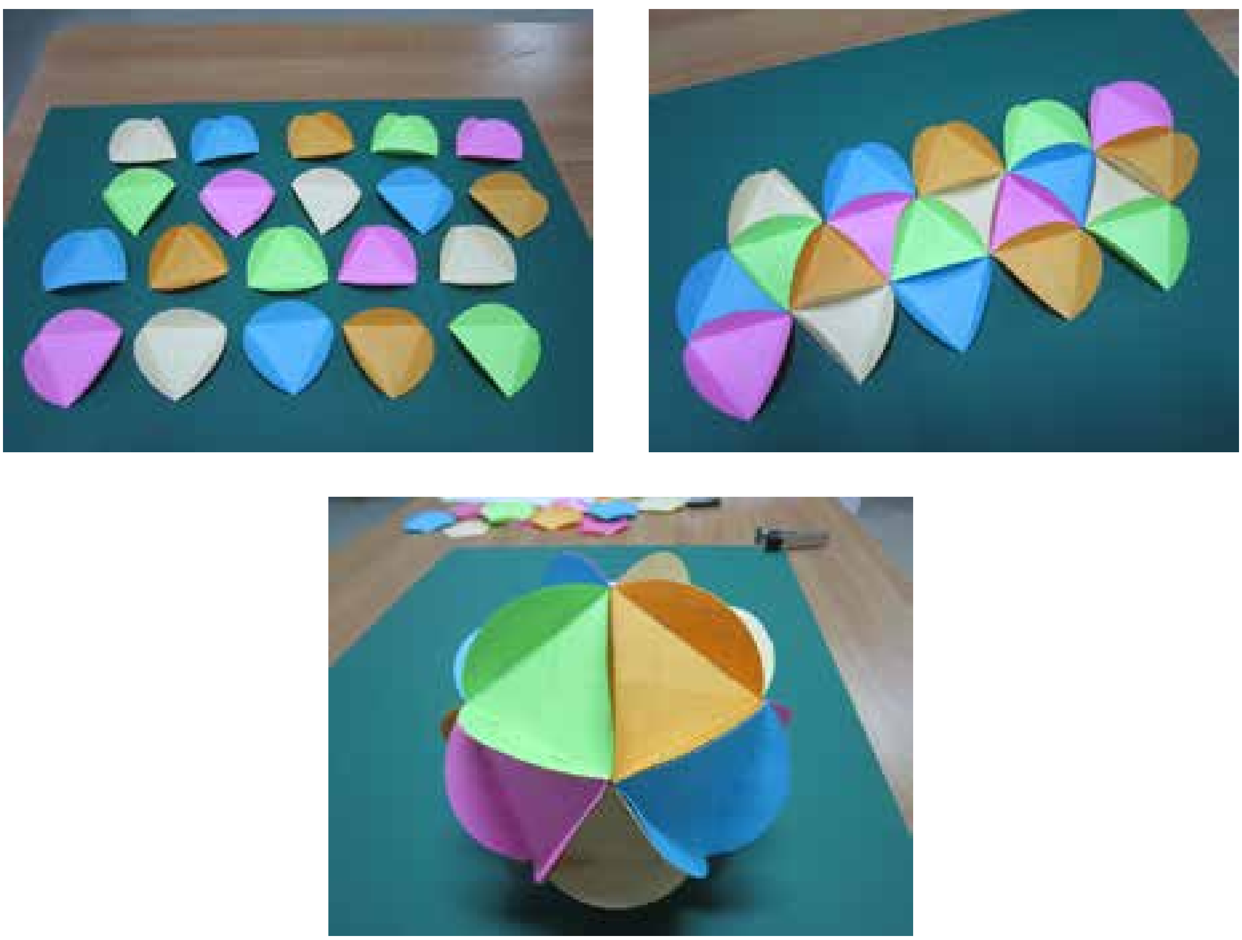
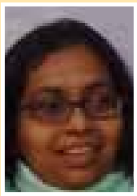


Figure. 9: And the grand finale: the assembling of 20 equilateral triangles to get the icosahedron. Notice the colour scheme; at least 5 colours are needed for no two circles at the same vertex to have the same colour.



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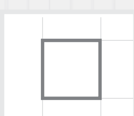
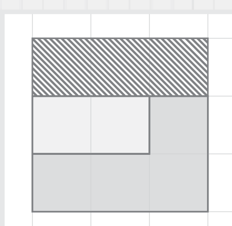
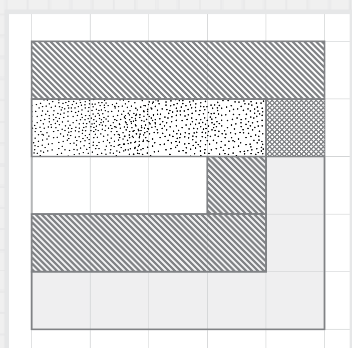
PICTURE SUMS FOR THE **ODD** SQUARES



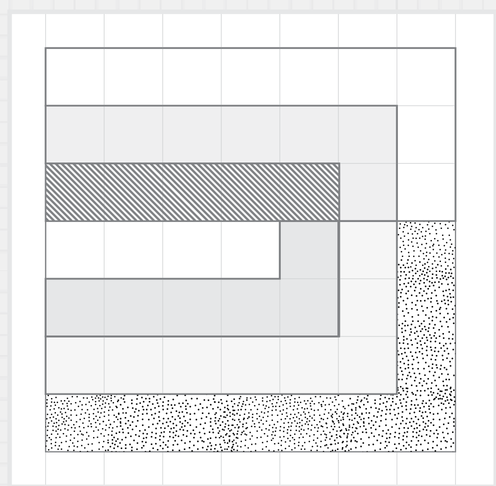
1



$2 + 3 + 4 = 3^2$



$3 + 4 + 5 + 6 + 7 = 5^2$



$4 + 5 + 6 + 7 + 8 + 9 + 10 = 7^2$

REFERENCES

Preebesh Kumar, https://www.facebook. com/photo.php?fbid=1809354995952471&set=gm.1215164325181388&type=3&theater, AtRiUM, June 19, 2016

ClassRoom

Middle School Mathematics

Some Reflections

A Ramachandran

Middle school maths is a convenient phrase used in textbook titles and faculty meetings and workshops for teachers. How do we distinguish the above from primary school maths, high school maths or higher secondary school maths? Primary level maths deals mostly with computation, whether it is with whole numbers or with fractions-and-decimals. Of course, a certain level of conceptualization takes place, but the emphasis is on drill and practice work so that the student will be able to calculate quickly and not get bogged down or daunted by numbers. One is trained to do the four basic operations with whole numbers as well as fractions and decimals. High school maths involves a good degree of abstraction, a lot of the content appearing to be far removed from daily life. Middle school maths in contrast is fairly grounded in the direct experience of students. There is less repetitive practice. It involves seeing patterns which can simplify situations and provide quicker pathways to solutions.

The teacher needs to be sensitive to the general temperament of students in this age group. Middle school students tend to be gregarious, enjoying social interaction, with peers as well as teachers.

The teacher should have a judicious mix of whole class discussions, group work, presentations, displays, home assignments, quiet individual work, etc., to transact the curriculum.

Keywords: *Middle school math, cooperative learning, activities, discussion, tables, LCM, GCD*

One of the key ideas/skills developed in middle school maths is the decomposition of numbers into factors, prime or otherwise – emphasizing multiplicative connections rather than additive ones. A simple way to start this is to revise the multiplication tables 'backwards'. Given a number, say 48, the student is asked to recollect multiplication facts featuring 48 as the product, in an oral exercise. The whole class could be involved in producing a factor table, where you go down the sequence of numbers, up to 100, 120 or 150, expressing each number as a product of two numbers in as many ways as possible. The students also get familiar with which numbers are prime, which numbers are composite, and which numbers are highly composite (i.e., having lots of factors).

Pedagogy: *The activity ensures drill and practice, but the approach is fresh and offers scope for discussion, is non-threatening (you may remember some tables but not all).*

This could be followed by an exercise in which one asks students to write down all the factors of a given number, including 1 and the given number, in ascending order. How does one ensure that all the factors have been taken? One could go stepwise, thus: Taking 60 to be the number in question, it is seen that 2 is a factor as also 30, 3 is a factor as also 20, 4 as also 15, 5 as also 12, and 6 as also 10. We seem to be moving to a central point from both sides. Now, since there are no factors of 60 between 6 and 10, we stop the search. This exercise leads to the observation (among others) that square numbers have an odd number of factors, while non-square numbers have an even number of factors.

Pedagogy: *It is good to emphasise the importance of individual written practice after discussion. Recording of learning helps both the student and the teacher; it helps the student reconstruct the concept or the subject matter, thus improving its registration in the mind; and it helps the teacher gauge if the student has followed the discussion.*

Pedagogy: *Building mathematical rigour and ensuring that all cases have been considered.*

Pedagogy: *Allow students to arrive at this observation – either overtly or in their practice.*

Another exercise that reinforces familiarity with numbers and their factors is based on the following observation. The product of two numbers x and y is the same as the product of the numbers ax and y/a. For instance, we could double one number and halve the other, and the product remains unchanged. Students could be encouraged to use this approach in multiplication. For instance, 48×75 may be replaced by 24×150 and then by 12×300, etc., as desired, to arrive at the product.

Pedagogy: *Strategy: laying the ground for algebraic thinking by pointing out the generalisation.*

These discussions and activities lead to a key topic at this level, LCM and GCD. Before we go for standard algorithms to obtain the LCM or GCD of two numbers, we could look at certain special cases. If the two numbers are mutually prime (this phrase needs to be defined now), then their LCM is simply their product and their GCD is 1. If one number is a factor of the other, then the greater number is the LCM and the smaller the GCD. In the general case, students can be encouraged to try to express the two given numbers as products of two numbers each, with one common factor which is as large as possible. For instance, given 36 and 48, we could express them as $36=12\times3$ and $48=12\times4$. Noting that 3 and 4 are mutually prime, 12 is the GCD and $12\times3\times4=84$ is the LCM.

Pedagogy: *Strategy of classification; understanding the difference between cases and the corresponding change in strategy.*

Middle school students also like to explore alternative pathways to the solution to a problem. They like to tackle multiple strategies, different groups or different individuals working on different lines and comparing results. For instance, the following problem is enjoyed by students who are often eager to try varied approaches. The task is to find the smallest number divisible by 1, 2, 3, …, 10. Some students interpret it straightaway as an exercise in finding LCM. They may take up the standard approach of expressing the numbers as products of primes. Others may go stepwise, thus: LCM of 2 and 3 is 6, LCM of 6 and 4 is 12, LCM of 12 and 5 is 60, and so on. Some others may try to cut out some numbers and simplify the situation thus: Since we have 6 in the list we can ignore 2 and 3; since we have 8 in the list we can ignore 4, and so on.

Pedagogy: *The teacher should give opportunities for students to justify their strategies thus building reasoning and logical thinking.*

An exercise with numbers (we confine ourselves to positive whole numbers for the present), which has applications in algebra is to find two numbers given their sum and product. After a few trial and error efforts one starts to choose between two approaches: splitting the 'sum' into two parts and finding their product, or expressing the 'product' as a product of two factors and finding their sum. One also notices a pattern. When we split the 'sum' into two parts and find their product, the product increases as the two parts get closer to each other. This ties up with the geometrical task of finding the rectangle with the greatest area for a given perimeter. This task could also be rounded off by preparing a graph of the observations, one of the linear dimensions (length or breadth) shown on the X-axis and the area shown on the Y-axis, to get the familiar parabolic shape.

Pedagogy: *Building HCK (horizon content knowledge), laying the foundation for concepts such as quadratics which will be taught in senior classes, also breaking the barriers between arithmetic, algebra and geometry, scope for visualisation and use of technology.*

If we think of prime numbers as atoms, then composite numbers are molecules. Getting to feel familiar with numbers and their factors and multiples is learning the composition or chemistry of numbers.

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ClassRoom

Dyscalculia:
A Little Known Learning Disability

Pooja Keshavan Singh

In recent years, Indian schools have become more sensitive to individual learning requirements, and teachers and psychologists have started to look more closely at individual learning trajectories. There have been attempts to explain differences in learning and response to teaching in spite of 'equal opportunities'. This explains why some schools have, on their panels, psychologists who can help plan instruction for individual needs. A lot has changed in recent years but much more needs to be done. This article dwells on a little known learning disability which affects around 5 percent of primary school children in India (see [5]), and 3% to 5% of the world school population according to an Australian study (see [8]). This means 1 to 2 children per class of 40 students.

For any child who is unable to do mathematics in spite of best instruction, a number of opportunities remain closed, but there are ways around the problem provided the child and her family have the awareness about the seriousness of the underlying condition. Literature in mathematics education has many terms that try to explain the 'learning difficulty in mathematics' and these range from 'learning disability in mathematics'

Keywords: *Mathematics difficulty, mathematics disability, Dyscalculia, Dysgraphia, Dyslexia*

to 'dyscalculia' to 'arithmetic disability' to 'maths disorder' to 'developmental dyscalculia', to 'acalculia' to 'anarithmetica', but there is no agreement on their use universally (see [7]), as we shall soon know why. Due to the different terminology used to describe difficulties in learning mathematics, there is also a lot of mismatch in the number of children reported in various studies having this condition in the normal classroom. Farham-Diggory (see [2]) in the US have reported that 80 percent of the children who are classified as 'learning disabled in mathematics' should not have been so classified in the first place (pp-56). I would like to make one important clarification though, that according to the available literature, 'learning difficulty' is a larger term and it includes students with learning disability in mathematics (Dyscalculia) and students whose low achievement may be a result of poor teaching methods. So, many students who are classified as 'learning disabled' could be having problems due to factors such as stressful environments or improper teaching and not due to a neurological condition.

Let us first try and understand how this learning disability – Dyscalculia – affects a child. I shall do so by drawing a summarised profile of a child coping with school at the expense of her carefree childhood. Ever since Solly started school, she had always had difficulty writing in straight lines. She would write at random places on a page and the teacher had to ask her where she had written what. After some time had passed, even Solly could not remember such details. She was slow to read in the primary grades and could not do addition and subtraction sums based on memory facts. For instance, most children by classes 2 and 3 can remember that 5+2 is 7, 3+4 is also 7, 5-2 is 3, and so on. Solly had to use her fingers over and over again and she would still make mistakes. She could not comprehend the passage of time in hours, minutes and seconds, that is, she also could not keep track of time. Sensing direction was also a problem; distinguishing left from right and east from west was difficult. These observations were recorded in Solly's portfolio maintained by her teachers over the years; thankfully, she had very observant teachers though they were not trained in identifying specific learning difficulties. Solly was referred to the school's child psychologist who put her through many standardised tests. The results showed that Solly had the IQ of a 'Slow learner', had mild dyslexia and that she needed help in mathematics. The school psychologist skipped a diagnosis on Dyscalculia. Had that condition been identified, Solly and her family would have been able to accept that her inability to do mathematics was due to a neurological condition, that is, a learning disability. As a result of this, Solly never improved much in mathematics but maths anxiety was added to her profile by the time she was in middle school. Clear indicators of her real problem were that though her IQ reached normal range, and she had no dyslexia three years later, numbers, operations, time and direction were very poorly comprehended by her.

Many individuals find it difficult to understand mathematics and it is so rampant and common that very often a learning difficulty is misread as learning disability in mathematics. According to Ginsburg (see [3]) there is a rampant misidentification of children with problems in mathematics due to two reasons. One, it is assumed that ordinary schools provide adequate learning opportunities to children (which may not be the case). Two, segregating children on the basis of IQ and achievement tests is overly broad because there are many reasons other than 'cognitive defects' that affect learning like motivation, self-concept, socio-cultural environment, etc. Shalev et al (see [6]) have reported that almost one half of the children who were identified with Dyscalculia in the fourth grade were still identified as having Dyscalculia 3 years later, clearly pointing to the tendency for misidentification or poor remediation.

A broad definition of Dyscalculia has been given by Chinn (see [1]): *Dyscalculia is a condition that impairs the ability to acquire mathematical skills. Dyscalculic learners can have difficulty in understanding number concepts, lack an intuitive grasp of numbers and have problems in remembering number facts and procedures.*

Each student has disabilities in mathematics which are unique, and they do not all exhibit the same traits. Some common characteristics listed by many researchers and summarised by Lerner and Kline (see [4]) can be classified as:

1. **Information Processing difficulties**

a. Attention: Difficulty maintaining attention to do steps in algorithms or problem solving.

b. Visual-spatial processing: Losing place in a worksheet, difficulty in seeing the differences between numbers, coins or operation symbols, writing in a straight line, problems with direction (up-down, left-right), aligning numbers, difficulty using a number line.

c. Auditory processing: Problems in 'counting on' from within a sequence, problems in following oral instructions.

d. Memory and retrieval: Difficulty remembering number facts, forgetting steps while doing problems, difficulty telling and remembering time, forgetting multiple step problems, poor sense of direction.

e. Motor problems: Writes numbers illegibly, slowly and inaccurately, difficulty in writing numbers in small spaces.

2. **Language and Reading difficulties:** Math word problems are difficult for students with reading disabilities because the child may not understand the underlying language structure.

3. **Math Anxiety:** Many children report that anxiety is their constant companion.

Dyscalculia is not a uniform phenomenon. By 'not uniform' we mean that the problem may be experienced differently by different individuals because of coexistence with other conditions like Dyslexia, ADHD and Autism. Also, since mathematics presents many facets like arithmetic, topology, probability, and so on, mathematical thinking is not unitary. It is thus necessary for us to look at the individual's problems in various domains and plan measures accordingly. Like other learning disabilities, Dyscalculia cannot be treated, but its effects can be mitigated so that a child can understand the basic ideas of mathematics even if she cannot perform mathematical tasks independently or express herself in the precise language of the subject. Some measures can be listed as follows.

First, it is important to consider the child's informal knowledge about mathematical concepts, because knowledge is an outcome of one's experiences with the real world; for instance, notions like bringing together, taking away, sequencing, equivalence, and so on. We should try to root all discussions in the child's external and internal world, especially problem solving, precisely because we all have an intuitive grasp of our personal problems and this helps in planning better solutions. It would also help to encourage variability in expressing a problem, such as drawing figures, animations, drama, songs and poems, etc. We should therefore accept solutions from children in varied modes as well. This way, the focus is on thinking rather than on the language or syntax of expression. This flexibility needs to be extended to the assessment of learning. Teachers can reduce abstraction of ideas by using manipulatives and multiple modes for introducing concepts to children and reduce their use gradually, based on individual needs. For instance, manipulatives like counters (beads, pebbles and marbles) can be used for counting and number operations. Involving whole body movements such as climbing stairs can be used to introduce integers. Many physical games can also be planned around mathematics concepts. Cues, primers and 'prosthetic aids' can be given to children for assistance. For instance, number charts and calculators for performing basic number operations, computer spreadsheets and word processors can help in writing. Reading software can help those with coexisting conditions because Dyslexia, Dysgraphia and Dyscalculia can exist together. Such a situation affects a student with Dyscalculia even more. Mnemonics can be devised to help remember steps in problem solving and markers can be developed for sensing time and direction. The individual students may benefit from one-to-one sessions so that they can cope with the whole class discussions. For older children, it is helpful to provide them with a checklist of possible strategies. This provides them with a structure. Most importantly, math anxiety should be minimised at all costs by encouraging students to develop alternate interests and hobbies for emotional growth and professional development. By developing other interests, students understand that Dyscalculia does not define them as a person. This helps in limiting a problem to a small part of one's persona rather than casting a shadow on one's existence.

Dyscalculia is not recognised by CBSE as a learning disability and that is why the relaxation given to children with Dyslexia is not extended to children with Dyscalculia. This must change because children with Dyscalculia need assistance in terms of extra time and writing support (option to use a scribe), just like children with Dyslexia.

Moreover, an option to leave mathematics after primary school also must be offered in special cases of Dyscalculia so that time and resources can be utilised more constructively. General awareness about Dyslexia is greater when compared with Dyscalculia but that does not make Dyscalculia any less important, because every individual has the right to special instruction in order to realise her potential. Given the 'halo' around mathematics in modern times, it is time we address the problem of mathematics disability for the sake of every individual.

Resources

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A New Test for Divisibility by 8

$\mathcal{C}\otimes\mathcal{M}\alpha\mathcal{C}$

The standard test for deciding whether a given number is divisible by 8 is the following:

> *Let N denote the given number. Examine only the last three digits of N. Regard these as making up a three-digit number; call this number M. Then:*
>
> - *If M is divisible by* 8*, then N is divisible by* 8*.*
> - *If M is not divisible by* 8*, then N is not divisible by* 8*.*
>
> *In short: N is divisible by* 8 *if and only if M is divisible by* 8*.*

This test allows us to decide on divisibility by 8 by considering only whether a particular three-digit number is divisible by 8. Since the original number may be arbitrarily large, the test simplifies our task by reducing our labour.

Could there be a test for divisibility by 8 which involves even less work than the above test? The surprising answer to this is: **Yes**. As per the report which appeared in a national newspaper ([1]), such a test has been devised by mathematics teacher **Sursinh Parmar**, of Kodinar Taluq, Saurashtra. On September 6, 2016, Parmar received the Best Teacher Award for this innovation. Reportedly, the discovery came about as a result of a query put by a child in class 5, who asked in effect whether the standard

Keywords: *Divisibility by 8, test, Saurashtra*

test for divisibility by 8 could be carried out with fewer divisions. In an interview to the newspaper, Parmar stated: "The query baffled me and forced me to think …. I had no answer to the query …." After wrestling with the problem for a while, he came up with the test described below.

Notation for divisibility. First we describe a convenient short-form notation that we shall use throughout this article: $a \mid b$ means that a is a divisor of b; i.e., b is divisible by a. (Examples: $4 \mid 12$; $5 \mid 35$.) The notation $a \nmid b$ means that a is *not* a divisor of b; i.e., b is not divisible by a. (Examples: $2 \nmid 5$; $3 \nmid 10$.)

Algorithm. Let the given number be N. As in the standard test, form the number M made by using its last three digits. Write M as abc where a is the Hundreds digit, b is the Tens digit, and c is the Units digit of M.

Step 1: Check whether or not $4 \mid bc$. If not, conclude that $8 \nmid N$.

Step 2: If bc is divisible by 4, compute the quotient $q = bc \div 4$.

Step 3: If a and q are both odd or both even, conclude that $8 \mid N$; else, that $8 \nmid N$.

Examples.

Ex 1: Let $N = 1,003,496$; then $M = 496$, $a = 4$, $bc = 96$, $bc \div 4 = 96 \div 4 = 24$. Since 24 and 4 are both even, we conclude that $8 \mid N$.

Ex 2: Let $N = 2,842,536$; then $M = 536$, $a = 5$, $bc = 36$, $bc \div 4 = 36 \div 4 = 9$. Since 5 and 9 are both odd, we conclude that $8 \mid N$.

Ex 3: Let $N = 6,042,586$; then $M = 586$, $a = 5$, $bc = 86$. We observe that $4 \nmid bc$; therefore $8 \nmid N$. (In this example, we did not even get past Step 1.)

Ex 4: Let $N = 6,042,588$; then $M = 588$, $a = 5$, $bc = 88$, $bc \div 4 = 88 \div 4 = 22$. Since 5 is odd whereas 22 is even, we conclude that $8 \nmid N$.

The proof of correctness of the algorithm is given at the end of the article. Further streamlining may be attempted by studying patterns in the multiples of 4.

Extension: Test for divisibility by 4. It is instructive to connect the above algorithm with the well-known test for divisibility by 4:

> *Let N denote the given number. Examine only the last two digits of N. Regard these as making up a two-digit number M. Then: (i) if $4 \mid M$, then $4 \mid N$; (ii) if $4 \nmid M$, then $4 \nmid N$. In short: $4 \mid N$ if and only if $4 \mid M$.*

We can streamline the algorithm in the light of Parmar's idea as follows. Let the given number be N. Form the number M made by using its last two digits. Write M as bc where b is the Tens digit, and c is the Units digit of M.

Step 1: Check whether $2 \mid c$. If not, conclude that $4 \nmid N$.

Step 2: If $2 \mid c$, compute the quotient $q = c \div 2$.

Step 3: If b and q are both odd or both even, conclude that $4 \mid N$; else that $4 \nmid N$.

This may be stated much more compactly as follows:

> An even number is divisible by 4 if and only if the Tens digit and half the Units digit are either both odd or both even.

Extension: Test for divisibility by 16. In the other direction, we may also generate, along much the same lines, a streamlined test for divisibility by 16.

Let the given number be N. Form the number M made by using the last four digits of N. Write M as $abcd$ where a is the Thousands digit, b is the Hundreds digit, c is the Tens digit, and d is the Units digit of M.

Step 1: Check whether or not $8 \mid bcd$. If not, conclude that $16 \nmid N$.

Step 2: If $8 \mid bcd$, compute the quotient $q = bcd \div 8$.

Step 3: If a and q are both odd or both even, conclude that $16 \mid N$; else, that $16 \nmid N$.

Proof of correctness of the algorithm for testing divisibility by 8. The proof hinges on the following simple observations which may be easily verified:

- $4 \mid 100$; $8 \nmid 100$; $8 \mid 200$. So 8 is a divisor of all even multiples of 100, but a non-divisor of all odd multiples of 100. Further, since 100 leaves remainder 4 on division by 8, it follows that every odd multiple of 100 leaves remainder 4 on division by 8.
- Let bc be a two-digit multiple of 4, and let q be the quotient in the division $bc \div 4$. If q is even, then it must be that bc is a multiple of 8; and if q is odd, then it must be that bc leaves remainder 4 on division by 8.

The proof of the test for divisibility by 8 follows from these observations. Let $M = abc$ be a three-digit multiple of 4, and let q be the quotient in the division $bc \div 4$. Note that $M = a00 + bc$. These two numbers ($a00$ and bc) are either both multiples of 8, or both leave remainder 4 on division by 8. If M is to be a multiple of 8, then one of the following must happen:

- Both $a00$ and bc are multiples of 8. This will be the case if a and q are both even.
- Both $a00$ and bc leave remainder 4 on division by 8. This will be the case if a and q are both odd.

The logic behind the algorithm should now be clear. □

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The COMMUNITY MATHEMATICS CENTRE (CoMaC) is an outreach arm of Rishi Valley Education Centre (AP) and Sahyadri School (KFI). It holds workshops in the teaching of mathematics and undertakes preparation of teaching materials for State Governments and NGOs. CoMaC may be contacted at shailesh.shirali@gmail.com.

Approximate Constructions of Certain Angles

ClassRoom

Shailesh Shirali

On the Facebook page of AtRiA, a reader (Surojit Shaw; see https://www.facebook.com/photo.php?fbid=787370044738665&set=p.787370044738665&type=3) posted the following comment in which he proposed constructions of certain angles (the words have been changed slightly, but the meaning is unaltered):

> *Using a compass, I construct a* 60° *angle EAB using* 6 *cm as the radius (see Figure 1;* $AB = AE = BE = 6$ *cm). Now for a* 40° *angle I take* 4 *cm as radius, put the compass point at B, draw an arc to cut the arc at C and join AC;* $\measuredangle CAB$ *will then be* 40°*. Similarly, for a* 50° *angle, I take* 5 *cm as the radius and repeat the same procedure (*$BD = 5$ *cm);* $\measuredangle DAB$ *will then be* 50°*. For a* 8° *angle, I take* 0.8 *cm, and so on. In this way we can construct other angles as well.*

The post invites us right away to try out the procedure using *GeoGebra*! We in turn invite the reader to do so and to explore the degree of accuracy of this construction.

The post is also instantly provocative; it seems to suggest that one can construct virtually any angle using a compass and a straightedge! Though the post mentions actual measurements (thus requiring a marked ruler), an unmarked straightedge would

Keywords: *approximate, angle, construction, compass*

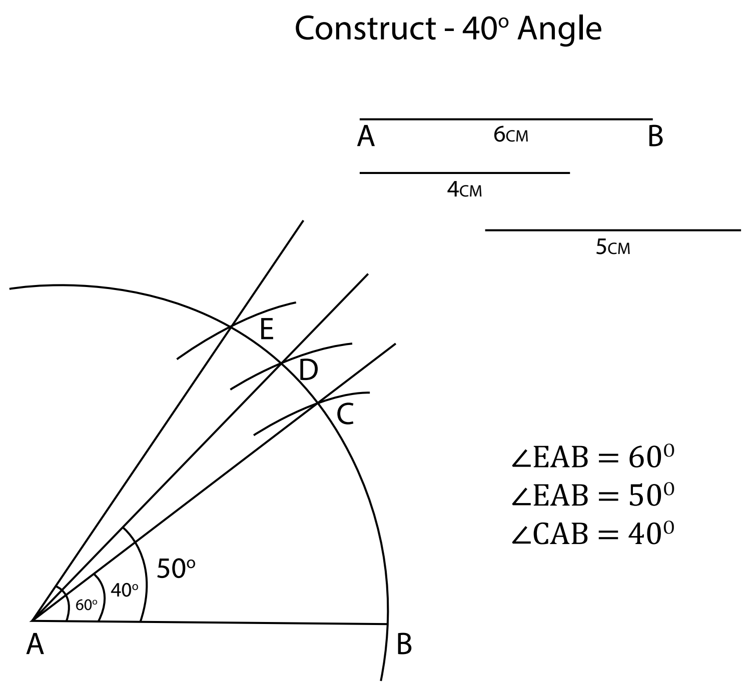

Figure 1

suffice. For: a 4 cm length is 2/3 of a 6 cm length; and one can construct 2/3 of a given line segment using only a compass and an unmarked straightedge.

In essence, the method may be described as follows. We draw a line segment AB with length a cm (in the FB post quoted above, we have $a = 6$) and then draw an arc centred at A, with the same radius a (Figure 2). We now wish to draw a ray AC such that $\measuredangle CAB$ has some desired measure t°. To do so, we measure off the length $t/10 \times a/6 = ta/60$ cm on the compass, lay the compass point at vertex B and draw an arc with this radius to cut the earlier arc at point C. (Why the fraction $t/10$? Examine the algorithm: for an angle of 50° he uses a radius of 5 cm, for an angle of 40° he uses a radius of 4 cm, and so on.) The claim then is that $\measuredangle CAB$ has the desired measure.

In Figure 2, the measure of $\measuredangle CAB$ is *supposed* to be t°. Let its actual measure be x°. To find the relationship between x and t, note that $\triangle CAB$ is isosceles, with $AB = AC$. Let M be the midpoint of segment CB; then AM is perpendicular to BC, so $\measuredangle MAB = x^\circ/2$, and $BM = ta/120$ cm. Hence we have:

$$\sin\frac{x^\circ}{2} = \frac{BM}{AB} = \frac{ta/120}{a} = \frac{t}{120},$$

$$\therefore\ \frac{x}{2} = \frac{180}{\pi}\ \arcsin\frac{t}{120}, \tag{1}$$

and so:

$$x = \frac{360}{\pi}\ \arcsin\frac{t}{120}. \tag{2}$$

(Note: In equality (1), the multiplicative factor $180/\pi$ has been inserted because we are measuring the angle in degrees and not radians.)

We have found the desired relationship: $x = (360/\pi)\ \arcsin(t/120)$. This allows us to compute x for any given t. The table below has been computed using this formula.

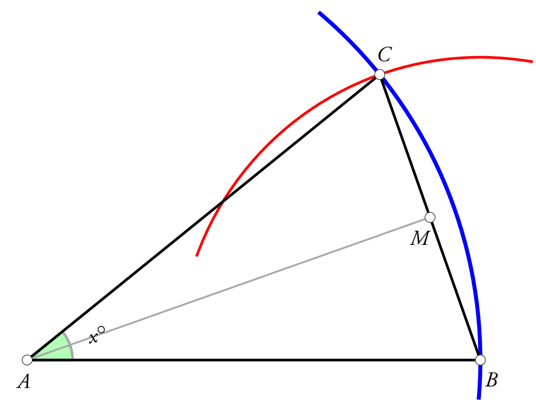

Figure 2

t	10	20	30	40	50	60	70	80	90
x	9.56	19.19	28.96	38.94	49.25	60	71.37	83.62	97.18

The results are striking. We observe that x is quite close to t for a good many values; and when $t = 60$, x and t are exactly equal to each other. For values of t beyond 60, however, the discrepancy between the two values grows steadily larger.

Figure 3 shows the graphs of both $x = (360/\pi)\ \arcsin(t/120)$ and $x = t$. Observe the closeness of the two graphs, especially for values of t between 0 and 60.

Mathematical essentials of the approximation

The approximation

$$t \approx \frac{360}{\pi} \arcsin \frac{t}{120} \qquad (0 \le t \le 60) \tag{3}$$

can be written in other ways that allow us to analyse it mathematically. We first write it as:

$$\sin \frac{\pi\, t}{360} \approx \frac{t}{120}, \qquad 0 \le t \le 60; \tag{4}$$

or, replacing t by $2t$ on both sides (the reason for doing this will become clear in a moment) and simplifying:

$$\sin \frac{\pi\, t}{180} \approx \frac{t}{60}, \qquad 0 \le t \le 30. \tag{5}$$

Now we have:

$$\frac{\pi\, t}{180} \text{ radians} = t^\circ.$$

Hence the proposed approximation is equivalent to the following assertion:

$$\sin t^\circ \approx \frac{t}{60}, \qquad 0 \le t \le 30; \tag{6}$$

or, switching back to radian measure:

$$\sin t \approx \frac{3t}{\pi}, \qquad 0 \le t \le \frac{\pi}{6}. \tag{7}$$

Note that the approximation is exact for $t = 0$ and for $t = \pi/6$ (i.e., for 0° and for 30°).

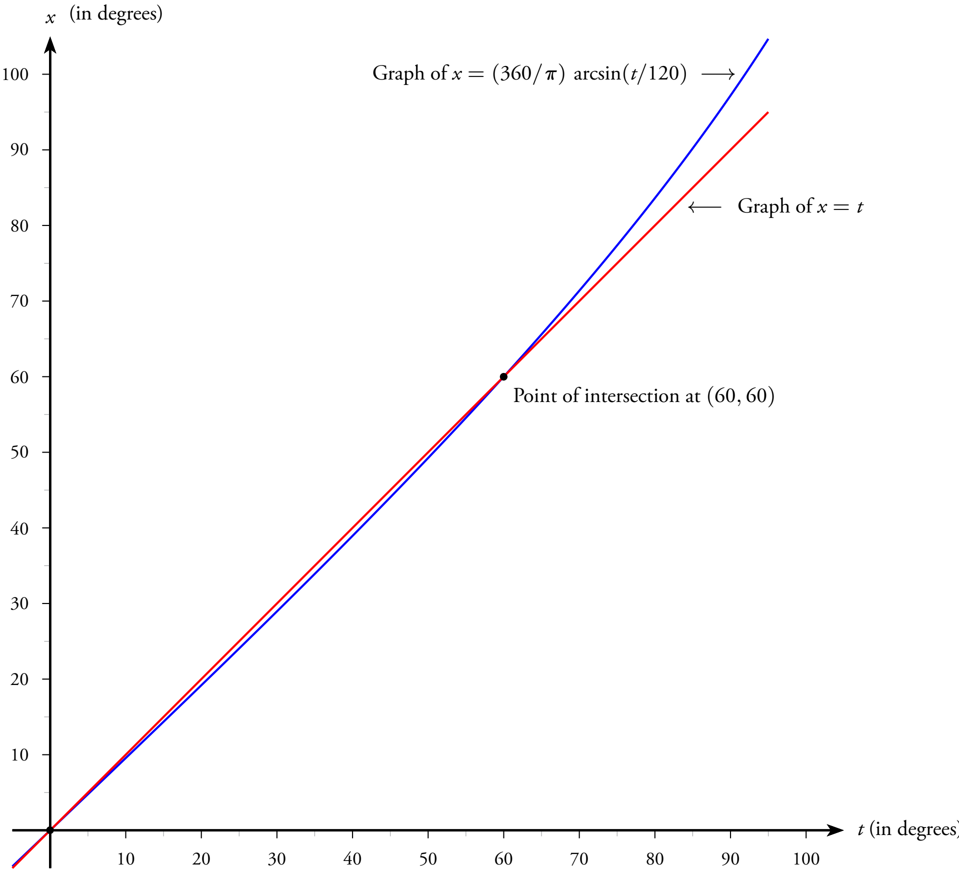

Figure 3

Error analysis. It is of interest to find out at which point in the interval I from 0 to $\pi/3$ the approximation is the worst. Let

$$f(t) = \sin t - \frac{3t}{\pi}, \quad 0 \leq t \leq \frac{\pi}{3}.$$

Then:

$$f'(t) = \cos t - \frac{3}{\pi}, \qquad f''(t) = -\sin t.$$

Using tables or a scientific calculator, we find that the acute angle whose cosine is $3/\pi$ is roughly 0.3014 radians, or roughly 17.27°; and, of course, $-\sin 17.27^\circ < 0$. For this value of t, therefore, $f(t)$ attains its maximum value within the interval I. The discrepancy between the two functions at this value of t is 0.00904. This represents a 3% error. Note that within I, $f(t)$ is consistently non-negative. So the function under study consistently overestimates the true value.

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How to Prove It

In this episode of "How To Prove It", we prove a striking theorem first discovered by Ptolemy. We then discuss some nice applications of the theorem.

Shailesh A Shirali

In this article we examine a famous and important result in geometry called *Ptolemy's Theorem*. Here is its statement (see Figure 1):

Theorem 1 (Ptolemy of Alexandria). *If ABCD is a cyclic quadrilateral, then we have the following equality:*

$$AB \cdot CD + BC \cdot AD = AC \cdot BD. \tag{1}$$

In words: "The sum of the products of opposite pairs of sides of a cyclic quadrilateral is equal to the product of the diagonals."

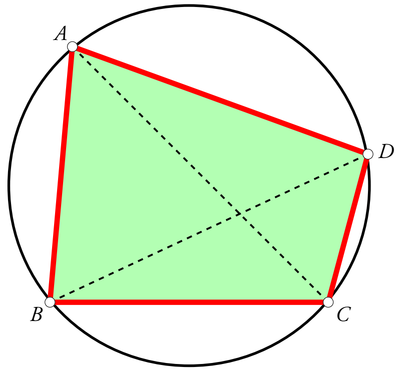

Figure 1. Cyclic quadrilateral and Ptolemy's theorem

Keywords: Ptolemy, similar triangle, power of a point

A 'pure geometry' proof. To prove the theorem presents a challenge. The difficulty lies in the fact that neither side of the equality $AB \cdot CD + AD \cdot BC = AC \cdot BD$ seems to *mean* anything. Terms like $AB \cdot CD$ and $AD \cdot BC$ suggest areas; but of what? There is nothing in the figure that yields a clue. So we try a different approach. We write the equality to be proved as

$$\frac{AB \cdot CD}{BD} + \frac{BC \cdot AD}{BD} = AC. \tag{2}$$

Have we made progress by writing it this way? Perhaps. Now the equality to be proved is a relation between *lengths*. Can we find or construct two segments whose lengths together yield the length of AC?

The expressions on the left ($AB \cdot CD/BD$ and $BC \cdot AD/BD$) suggest that we must look for or construct suitable pairs of similar triangles. Indeed, the form $AB \cdot CD/BD$ suggests that we should construct a triangle similar to $\triangle ABD$, and moreover that this (yet to be constructed) triangle should have CD for a side. Noting that $\angle ABD = \angle ACD$ we ask: what if we locate a point E on AC such that $\triangle ABD \sim \triangle ECD$? Then we would have $AB/BD = EC/CD$, giving $EC = AB \cdot CD/BD$; just what we want! Now we have a clue on how to proceed. Figure 2 shows the construction.

Locate a point E on diagonal AC such that $\angle CDE = \angle ADB$ (the two angles are marked with a bullet in Figure 2). Now consider $\triangle CDE$ and $\triangle ADB$. Since $\angle ECD = \angle ABD$ by the angle property of a circle, and $\angle CDE = \angle ADB$ by design, we have $\triangle CDE \sim \triangle ADB$. Hence:

$$\frac{EC}{CD} = \frac{AB}{BD}, \qquad \therefore\ EC = \frac{AB \cdot CD}{BD}. \tag{3}$$

With reference to the same figure (redrawn as Figure 3) we also have $\triangle DAE \sim \triangle DBC$, because $\angle DAE = \angle DBC$ and $\angle ADE = \angle BDC$.

Hence:

$$\frac{BC}{BD} = \frac{AE}{AD}, \qquad \therefore\ AE = \frac{BC \cdot AD}{BD}. \tag{4}$$

Adding (3) and (4) we get, since $AE + EC = AC$:

$$AC = \frac{AB \cdot CD}{BD} + \frac{BC \cdot AD}{BD}, \qquad \therefore\ AC \cdot BD = AB \cdot CD + BC \cdot AD, \tag{5}$$

as was to be proved. $\square$

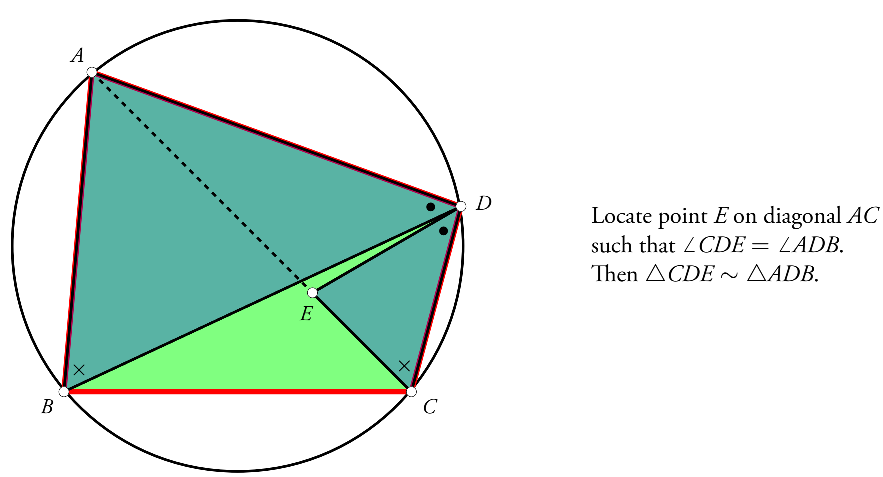

Figure 2. Construction of an appropriate point E on diagonal AC

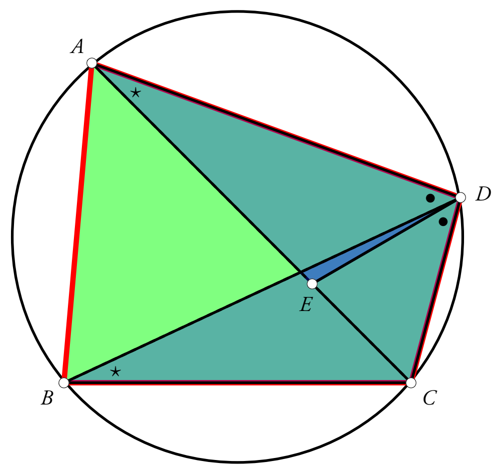

Figure 3. Another pair of similar triangles

You will agree that this is a very elegant proof (it is the proof given by Ptolemy), but it would not be easy to find it on one's own.

It turns out that Ptolemy's theorem can be proved in many different ways. Of particular interest are the following: (i) a proof using complex numbers, (ii) a proof using vectors, (iii) a proof based on a geometrical transformation called 'inversion'. We will have occasion to study these different ways in later articles.

A few elegant applications of Ptolemy's theorem

We showcase below three pleasing applications of the theorem proved above. The first one is an elegant result relating to an equilateral triangle.

Theorem 2. *Let ABC be an equilateral triangle, and let P be any point on the circumcircle of the triangle. Then the largest of the distances PA, PB, PC is equal to the sum of the other two distances.*

The theorem is illustrated in Figure 4. Note that P is located on the minor arc BC, i.e., it lies between the points B and C. The theorem now asserts that $PA = PB + PC$.

The proof is simplicity itself. Consider the cyclic quadrilateral $PBAC$. Apply Ptolemy's theorem to it; we get:

$$PA \cdot BC = PB \cdot AC + PC \cdot AB. \tag{6}$$

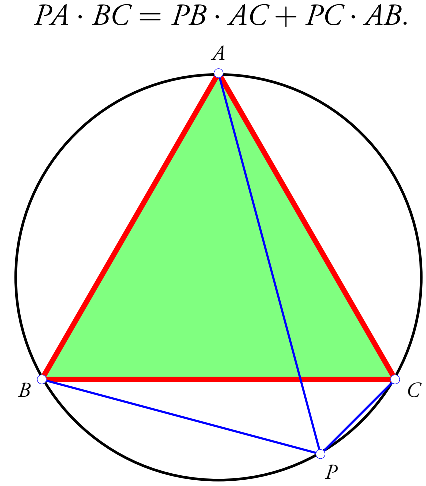

Figure 4. Application of Ptolemy's theorem to an equilateral triangle

But $AB = BC = CA$. The equal factor present in all three products in (6) can be cancelled, and we are left with the desired relation, $PA = PB + PC$. □

Remark. Theorem 2 can also be proved using the following trigonometric identity: for all angles θ (measured in degrees),

$$\sin(60 - \theta) + \sin\theta = \sin(60 + \theta). \tag{7}$$

The next result that we describe refers to a regular pentagon. If you examine such a pentagon, you will notice that it has five diagonals all of which have the same length.

Theorem 3. *Given a regular pentagon with side a, let its diagonals have length d. Then we have the following relation:*

$$a^2 + ad = d^2. \tag{8}$$

In Figure 5, $ABCDE$ is a regular pentagon; its sides have length a and its diagonals have length d. We apply Ptolemy's theorem to the inscribed quadrilateral $BCDE$; we get:

$$BC \cdot DE + BE \cdot CD = BD \cdot CE. \tag{9}$$

That is, $a^2 + ad = d^2$, as required. □

If we write $x = d/a$ (i.e., x is the ratio of the diagonal to the side of a regular pentagon), then the above equation yields: $x^2 = x + 1$. Solving this we get:

$$x = \frac{1 \pm \sqrt{5}}{2}.$$

The negative sign clearly cannot hold, since x is positive; hence we have:

$$x = \frac{\sqrt{5} + 1}{2}. \tag{10}$$

So the ratio of the diagonal to the side of a regular pentagon is the Golden Ratio!

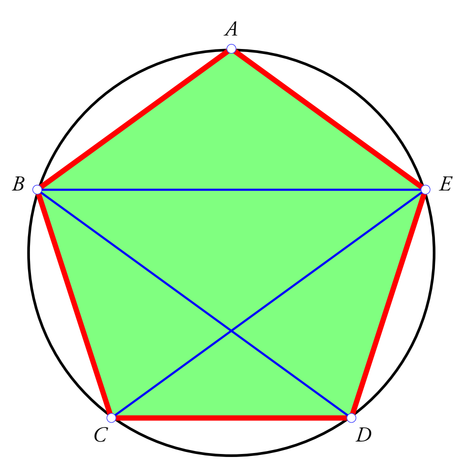

Figure 5. Application of Ptolemy's theorem to a regular pentagon

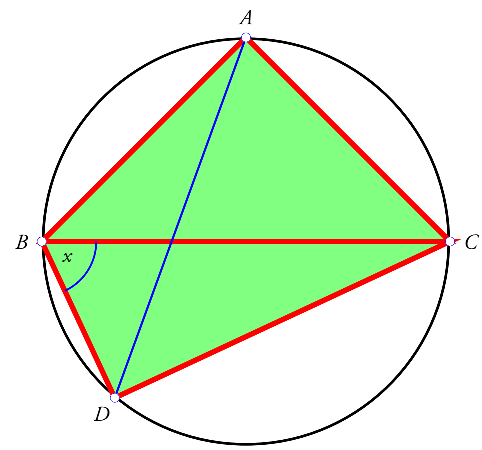

Figure 6. Application of Ptolemy's theorem to a trig inequality

Our third result is a trigonometric inequality which would seem difficult to prove using purely geometric methods. We shall show that for any acute angle x,

$$\sin x + \cos x \leq \sqrt{2}. \tag{11}$$

Figure 6 shows a circle with unit radius, a diameter BC, an isosceles right-angled triangle ABC with BC as hypotenuse, and a right-angled triangle DBC with BC as hypotenuse and with one acute angle equal to x; vertices A and D lie on opposite sides of BC.

We apply Ptolemy's theorem to the quadrilateral $ABDC$:

$$AB \cdot CD + AC \cdot BD = AD \cdot BC.$$

Since $AB = AC = \sqrt{2}$, $BD = 2\cos x$, $CD = 2\sin x$, $BC = 2$, we get:

$$2\sqrt{2}\,\sin x + 2\sqrt{2}\,\cos x = 2\,AD.$$

Now note that AD is a chord of the circle and so does not exceed in length the diameter of the circle; hence $AD \leq 2$. This yields $2\sqrt{2}\,\sin x + 2\sqrt{2}\,\cos x \leq 4$, and therefore:

$$\sin x + \cos x \leq \sqrt{2}, \tag{12}$$

as claimed. □

In the next episode of "How To Prove It" we shall showcase a few more applications of Ptolemy's theorem, and also prove an inequality version of the theorem.

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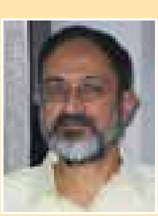

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A REVERSE GOLDBACH THEOREM!

$\mathcal{C}\otimes\mathcal{M}\alpha\mathcal{C}$

Almost everyone has heard of the Goldbach conjecture, first stated by the mathematician Christian Goldbach in a letter to the great Leonhard Euler written in 1742, in which he made the following statement.

Conjecture: Every integer greater than 2 can be written as the sum of three primes.

The modern version of this conjecture is the following (see reference 1 noted below):

Conjecture: Every even integer greater than 2 can be written as the sum of two primes.

The statement has an astonishingly simple appearance; yet it remains unproved to this day, though slightly weaker forms of the statement have been proved.

Here we state and prove a reverse Goldbach theorem!

Theorem: Every prime number exceeding 12 is a sum of two composite numbers.

Considering that the Goldbach conjecture remains unproved, this theorem is surprisingly easy to prove. Here is one such proof, given by LinkedIn member Anders Dahlner; see reference 3.

Proof of the theorem: Suppose we have a prime number $p > 11$; obviously, p is odd. Consider the number $m = p - 9$. Obviously, m is an even number exceeding 3. Hence m is a composite number. And 9 too is a composite number. Hence $p = 9 + m$, where both 9 and m are composite. So we have expressed the prime number p as the sum of two composite numbers. It follows that every prime number exceeding 11 can be expressed as the sum of two composite numbers. QED!

Remark: It may be easily checked that the prime numbers 2,3,5,7,11 cannot be written as a sum of two composite numbers. This means that a prime number can be expressed as the sum of two composite numbers if and only if it exceeds 11.

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Fractal Constructions Leading to Algebraic Thinking

TechSpace

This article describes how pre-service teachers explored fractal constructions using pictorial, numerical, symbolic and graphical representations while studying the topic geometric sequences in the Algebra unit of their mathematics course. By engaging with meaningful generalization tasks which required both explicit and recursive reasoning, they gained an insight into fractal geometry and also developed their algebraic thinking.

Jonaki B Ghosh

Introduction

One of the foundational aspects of developing algebraic thinking is the ability to generalize. Research describes two kinds of generalization (Kinach, 2014), namely, *generalization by analogy* and *generalization by extension*. Generalization by analogy refers to observing a pattern, extending a sequence to the next few terms and being able to relate a particular term of the sequence to its previous terms. This kind of generalization requires *recursive thinking*. Generalization by extension, on the other hand, refers to writing a formula for the nth term of a sequence – which requires *explicit thinking*. Both kinds of generalization require abstraction and form the core of algebraic thinking. In fact generalization is a skill which is required at various stages of the school mathematics curriculum. However, guiding students through the process of generalization can be quite challenging and teachers must be familiar with tasks which can create a context for generalization.

Keywords: *Algebra, generalization, analogy, extension, recursive, explicit, Sierpinski triangle, self-similarity, fractal construction, infinite, geometric progression*

In this article we shall highlight that the topic of fractals provides an authentic context for engaging in generalization tasks giving ample opportunity for developing recursive and explicit thinking. According to Kinach (2014), *calculating the area and perimeter of the growing pattern of red triangles in the famous Sierpinski triangle and then expressing a formula for the area and perimeter for any iteration …are more advanced examples of generalization by analogy and extension.* (p.443)

We shall describe a module where 30 students of a pre-service teacher education programme explored fractals as a part of the Algebra unit in their *Core Mathematics* course. The primary goal was to engage them in exploring various patterns within fractal constructions through pictorial, tabular, symbolic and graphical representations, and to make connections between these representations. We will highlight that while going through the module they developed an insight into the nature of fractal geometry and engaged in meaningful generalization tasks emerging from the construction process. Mathematics curricula in many countries have emphasised the importance of developing algebraic thinking and the same has been articulated in the *Principles and Standards for School Mathematics* as 'expectations in the Algebra Standard' that students in grades 9 – 12 should be able to

- Generalize patterns using explicitly and recursively defined functions,…use symbolic algebra to represent and explain mathematical relationships;…[and] use symbolic expressions, including iterative and recursive forms, to represent relationships arising from various contexts. (NCTM 2000, p. 296)

The position paper *Teaching of mathematics* of the National Curriculum framework (NCF) 2005 (National Council for Educational Research and Training [NCERT], 2005) also articulates the importance of developing algebraic skills in the secondary school stage

- Algebra…is developed at some length at this stage. Facility with algebraic manipulation is essential, not only for applications of mathematics, but also internally in mathematics. Proofs in geometry and trigonometry show the usefulness of algebraic machinery. It is important to ensure that students learn to geometrically visualize what they accomplish algebraically. (NCF 2005, p. 17)

The fractal investigations – a background

The module was conducted by the author with 30 first year students of a pre-service teacher education programme, as a part of the algebra unit of their *Core Mathematics* course. The focus of this course is to enable the student teacher to enhance her content knowledge of the school mathematics curriculum. 12 out of the 30 students who went through this module had studied mathematics in school up to grade 10 and the rest had studied mathematics up to grade 12. Prior to the module, students had recapitulated their knowledge of arithmetic sequences, exponents and had been introduced to the concept of geometric sequences. The author (who was also their teacher) decided to use fractal constructions to enhance their understanding of geometric sequences.

Understanding fractal constructions through multiple representations

Being able to work with a variety of representations such as tables, pictures, graphs and abstracting their interrelationships are an essential aspect of developing algebraic reasoning. In this section we shall describe how students used pictorial, tabular, symbolic and graphical representations to explore fractal constructions.

Pictorial representations led to understanding of self-similarity

In the very first session of the module, students were introduced to the Sierpinski triangle construction. The construction process was briefly explained by the teacher. An equilateral triangle (stage 0) was drawn and cut out from a sheet of paper. The mid-points of the sides were joined, to obtain four smaller triangles and the centre triangle was removed. This piece with a triangular 'hole' was referred to as stage 1. Students observed that stage 1 comprised three identical smaller copies of stage 0 (each copy was a smaller

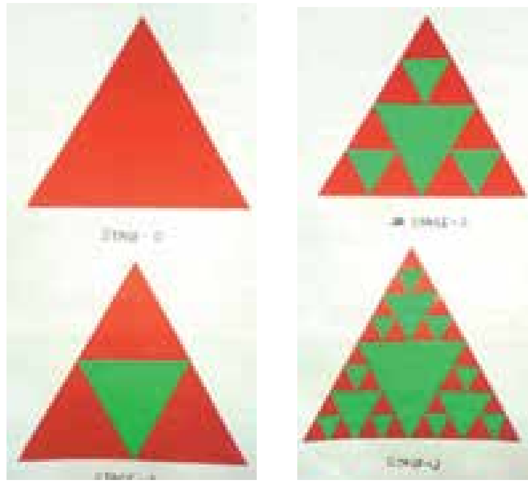


Figure 1. Stages 0 to 3 of the Sierpinski triangle as depicted by a student.

equilateral triangle). The process of creating smaller equilateral triangles and removing the centre triangle was repeated on the three smaller triangles of stage 1 to obtain stage 2. Figure 1 shows stages 0,1,2 and 3 as obtained by a student who preferred to use a combination of red and green triangles. The green triangular portions represent the triangular 'holes'.

After the construction process was over, some time was spent on discussing students' observations. A few students said that the process could 'go on forever' although many could not describe what higher stages would look like. One student commented that 'the number of triangular holes will go on increasing' referring to the parts which are being removed. A majority of students agreed that the number of triangles 'will increase at every stage and each triangle will also get smaller in size'.

To give a direction to their observations, students were assigned two tasks. The first task required them to count the number of shaded triangles in stages 0 to 3 and predict the number for stages 4 and 5. They were required to find a rule for the number of shaded triangles at the nth stage.

In the second task, they had to find a rule for shaded area at the various stages and also at the nth stage (given that the area of the equilateral triangle at stage 0 is 1 square unit). At this point the teacher helped students to make the observation that stage 1 has three smaller copies of stage 0. Similarly stage 2 has three smaller copies of stage 1 and nine still smaller copies of stage 0. This idea of identifying smaller copies of previous stages in subsequent stages was introduced as self-similarity. Figure 2 was used by the teacher to explain this idea.

Numerical representations led to generalization by extension

Task 1 was easily done by all students as they observed that the number of shaded triangles at each stage was 'a power of 3' and using a

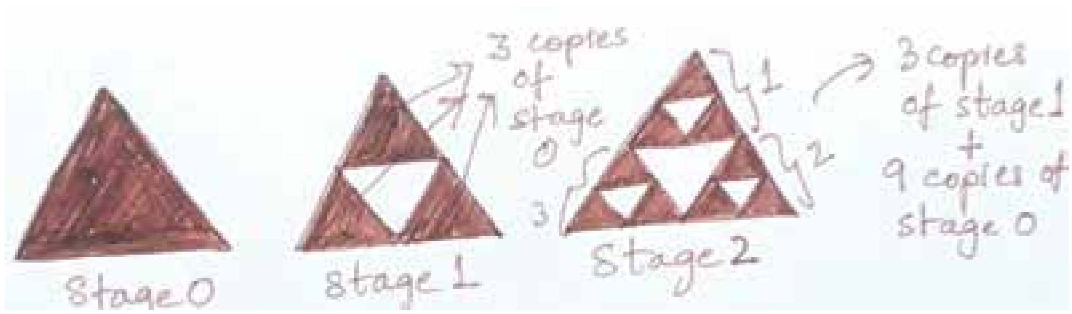


Figure 2. The idea of self-similarity -- finding scaled down copies of previous stages in a given stage.

multiplying factor of 3, they came up with the geometric sequence $1, 3, 3^2, 3^3, \ldots\ldots$. However, the second task posed a challenge for a few students. While they concluded that the shaded area at stage 1 is $3/4$ units (since only three of the 4 smaller equilateral triangles were shaded), they were unable to extend the idea to stage 2. A few students pointed out 'the shaded area at stage 1 is being divided into 4 equal parts in stage 2 and one of these parts is being removed' thus leading them to conclude that the shaded area at stage 2 is $3/4$ of $3/4$, that is, $9/16$ or $(3/4)^2$. This idea was taken up by others and extended to the fact that the multiplying factor in the sequence of shaded areas was 3/4. Finally a majority of the class obtained the geometric sequence $1, \frac{3}{4}, \left(\frac{3}{4}\right)^2, \left(\frac{3}{4}\right)^3, \ldots\ldots\ldots$ to represent the shaded area at various stages. Students worked in pairs, reasoned about the number of shaded triangles and shaded area using their pictorial and tabular representations and arrived at the geometric sequences. These may be considered as examples of generalization by analogy. However, finding the formula for the nth stage entails generalization by extension. This required them to observe that the exponents of 3 and $3/4$ in the two sequences coincide with the stage number. With facilitation, students were able to conclude that the nth terms of the sequences were 3^n and $\left(\frac{3}{4}\right)^n$ respectively. This exercise led to two geometric sequences, one with common ratio 3 (greater than 1) and the other with common ratio $3/4$ (less than 1).

Symbolic representations

At this stage, the teacher tried to help students to make connections between their recursive and explicit reasoning. She introduced the following symbols and asked them to write the nth terms of the two sequences using these

S_n= number of shaded triangles at stage n,
A_n= Shaded area at stage n

Students had to relate the formula of stage n with that of stage n-1 for both sequences. The aim was to help them see the recursive relationships within each attribute (number of shaded triangles and shaded area) and to think of the nth terms of the sequences as independent expressions which they could manipulate.

For the number of shaded triangles at every stage, students obtained the generalized formula $S_n = 3^n$. Writing the recursive relation $S_n = 3 \times S_{n-1}$ however, took some scaffolding. The teacher had to emphasise that $S_1 = 3 \times S_0$ and $S_2 = 3 \times S_1$ to help them see the relation.

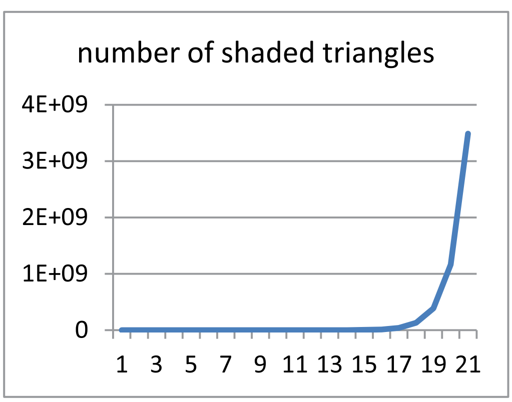
For shaded area, students came up with the recursive and explicit formulae, $A_n = \frac{3}{4} \times A_{n-1}$ and $A_n = \left(\frac{3}{4}\right)^n$ more easily. At this point, the teacher asked them to express the self-similarity of the Sierpinski triangle using the same ideas. After some facilitation, many students could articulate the idea that stage n has three copies of stage n-1, 9 copies of stage n-2, etc. For the teacher, this was a high point of the class, as it convinced her that students had succeeded in generalizing the Sierpinski triangle construction through multiple representations.

A spreadsheet exploration of the fractal constructions

Finally students were assigned the task of describing what would happen as n, the number of stages, approached infinity. They conjectured that the number of shaded triangles would 'become very large' and some used the phrase 'will approach infinity'. For the shaded area, many students said it would get 'smaller and smaller'. To help them visualise this numerically, the students were encouraged to explore these sequences on MS Excel by generating values up to stage 20 (see figure 3). For example, the first column shows 'n' the stage number (up to 20); the second column shows the number of shaded triangles, obtained by entering 1 in the first cell (say C3) and $\boxed{= \text{C3*3}}$ in cell C4. The sequence of shaded areas was similarly obtained in the third column. Graphing the sequences revealed that the number of shaded triangles was growing very rapidly whereas the shaded area was approaching 0. Thus Excel played a pivotal role in helping students visualize the fractal construction process, numerically and graphically. It is impossible to draw the Sierpinski triangle after stage 4 or 5. However, using Excel students could visualize the growth process at higher stages.

	stage	no of shaded triangles	shaded area
	0	1	1
	1	3	0.75
	2	9	0.5625
	3	27	0.421875
	4	81	0.31640625
	5	243	0.237304688
	6	729	0.177978516
	7	2187	0.133483887
	8	6561	0.100112915
	9	19683	0.075084686
	10	59049	0.056313515
	11	177147	0.042235136
	12	531441	0.031676352
	13	1594323	0.023757264
	14	4782969	0.017817948
	15	14348907	0.013363461
	16	43046721	0.010022596
	17	129140163	0.007516947
	18	387420489	0.00563771
	19	1162261467	0.004228283
	20	3486784401	0.003171212

(i)



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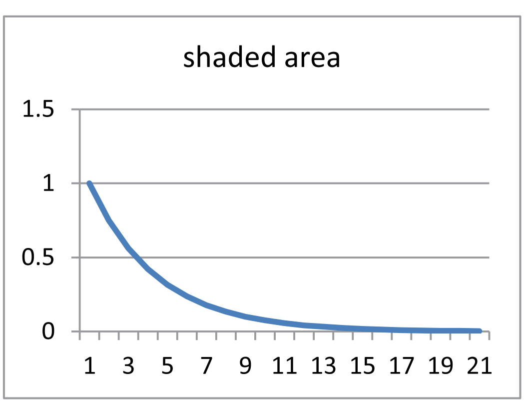


(iii)

Figure 3. Numerical and graphical representations of the geometrical sequences arising from the Sierpinski triangle construction in MS Excel.

By the end of the first two-hour session, students had been introduced to the nature of fractal constructions, meaning of self – similarity and had quantified the patterns emerging from the construction process. They had succeeded in exploring the Sierpinski triangle using multiple representations, made connections between these representations and had engaged in explicit as well as recursive reasoning. The exploration in MS Excel led to a 'big picture' understanding of the Sierpinski triangle.

In the second session, students asked if they could extend the idea of the Sierpinski triangle construction to a square. Their efforts, interestingly, led to the Sierpinski square carpet (see figure 4). Here the construction process requires each side of the square piece (stage 0) to be trisected. When points of trisection of opposite sides are joined, 9 smaller squares are created. To obtain stage 1, the centre square is removed and 8 shaded squares are obtained. The same process is repeated on the 8 smaller shaded squares to obtain stage 2. The construction process was done quite easily by most students. They used a dotted grid paper, so as to make the trisection process easy. Without even being asked, students tried to predict the geometric sequences which would emerge by counting the number of shaded squares and shaded area at each stage. It was not difficult for them to conclude that the number of shaded squares led to the geometric sequence $1, 8, 8^2, 8^3, \ldots\ldots.$ which could be represented explicitly and recursively using the relations $S_n = 8^n$ and $S_n = 8 \times S_{n-1}$. For the shaded area, a discussion among students led to the conclusion that the multiplying factor was 8/9. The geometric sequence $1, \frac{8}{9}, \left(\frac{8}{9}\right)^2, \left(\frac{8}{9}\right)^3, \ldots\ldots\ldots$ with explicit and recursive relations $A_n = \left(\frac{8}{9}\right)^n$ and $A_n = \frac{8}{9} \times A_{n-1}$ was obtained. Students identified self-similarity within the Sierpinski carpet by extending the idea from the Sierpinski triangle.

Students' explorations took an interesting turn at this stage in the module. Those who had studied geometric sequences in grade 11 (*Sequences and Series* is a topic in the grade 11 syllabus as prescribed by the Central Board of Secondary

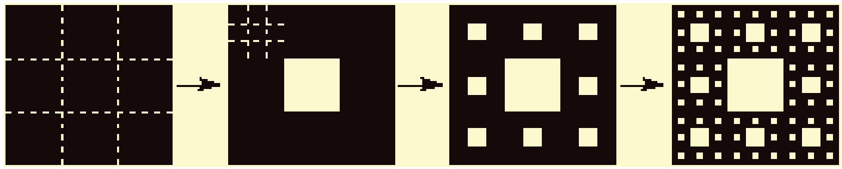
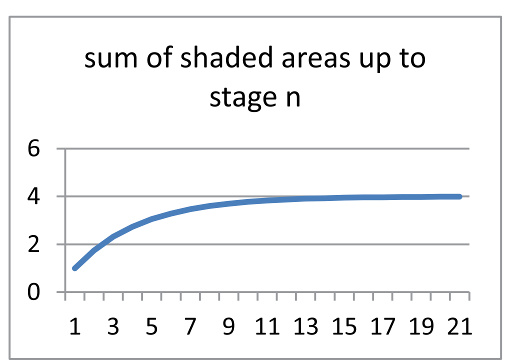


Figure 4. Stages 0 to 3 of the Sierpinski square carpet construction (retrieved from http://www2.edc.org/makingmath/mathprojects/pascal/pascal_warmup.asp)

Education (CBSE)) during their school days, wanted to know if the meaning of the formula for S_∞ could be visualised in the context of the geometric sequences arising out of the Sierpinski constructions. What would happen if, for example, they took the sum (to infinity) of the geometric progression $1 + \frac{3}{4} + \left(\frac{3}{4}\right)^2 + \left(\frac{3}{4}\right)^3 + \ldots \infty$? The teacher found this to be an exciting opportunity. She encouraged students to explore the cumulative sum of shaded areas represented by the above progression in MS Excel. Figure 5 shows the Excel output where the first column represents the stages, the second column, terms of the sequence of shaded areas, and third column, the cumulative sum of areas. Indeed by the 20^{th} term the cumulative sum approaches a fixed value, 4. This was verified by students using the formula $S_\infty = \frac{a}{1-r}$, where a is the initial term and r, the common ratio of the geometric progression. Of course, this is applicable only when $|r| < 1$. Thus $S_\infty = \frac{1}{1-3/4} = \frac{1}{1/4} = 4$. Graphing the second and third columns (see figures 3 (iii) and 5 (ii) respectively) helped them to visualize the process. While the sequence of shaded areas was approaching 0 as n approached infinity, the cumulative sum of areas was approaching 4. The graphical representations led to an interesting discussion in the class. 'But how can we explain the infinite process leading to a fixed value?' some students asked. Another group of students, after some discussion, explained - 'the amount of area getting added at each successive stage is decreasing, so effectively, the total area is approaching a fixed value.' One student commented 'I had used the formula for S_∞ in school, but I never knew what it meant. Today it makes sense!' This was indeed the high point of the class! It was very satisfying to see that students, who had studied geometric sequences in school, now actually understood them and those who had not studied this topic earlier, had also learnt it in a meaningful way. In the beginning of the module students had relied more on pictorial and tabular representations to understand the fractal

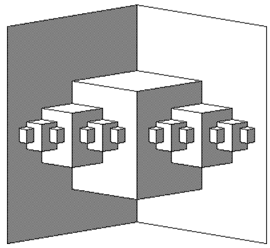
stage	shaded area	sum of area upto stage n
0	1	1
1	0.75	1.75
2	0.5625	2.3125
3	0.421875	2.734375
4	0.31640625	3.05078125
5	0.237304688	3.288085938
6	0.177978516	3.466064453
7	0.133483887	3.59954834
8	0.100112915	3.699661255
9	0.075084686	3.774745941
10	0.056313515	3.831059456
11	0.042235136	3.873294592
12	0.031676352	3.904970944
13	0.023757264	3.928728208
14	0.017817948	3.946546156
15	0.013363461	3.959909617
16	0.010022596	3.969932213
17	0.007516947	3.97744916
18	0.00563771	3.98308687
19	0.004228283	3.987315152
20	0.003171212	3.990486364

(i)



(ii)

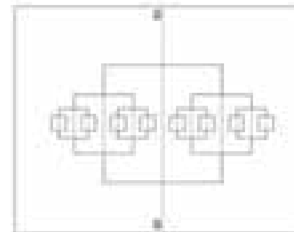
Figure 5. Numerical and graphical representations of the cumulative sum of shaded areas of the various stages of the Sierpinski triangle in MS Excel.



(i)

(ii)

Figure 6. A fractal card obtained by repeated cutting and folding.



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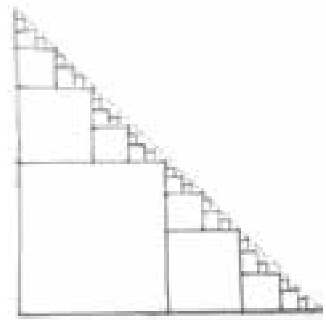

(ii)

Figure 7. The fractal card when flattened (i) and its cross-section (ii).

constructions, but later they made the transition to representing the same ideas using symbols, thus obtaining recursive and explicit formulae.

In the remaining part of the module, students learnt to make fractal cards and explored number patterns within them. For example, Figure 6 (i) shows a drawing of a fractal card obtained by repeated cutting and folding a rectangular sheet of paper and pushing out the 'cells'. Figure 6 (ii) shows a photograph of an actual card made by a student.

While exploring the card they identified several geometric sequences within the card. The number of 'pop up' cells at the various stages led to the sequence, $1,2,2^2,2^3,\ldots\ldots.$ When the card is flattened (figure 7 (i)), the lengths of the cuts (horizontal lines) leads to the sequence $\frac{l}{2}, \frac{l}{4}, \frac{l}{8}, \frac{l}{16}, \ldots.$ with the nth term $\frac{l}{2^n}$, where 'l' denotes the length of the rectangular sheet. The cross section of the card (figure 7 (ii)) reveals squares of reducing size. If the process is continued to infinity and the areas of the squares are added, we get the geometric progression $\frac{l^2}{16} + \frac{l^2}{32} + \frac{l^2}{64} + \cdots.$ whose sum to infinity is $\frac{l^2}{8}$. In fact, as the process of cutting and folding is continued, smaller squares appear with each successive stage and these approach the hypotenuse of the right angled triangle whose area is $\frac{l^2}{8}$. Once again students encountered a process, where the sum of infinitely many terms of a geometric sequence actually approaches a fixed value. The output of the fractal card activity led to much excitement as the cards were very attractive and students tried to think of other kinds of cuts and folds which could lead to cards with fractal structure.

Conclusion

The module described in this article provided the pre-service teachers with the opportunity to visualize and explore geometric sequences through fractal constructions. Using multiple representations – pictures, tables, symbols and graphs, they generalised various attributes of fractals such as the Sierpinski triangle, Sierpinski carpet and fractal cards. The fractal constructions provided an authentic context to engage in recursive as well as explicit reasoning thus leading to meaningful generalisation of the fractals at higher stages. Further, Excel helped them to visualize the generalization process numerically, by generating values of the sequences at higher stages which could not be calculated manually and also by producing graphs which illustrated the behaviour of the attributes in the long run. It helped them to see that as 'n' increased, the sum of terms of particular geometric sequences approach fixed values, a concept which they were unable to visualize earlier. To summarise, the fractal explorations in the module helped the pre-service teachers to gain an insight into the nature of fractal geometry and familiarised them with a range of activities which can be easily integrated into classroom teaching at the secondary level. The module also highlighted the power of generalization in leading to algebraic reasoning.

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Octagon in a Square: Another Solution

C⊗MαC

Problem Corner

In the November 2015 issue of AtRiA, the following geometrical puzzle had been posed. An octagon is constructed within a square by joining each vertex of the square to the midpoints of the two sides remote from that vertex. Eight line segments are thus drawn within the square, creating an octagon (shown shaded). The following two questions had been posed: (i) Is the octagon regular? (ii) What is the ratio of the area

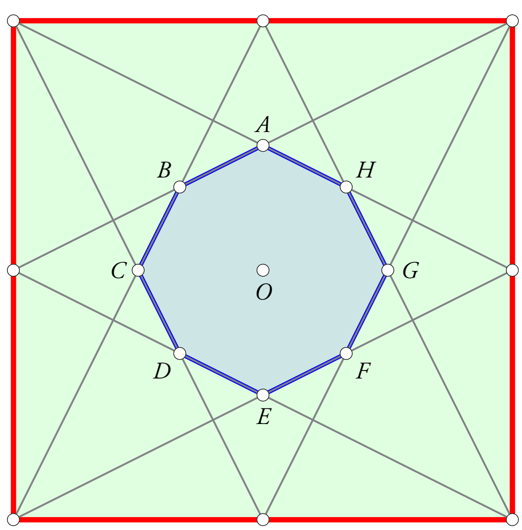

Figure 1

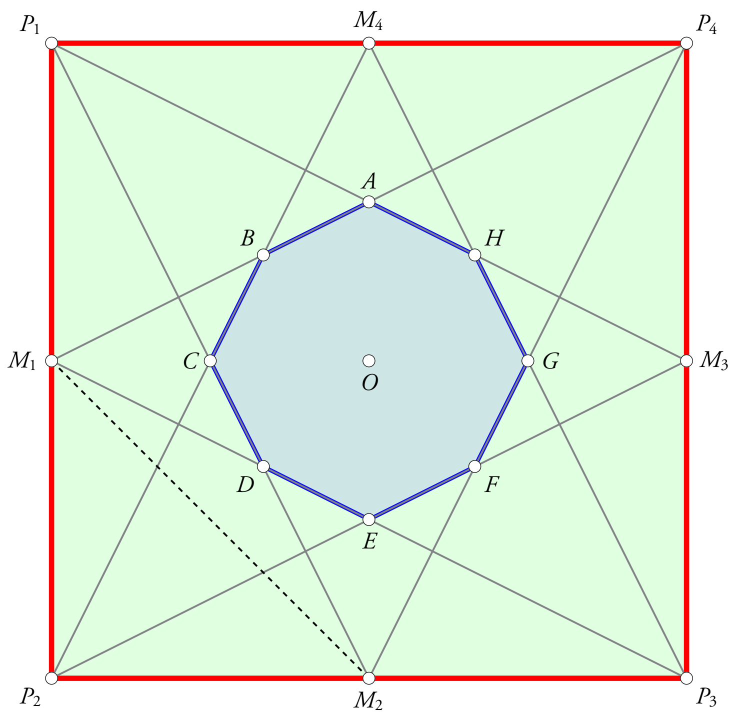

Figure 2

of the octagon to that of the square? We had given a solution in the March 2016 issue. Now we feature another solution to this problem, sent in by a reader: **R Desai**, of Gujarat.

Label the vertices of the octagon A, B, C, D, E, F, G, H as shown. Let O be the centre of the square. We shall show that the octagon is not regular by showing that $\measuredangle BAH < \measuredangle AHG$. (If the octagon were regular, these two angles would have had the same measure.)

Label the vertices of the square and the midpoints of the sides as shown in Figure 2. Join M_1M_2.

Observe that triangles $M_4P_2P_3$ and $P_4M_1M_2$ are both isosceles, and further have equal length for the congruent pairs of sides ($M_4P_2 = M_4P_3 = P_4M_1 = P_4M_2$). However, their bases are unequal; indeed, $P_2P_3 > M_1M_2$. It follows from this that their apex angles are unequal; indeed that $\measuredangle P_2M_4P_3 > \measuredangle M_1P_4M_2$.

Since $M_4P_2 \perp P_1M_3$ and $P_4M_2 \perp P_1M_3$, it follows that $\measuredangle AHG = 90^\circ + \measuredangle P_2M_4P_3$ and $\measuredangle BAH = 90^\circ + \measuredangle M_1P_4M_2$. Hence $\measuredangle AHG > \measuredangle BAH$. Thus the octagon is not regular, as claimed. $\square$

Computation of area. It remains to compute the area of the octagon relative to the area of the square. There are many ways of obtaining the result. Desai's solution avoids the use of trigonometry; it uses only pure geometry.

If all the vertices of the octagon are joined to the centre O, the octagon is divided into 8 congruent triangles. One of these triangles is $\triangle ODE$. Extend OD to P_2 and OE to M_2 as shown. Observe that $OE = OM_2/2$. Also, $\triangle ODE \sim \triangle P_2DM_1$, the ratio of similarity being $1/2$ since $OE = OM_2/2$. Hence $OD = OP_2/3$. It follows that

$$\frac{\text{Area of } \triangle ODE}{\text{Area of } \triangle OP_2M_2} = \frac{OD}{OP_2} \cdot \frac{OE}{OM_2} = \frac{1}{3} \cdot \frac{1}{2} = \frac{1}{6}.$$

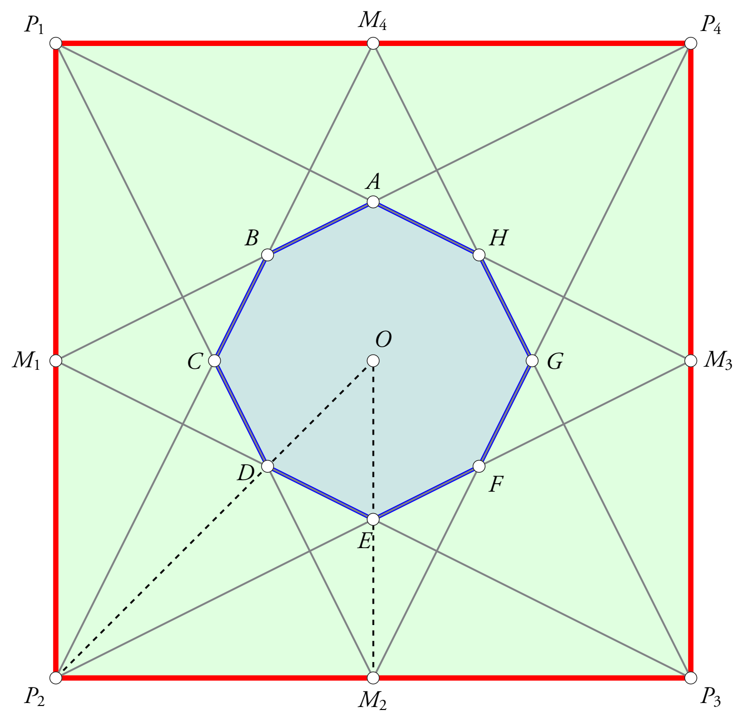

Figure 3

Since the area of $\triangle ODE$ is $1/8$ of the area of the octagon, and the area of $\triangle OP_2M_2$ is $1/8$ of the area of the square, it follows that the area of the octagon is $1/6$ of the area of the square. $\square$

The COMMUNITY MATHEMATICS CENTRE (CoMaC) is an outreach arm of Rishi Valley Education Centre (AP) and Sahyadri School (KFI). It holds workshops in the teaching of mathematics and undertakes preparation of teaching materials for State Governments and NGOs. CoMaC may be contacted at shailesh.shirali@gmail.com.

Solution to 'A Circular Challenge'

C⊗MαC

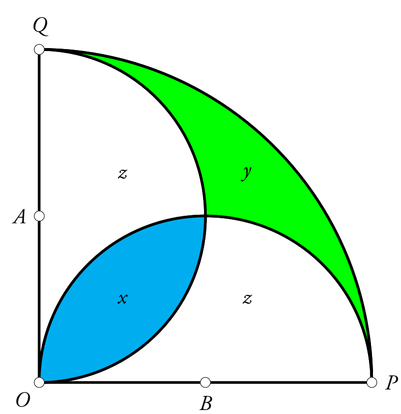

Figure 1

Shown in Figure 1 is a portion of Figure 1 in the original article (July 2016 issue): two semi-circles centred at A and B, with radii 1 unit each, and a quarter-circle OPQ, with radius 2 units. The portions of the semi-circles minus the blue region (with area x) have area z each. (They obviously have equal area.) Clearly:

$$x + z = \frac{1}{2}\pi \times 1^2 = \frac{\pi}{2}, \qquad x + 2z + y = \frac{1}{4}\pi \times 2^2 = \pi,$$

hence $x + 2z + y = 2(x + z)$, therefore $x + y = 2x$, therefore $x = y$. Hence $x : y = 1 : 1$. □

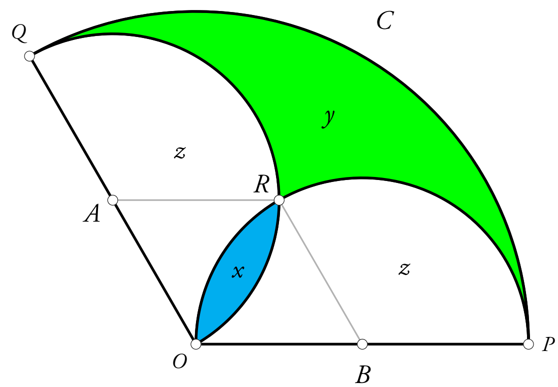

Figure 2

Remark. We did not bother to find x and y individually, as we had only been asked to find $x : y$. But if we need these as well, then:

$$x = 2\left(\frac{1}{2}\,1^2 \times \frac{\pi}{2} - \frac{1}{2}\,1^2 \times \sin\frac{\pi}{2}\right) = \frac{\pi}{2} - 1,$$

and of course, y has the same value. □

Generalization

Now suppose that $\measuredangle POQ = t$, where $0 \le t \le \pi$; see Figure 2. As earlier, let z denote the areas of the semi-circles minus the blue region. Let A and B denote the centres of the two circles, and R the point of intersection of the two small circles other than O (the center of the large circle). Since $OARB$ is a rhombus, $BR \parallel OA$, hence $\measuredangle PBR = t$, hence $\measuredangle OBR = \pi - t$. Hence:

$$\begin{aligned} x &= 2\,(\text{Area of sector } OBR - \text{Area of } \triangle OBR) \\ &= 2\left(\frac{1^2\cdot(\pi - t)}{2} - \frac{1^2\cdot\sin(\pi - t)}{2}\right) = \pi - t - \sin t. \end{aligned}$$

Again, we have:

$$x + z = \frac{1}{2}\,1^2 \times \pi = \frac{\pi}{2}, \qquad x + 2z + y = \frac{1}{2}\,2^2 \times t = 2t.$$

Hence by subtraction $y - x = 2t - \pi$, and therefore $y = x + 2t - \pi$. Hence the desired ratio $x : y$ is:

$$\frac{x}{y} = \frac{\pi - t - \sin t}{t - \sin t}.$$

For the particular value $t = \pi/2$ we have $x : y = (\pi/2 - 1) : (\pi/2 - 1) = 1 : 1$, as earlier. □

Note. Reader **Tejash Patel** of Gujarat sent in correct solutions to both parts of the problem.

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Problem Corner

Problems for the Senior School

Problem Editors: Prithwijit De & Shailesh Shirali

PROBLEMS FOR SOLUTION

Problem V-3-S.1
Let $ABCD$ be a convex quadrilateral. Let P, Q, R and S be the midpoints of AB, BC, CD and DA respectively. What kind of a quadrilateral is $PQRS$? If $PQRS$ is a square, prove that the diagonals of $ABCD$ are perpendicular to each other.

Problem V-3-S.2
Let $ABCD$ be a convex quadrilateral. Let P, Q, R and S be the midpoints of AB, BC, CD and DA respectively. Let U and V be the midpoints of AC and BD, respectively. Prove that the lines PR, QS and UV are concurrent.

Problem V-3-S.3
There are 12 lamps, initially all OFF, each of which comes with a switch. When a lamp's switch is pressed, it's state is reversed, i.e., if it is ON, it will go OFF, and vice-versa. One is allowed to press exactly 5 different switches in each round. What is the minimum number of rounds needed so that all the lamps will be turned ON? (Hong Kong Preliminary Selection Contest 2015)

Problem V-3-S.4
The greatest altitude in a scalene triangle has length 5 units, and the length of another altitude is 2 units. Determine the length of the third altitude, given that it is integer valued.

Problem V-3-S.5
If the three-digit number $\overline{ABC}$ is divisible by 27, prove that the three-digit numbers $\overline{BCA}$ and $\overline{CAB}$ are also divisible by 27.

SOLUTIONS OF PROBLEMS IN ISSUE-V-2 (JULY 2016)

Call a convex quadrilateral *tangential* if a circle can be drawn tangent to all four sides. All the problems studied below have to do with this notion.

Solution to problem V-2-S.1
Let ABCD be a tangential quadrilateral. Prove: $AD + BC = AB + CD$.

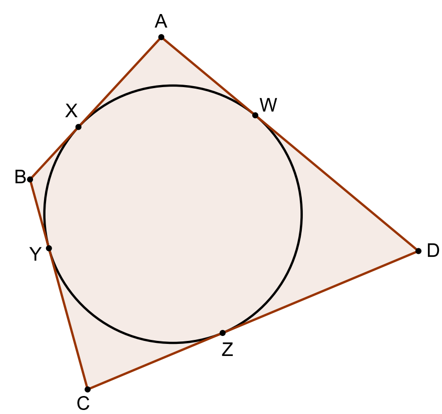

Figure 1

Let the circle touch the sides AB, BC, CD, DA at X, Y, Z, W respectively (Figure 1). Then $AX = AW$, $BX = BY$, $CY = CZ$ and $DZ = DW$. Therefore

$$\begin{aligned} AD + BC &= (AW + WD) + (BY + YC) = (AX + DZ) + (XB + ZC) \\ &= (AX + XB) + (DZ + ZC) = AB + CD. \end{aligned}$$

Solution to problem V-2-S.2
Let ABCD be a convex quadrilateral with $AD + BC = AB + CD$. Prove that ABCD is tangential.

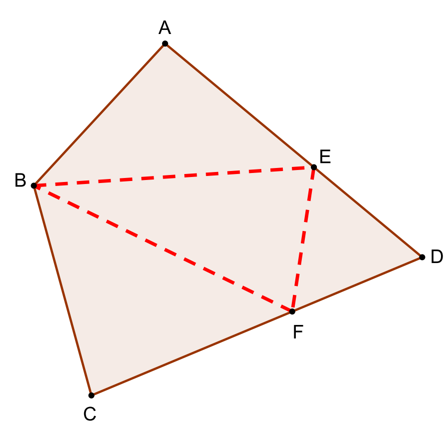

Figure 2

Suppose that $AD > AB$. Then $CD > BC$. Let E be a point on the segment AD such that $AE = AB$ and let F be a point on the segment CD such that $BC = CF$. Then it follows that $DE = DF$ and the triangles ABE, BCF and DEF are isosceles. The perpendicular bisectors of the sides BE, BF and EF of triangle BEF are concurrent at the circumcentre of BEF. But the perpendicular bisectors of BE, BF and EF are also the angle bisectors of $\measuredangle BAE$, $\measuredangle BCF$ and $\measuredangle EDF$. Thus three angle bisectors of the quadrilateral $ABCD$ are concurrent. This point of concurrency is equidistant from all four sides of the quadrilateral and hence is the centre of its inscribed circle. Therefore $ABCD$ is tangential. The case when $AD = AB$ is left as an exercise to the reader.

Solution to problem V-2-S.3

Place four coins of different sizes on a flat table so that each coin is tangent to two other coins. Prove that the quadrilateral formed by joining the centres of the coins is tangential. Prove also that the convex quadrilateral whose vertices are the four points of contact is cyclic. Is the circle passing through the four points of contact tangent to the sides of the tangential quadrilateral formed by joining the four centres of the coins?

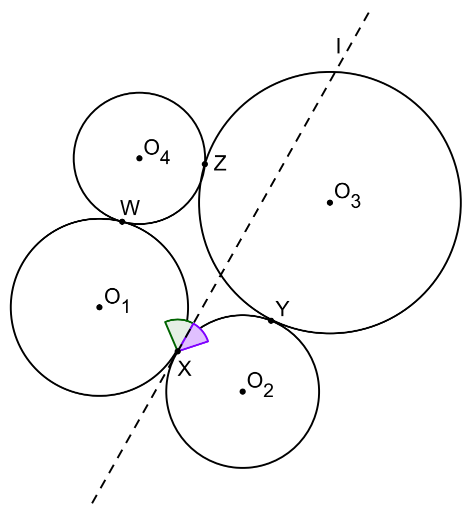

Figure 3

Let the radii of the coins be r_1, r_2, r_3, r_4 respectively. One pair of opposite sides of the quadrilateral formed by joining the centres of the coins are $r_1 + r_2$ and $r_3 + r_4$ whereas the other pair of opposite sides are $r_1 + r_4$ and $r_2 + r_3$. Evidently the sums of the lengths of the pairs of opposite sides are same and equal to $r_1 + r_2 + r_3 + r_4$. Thus the quadrilateral is tangential.

Name the coins C_1, C_2, C_3, C_4. Let the centres of the coins be O_1, O_2, O_3 and O_4 and let the point of contact of C_1 and C_2 be X, that of C_2 and C_3 be Y. Let Z and W be the points of contact of C_3 and C_4, and C_4 and C_1 respectively. Let l be the common tangent of C_1 and C_2 passing through X. Then the angle between WX and l is $\frac{1}{2}\measuredangle WO_1X$ and the angle between YX and l is $\frac{1}{2}\measuredangle XO_2Y$. Therefore

$$\measuredangle WXY = \frac{\measuredangle WO_1X + \measuredangle XO_2Y}{2}.$$

Similarly

$$\measuredangle WZY = \frac{\measuredangle WO_4Z + \measuredangle YO_3Z}{2}.$$

Therefore

$$\measuredangle WXY + \measuredangle WZY = \frac{\measuredangle WO_1X + \measuredangle XO_2Y + \measuredangle WO_4Z + \measuredangle YO_3Z}{2} = 180^\circ.$$

Hence the quadrilateral $XYZW$ is cyclic.

As regards the question of whether the circle through X, Y, Z, W is the incircle of quadrilateral $O_1O_2O_3O_4$, the answer is: *it need not be*. For "proof" please see Figure 4. (Actually, a diagram can never serve as a proof! So there is rather more to this question than meets the eye. We will offer a fuller analysis of the matter in the next issue, March 2017.)

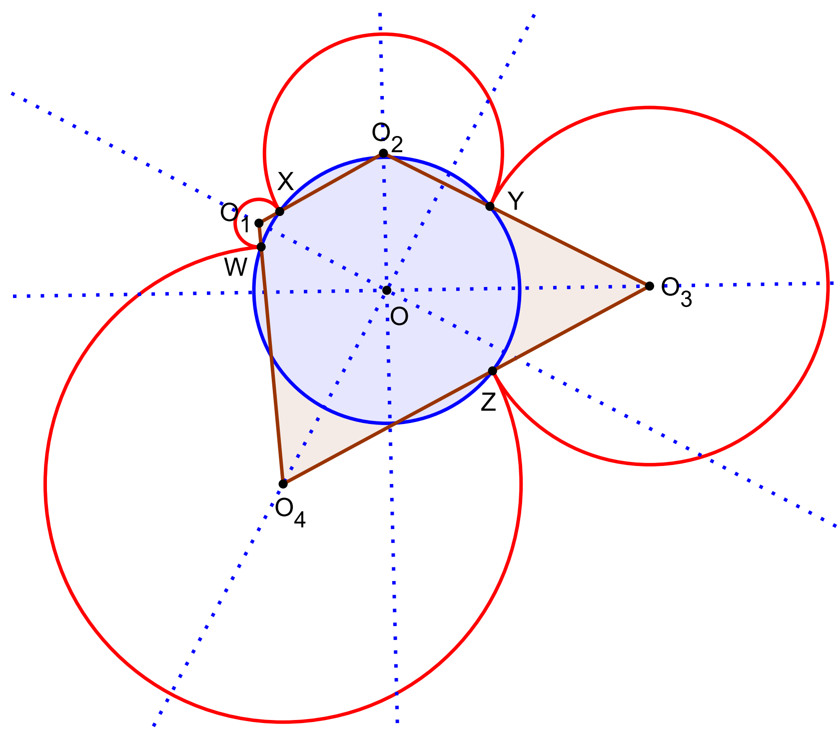

Figure 4

Solution to problem V-2-S.4

Let ABCD be a cyclic quadrilateral and let X be the intersection of diagonals AC and BD. Let P_1, P_2, P_3 and P_4 be the feet of the perpendiculars from X to BC, CD, DA and AB respectively. Prove that quadrilateral $P_1P_2P_3P_4$ is tangential.

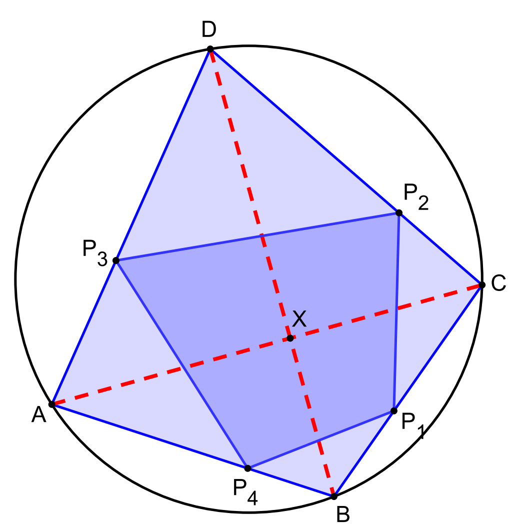

Figure 5

See Figure 5. Observe that the quadrilaterals P_4BP_1X and P_1CP_2X are cyclic. In quadrilateral P_4BP_1X, we have $\measuredangle P_4BX = \measuredangle P_4P_1X$. In quadrilateral P_1CP_2X, we have $\measuredangle P_2CX = \measuredangle P_2P_1X$. As $ABCD$ is cyclic,

$$\measuredangle P_4BX = \measuredangle ABD = \measuredangle ACD = \measuredangle P_2CX.$$

Therefore $\measuredangle P_4P_1X = \measuredangle P_2P_1X$. Thus XP_1 bisects $\measuredangle P_4P_1P_2$. Similarly it can be shown that XP_2, XP_3 and XP_4 bisect $\measuredangle P_1P_2P_3$, $\measuredangle P_2P_3P_4$ and $\measuredangle P_3P_4P_1$ respectively. Thus X is the point of concurrency of the angle bisectors $P_1P_2P_3P_4$ and is therefore equidistant from all four sides of the quadrilateral. Therefore X is the centre of the inscribed circle and $P_1P_2P_3P_4$ is tangential.

SOLUTIONS

NUMBER CROSSWORD

Page 23

(1) 2	(2) 3	4	0	■	(3) 1	0	(4) 8	9
■	1	■	■	(5) 2	■	■	4	■
■	(6) 2	9	■	9	■	(7) 2	5	■
(8) 6	■	■	(9) 4	4	4	■	■	(10) 1
(11) 5	2	8	■	■	■	(12) 5	9	4
0	■	■	(13) 5	(14) 1	2	■	■	4
■	(15) 6	5	■	0	■	(16) 1	(17) 2	■
■	2	■	■	1	■	■	1	■
(18) 3	1	7	7	■	(19) 9	8	0	1

Problems for the Middle School

Problem Editors: SNEHA TITUS AND ATHMARAMAN R

Problem Corner

Since our first issue in July 2012, we have been carrying problems for the Middle School Section in Problem Corner. We hope that you have enjoyed attempting and solving them. We are taking a brief hiatus with this issue and departing from our usual format to provide all those who would like to improve their problem solving skills at this level with a Handy Reference Sheet for Middle Problems.

As we said in our very first issue, *problems are the life blood of mathematics*. Here is a transfusion: a quick boost for your problem solving cells. We have grouped them according to topic and provided a few problems from past issues which were solved with the help of these facts. In the reference sheet for this issue we have focused on **parity** and **divisibility rules**.

Parity

P1. Every even integer can be written as $2k$, where k is an integer.

P2. Every odd integer can be written as $2k+1$, where k is an integer.

P3. The square of every even integer is a multiple of 4 and can be written as $0 \pmod 4$.

P4. The square of every odd integer is 1 more than a multiple of 4 and can be written as $1 \pmod 4$.

P5. The sum of the squares of two odd numbers cannot be a perfect square.

Keywords: *divisibilty, parity, squares, multiples, factors, Pythagorean triplets*

P6. When a and b are both odd or both even, the difference of their squares is always a multiple of 4.

P7. For any two integers a and b, the integers $a+b$ and $a-b$ have the same parity, i.e., are either both odd or both even.

Divisibility
These are in addition to the commonly used divisibility rules (by 4, 9 and 11).

D1. The remainder that a number leaves on division by 3 is equal to the remainder that its sum of digits leaves on division by 3.

D2. If k is an odd positive integer, then $a^k + b^k$ is divisible by $a+b$.

D3. Every perfect square is either a multiple of 3 or 1 more than a multiple of 3 and can be written as $0 \pmod 3$ or $1 \pmod 3$.

D4. Every perfect square is either a multiple of 5 or 1 more or 4 more than a multiple of 5 and can be written as $0 \pmod 5$, $1 \pmod 5$ or $4 \pmod 5$.

As a first exercise, you could try to prove P3 to P7 and D3 and D4 using P1 and P2.

Pythagorean triplets
A *Pythagorean triplet* is a triple (a, b, c) of positive integers such that $a^2 + b^2 = c^2$. Examples: $(3, 4, 5)$, $(6, 8, 10)$, $(5, 12, 13)$. The commonly used short form for a Pythagorean triplet is **PT**.

A PT (a, b, c) is called *primitive* if $\text{GCD}(a, b, c) = 1$. The commonly used short form for a primitive Pythagorean triplet is **PPT**.

Now try to prove these facts about PTs using the given rules.

PT1. It is not possible to have a PT in which exactly two of the numbers are even.

PT2. It is not possible to have a PT in which just two of the numbers are multiples of 3.

PT3. It is not possible to have a PT in which none of the numbers are even.

PT4. It is not possible to have a PT in which none of the numbers are multiples of 3.

Some problems from past issues
Here are some problems from past issues which you can solve using the given facts.

II-3 M.2. It is easy to find a pair of perfect squares that differ by 2013; for example, $47^2 - 14^2 = 2013$.

Now that the new year (2014) is close upon us, we ask: Can you find a pair of perfect squares that differ by exactly 2014?

III-2 M.1. What is the least multiple of 9 which has no odd digits?

III-3-M. 4. Find the digits A and B if the product $2AA \times 3B5$ is a multiple of 12. (Find all the possibilities.)

IV-2 M.2. The sum of the digits of a natural number n is 2015. Can n be a perfect square?

SOLUTIONS OF PROBLEMS IN ISSUE-V-2 (JULY 2016)

The problems studied below are all based on the fundamental concept of divisibility. They require for their solution only a basic understanding of the rules of divisibility.

Solution to problem V-2-M.1

What is the largest prime divisor of every three-digit number with three identical non-zero digits?

Any three-digit number with three identical digits is a multiple of 111. Hence it suffices to focus attention on the number 111. The prime factorisation of this number is $111 = 3 \times 37$. Hence 37 is a prime factor of every number with three identical digits. Therefore the required answer is 37.

Solution to problem V-2-M.2

Given any four distinct integers a, b, c, d, show that the product $k = (a-b)(a-c)(a-d)(b-c)(b-d)(c-d)$ is divisible by 12.

An integer is divisible by 12 if and only if it is divisible by both 3 and 4. So let us prove that the product k is divisible by both 3 and 4. We now make use of the well-known pigeonhole principle.

Divide the numbers a, b, c, d by 3; each division must leave remainder 0, 1 or 2. There being just three possible remainders, some two of the numbers a, b, c, d must leave the same remainder under division by 3. For these two numbers, their difference is a multiple of 3. Hence k is a multiple of 3.

Next, suppose that two of the four integers a, b, c, d are even and two are odd. Then the differences for both the pairs are even, so k is divisible by 4. If three of the four integers a, b, c, d are even and just one is odd, then three of the differences between the numbers are even, hence k is divisible by 8. The argument is similar when three of the integers a, b, c, d are odd and just one is even. The case when all of the integers a, b, c, d are even (or all four are odd) is trivial.

Editor's remark. The above proposition cannot be strengthened. That is, the 12 in the proposition cannot be replaced by any larger number. To see this, it suffices to consider the four numbers $1, 2, 3, 4$. For this selection, we get $k = 12$. This immediately shows why 12 cannot be replaced by any higher number.

Solution to problem V-2-M.3

Let n be a natural number, and let $d(n)$ denote the sum of the digits of n. Show that if $d(n) = d(3n)$, then 9 divides n. Show that the converse statement is false.

We make repeated use of the tests for divisibility by 3 and 9, namely: a positive integer is divisible by 3 if and only if the sum of its digits is divisible by 3; and likewise for divisibility by 9. We also make repeated use of the notation $a \mid b$ which means that a is a divisor of b.

Assume that $d(n) = d(3n)$. Since $3 \mid 3n$, it follows that $3 \mid d(3n)$. Since $d(n) = d(3n)$, it follows that $3 \mid d(n)$, and therefore that $3 \mid n$. This implies that $9 \mid 3n$, which implies that $9 \mid d(3n)$, and this in turn implies that $9 \mid d(n)$. Now recalling the test for divisibility by 9, we see that $9 \mid n$.

The converse statement is false; e.g., take $n = 126$. Then $9 \mid n$, $3n = 378$, $d(n) = 9$, $d(3n) = 18$, so $d(n) \neq d(3n)$.

Solution to problem V-2-M.4

Let n be an arbitrary positive integer. Show that: (a) $n^5 - n$ is divisible by 5; (b) $n^7 - n$ is divisible by 7; (c) $n^9 - n$ is not necessarily divisible by 9.

(a) We shall use the method of mathematical induction. Let f be the function defined as follows: $f(n) = n^5 - n$. We must prove that $f(n)$ is divisible by 5 for all positive integers n. We observe the following: $f(1) = 0; f(2) = 30; f(3) = 240; f(4) = 1020; f(5) = 3120$. All these numbers are divisible by 5. Next we show the inductive step: namely, if $f(n)$ is divisible by 5, then so is $f(n+5)$. For this, we only need to simplify the expression for $f(n+5) - f(n)$. We find that:

$$f(n+5) - f(n) = 25n^4 + 250n^3 + 1250n^2 + 3125n + 3120,$$

and this is clearly a multiple of 5. By the principle of induction, it follows that $f(n)$ is a multiple of 5 for any positive integer n.

(b) The same approach works for the function $g(n) = n^7 - n$. We first verify that $g(n)$ is a multiple of 7 for $n = 1, 2, 3, \ldots, 7$. Next we simplify the expression $g(n+7) - g(n)$ and verify that each of its coefficients is a multiple of 7; this implies that $g(n+7) - g(n)$ is always a multiple of 7. These two statements in combination show that $g(n)$ is a multiple of 7 for every positive integer n.

(c) It suffices to observe that $2^9 - 2 = 510$ is not a multiple of 9.

Solution to problem V-2-M.5

Find all positive integers $n > 3$ such that $n^3 - 3$ is divisible by $n - 3$.

We first divide $n^3 - 3$ by $n - 3$; we get:

$$n^3 - 3 = (n-3) \cdot \left(n^2 + 3n + 9\right) + 24.$$

It follows that if $n - 3$ divides $n^3 - 3$, then $n - 3$ divides 24 as well. Hence $n - 3$ is a divisor of 24, i.e., $n - 3$ can be any of the numbers $-2, -1, 1, 2, 3, 4, 6, 8, 12, 24$. Hence the possible values of n are: $1, 2, 4, 5, 6, 7, 9, 15, 27$.

Solution to problem V-2-M.6

Show that there cannot exist three positive integers $a, b, c > 1$ such that the following three conditions are simultaneously satisfied: $a^2 - 1$ is divisible by b and c; $b^2 - 1$ is divisible by c and a; $c^2 - 1$ is divisible by a and b.

The conditions imply that a, b, c are mutually coprime. To see why, assume that this is not the case. Suppose that a, b have a common factor exceeding 1; then it would not be possible for $a^2 - 1$ to be divisible by b. The same reasoning works for each pair from a, b, c. It follows that a, b, c are mutually coprime. Hence the given conditions may be replaced by the following: $a^2 - 1$ is divisible by bc; $b^2 - 1$ is divisible by ca; $c^2 - 1$ is divisible by ab.

Since a, b, c are mutually coprime and $a, b, c > 1$, it follows that a, b, c are unequal. Without any loss of generality, we may suppose that $a < b < c$. But in this case we would have $a^2 < bc$, and the requirement that $a^2 - 1$ is divisible by bc is not possible. Hence there cannot exist three positive integers all exceeding 1 which satisfy the stated conditions.

Solution to problem V-2-M.7

Using the nine nonzero digits $1, 2, 3, 4, 5, 6, 7, 8, 9$, *form a nine-digit number in which each digit occurs exactly once, such that when the digits are removed one at a time starting from the units end (i.e., from the "right side"), the resulting numbers are divisible respectively by* $8, 7, 6, 5, 4, 3, 2, 1$. (So if the nine-digit number is $\overline{ABCDEFGHI}$, then we must have:

$$8 \mid \overline{ABCDEFGH}; \qquad 7 \mid \overline{ABCDEFG}; \qquad 6 \mid \overline{ABCDEF}; \qquad 5 \mid \overline{ABCDE};$$

and so on. Here the notation $a \mid b$ means: "a divides b".)

We argue as follows:

- $5 \mid \overline{ABCDE}$, so $E = 5$.
- Divisibility by 4, 6, 8 implies that B, D, F, H are $2, 4, 6, 8$ in some order.
- Similarly, A, C, G, I are $1, 3, 7, 9$ in some order.
- $3 \mid \overline{ABC}$, hence $3 \mid A + B + C$.
- $6 \mid \overline{ABCD5F}$, hence $3 \mid A + B + C + D + 5 + F$; hence $3 \mid D + 5 + F$; hence $D + F$ leaves remainder 1 on division by 3; hence $D + F = 7$, 10 or 13 (since $D + F$ lies between $2 + 4 = 6$ and $6 + 8 = 14$). Since D, F are even, one of them must be 6 and the other one 4.
- $9 \mid \overline{ABCD5FGHI}$, hence $3 \mid A + B + C + D + 5 + F + G + H + I$, hence $3 \mid G + H + I$.
- $4 \mid \overline{ABCD}$, hence $4 \mid \overline{CD}$. As C is odd, and in any multiple of 4 where the tens digit is odd, the units digit is either 2 or 6, it follows that $D = 2$ or 6. Hence $D = 6$ and $F = 4$.
- Hence among B, H, one must be 2 and the other must be 8.
- The number is now $\overline{ABC654GHI}$. Now $8 \mid \overline{ABCDEFGH}$, therefore $8 \mid \overline{4GH}$, therefore $8 \mid \overline{GH}$ (since $8 \mid 400$). Since G is odd, $H = 2$ or 6. But 6 has already been used, so $H = 2$, and the number is $\overline{ABC654G2I}$.
- Since B is even and $B \neq 2, 4, 6$, it follows that $B = 8$. So the number is $\overline{A8C654G2I}$.
- As stated earlier, A, C, G, I are $1, 3, 7, 9$ in some order.
- Since $3 \mid \overline{A8C}$, it follows that $A + C \equiv 1 \pmod 3$, so $\{A, C\} = \{1, 3\}$ or $\{3, 7\}$ or $\{1, 9\}$.
- The requirement of divisibility by 7 means that $7 \mid \overline{A8C654G}$. We now consider the different possibilities for A, C.
- Suppose that $(A, C) = (1, 3)$; then since $1836540 \equiv 6 \pmod 7$, we must have $G \equiv 1 \pmod 7$, i.e., $G = 1$ or 8. However, 1 has been used up, and G is odd. Hence this possibility cannot happen.
- Suppose next that $(A, C) = (3, 1)$; then since $3816540 \equiv 0 \pmod 7$, we must have $G \equiv 0 \pmod 7$, i.e., $G = 7$ (as 0 is not available). Hence the number is 381654729.
- Suppose that $(A, C) = (3, 7)$; then since $3876540 \equiv 3 \pmod 7$, we must have $G \equiv 4 \pmod 7$, i.e., $G = 4$. However, G is odd. Hence this possibility cannot happen.
- Suppose that $(A, C) = (7, 3)$; then since $7836540 \equiv 5 \pmod 7$, we must have $G \equiv 2 \pmod 7$, i.e., $G = 2$ or 9. Since G is odd, we get $G = 9$. This yields the number 783654921. But this fails the test for divisibility by 8; indeed, $78365492 \equiv 4 \pmod 8$.
- Suppose that $(A, C) = (1, 9)$; then since $1896540 \equiv 2 \pmod 7$, we must have $G \equiv 5 \pmod 7$, i.e., $G = 5$. However, 5 has been used up. Hence this possibility cannot happen.
- Suppose that $(A, C) = (9, 1)$; then since $9816540 \equiv 6 \pmod 7$, we must have $G \equiv 1 \pmod 7$, i.e., $G = 1$ or 8. However, 1 has been used up, and 8 is even. Hence this possibility cannot happen.
- *Hence* 381654729 *is the only number that satisfies all the requirements.*

Problem Corner

Two Problem Studies

C⊗MαC

In this edition of *Adventures in Problem Solving*, we continue the theme of studying two problems in some detail. Both problems studied here are from training sessions of the famous Tournament of the Towns (see https://en.wikipedia.org/wiki/Tournament_of_the_Towns); both are accessible to students of classes 9 and 10. We state the problems first, so that you have an opportunity to tackle them before seeing the solutions.

Problem 1: Determine all positive integers n for which there exist n consecutive positive integers whose sum is a prime number.

Problem 2: The sum of all terms of a finite arithmetic progression of integers is a power of 2. Prove that the number of terms is also a power of 2.

Solution to Problem 1: *Determine all positive integers n for which there exist n consecutive positive integers whose sum is a prime number.*

Suppose that the sum of the n consecutive positive integers

$$a,\quad a+1,\quad a+2,\quad \ldots,\quad a+n-1$$

is a prime number; here $a \geq 1$ and $n \geq 2$. (The possibility $n = 1$ clearly need

Keywords: *series, arithmetic progression, parity, sum to n terms, consecutive, prime, powers of 2*

not be considered, as it leads to a triviality.) The sum s of these n integers, using the standard formula for the sum of an arithmetic progression, is:

$$s = \frac{n(2a+n-1)}{2}.$$

It is convenient to divide the analysis into two cases, depending upon whether n is odd or even.

- Suppose that n is odd; let $n = 2k+1$, where $k \geq 1$ (since $n > 1$). We now have:

  $$s = \frac{(2k+1)(2a+2k)}{2} = (2k+1)(a+k).$$

  In the above expression for s, note that both factors are greater than or equal to 2. Hence s cannot be prime in this case.

- Suppose that n is even; let $n = 2k$, where $k \geq 1$. We now have:

  $$s = \frac{2k(2a+2k-1)}{2} = k(2a+2k-1).$$

  Since $a \geq 1$ and $2k-1 \geq 1$, it follows that $2a+2k-1 \geq 2$. Hence if the product $k(2a+2k-1)$ is to be a prime number, it must be that $k=1$, i.e., $n=2$. It can obviously happen that the sum of the 2 consecutive integers $a, a+1$ is a prime number. Indeed, every odd prime can be thus expressed; e.g., $7 = 3+4$. The only prime number which *cannot* be expressed in this way is 2.

So the answer to the stated problem is $n = 2$.

Solution to Problem 2: *The sum of all terms of a finite arithmetic progression of integers is a power of* 2. *Prove that the number of terms is also a power of* 2.

Let the arithmetic progression have first term a and common difference d, and let it have n terms:

$$a,\ \ a+d,\ \ a+2d,\ \ \ldots,\ \ a+(n-1)d.$$

The sum s of these terms is given by

$$s = \frac{n(2a+(n-1)d)}{2}.$$

As earlier, it is convenient to divide the analysis into two cases, depending upon whether n is even or odd.

- Suppose that n is odd; let $n = 2k+1$, where $k \geq 1$ (since $n > 1$). We now have:

  $$s = \frac{(2k+1)(2a+2kd)}{2} = (2k+1)(a+kd).$$

  In the above expression, note that s has the odd factor $2k+1$, which strictly exceeds 1. Hence s cannot be a power of 2 in this case. (**Remark.** The fact that a power of 2 does not possess an odd factor exceeding 1 is a useful one to record in one's 'math memory'.)

- Suppose that n is even; let $n = 2k$, where $k \geq 1$. We now have:

  $$s = \frac{2k(2a+d(2k-1))}{2} = k(2a+d(2k-1)).$$

We are told that s is a power of 2. Hence it must be that the factors k and $(2a + d(2k - 1))$ are both powers of 2. But since $n = 2k$, it follows that n is a power of 2 as well. (**Remark.** The property that the divisors of a power of 2 are all powers of 2 is another useful fact to record in one's math memory.)

Remark. In connection with the second problem, you may wonder whether every power of 2 can be so expressed, i.e., as the sum of two or more terms of an integer arithmetic progression. The answer is: Yes, but in a rather trivial and none-too-exciting manner. The following array should give an idea of how this is done.

$$4 = 1 + 3,$$
$$8 = 3 + 5,$$
$$16 = 7 + 9,$$
$$32 = 15 + 17,$$

and so on. Observe that in each case, we have used only 2 terms in the expression. Can "more interesting expressions" be found, in which the number of terms is 4 or 8 or 16 or some higher power of 2? Indeed we can, for example:

$$16 = 1 + 3 + 5 + 7,$$
$$32 = 2 + 6 + 10 + 14,$$
$$64 = 1 + 3 + 5 + 7 + 9 + 11 + 13 + 15,$$
$$128 = 2 + 6 + 10 + 14 + 18 + 22 + 26 + 30,$$

and so on. But we leave the exploration of this to the reader.

Two Often Used Facts…

We have come across two often used facts in these solutions:

- A power of 2 does not possess any odd factor exceeding 1. Moreover, if a positive integer does not possess any odd factors exceeding 1, then it must be a power of 2.
- The only divisors of a power of 2 are smaller powers of 2. Such a statement can be madefor the power of any prime number: the only divisors of a prime power are smaller powersof that same prime number.

As noted above, it is useful to store away these two facts in one's math memory.

The COMMUNITY MATHEMATICS CENTRE (CoMaC) is an outreach arm of Rishi Valley Education Centre (AP) and Sahyadri School (KFI). It holds workshops in the teaching of mathematics and undertakes preparation of teaching materials for State Governments and NGOs. CoMaC may be contacted at shailesh.shirali@gmail.com.

Adventures with Triples

Part II

Problem Editor: Ranjit Desai

Problem Corner

In Part I of this two-part note, we described the following two problems. Given three positive integers a, b, c, we say that the triple (a, b, c) has the **linear property** if the sum of some two of the three numbers equals the third number (i.e., either $a + b = c$ or $b + c = a$ or $c + a = b$); and we say that the triple has the **triangular property** if the sum of any two of the three numbers exceeds the third number (i.e., $a + b > c$ and $b + c > a$ and $c + a > b$). Next, we fixed an upper limit n, and let a, b, c take all possible positive integer values between 1 and n (i.e., $1 \leq a, b, c \leq n$); we obtained a total of n^3 triples as a result. Then we asked: *How many of these triples possess the linear property?* We went on to answer this question fully.

Now we ask the same question, but about the triangular triples. However, there are two different contexts in which we can ask this question. We could study only **ordered triples**, in which for example, the triples $(2, 4, 3)$ and $(2, 3, 4)$ are considered to be different from each other; or we could study only **unordered triples**, in which triples such as $(2, 3, 4)$ and $(2, 4, 3)$ are considered to be the same (i.e., we do not distinguish between them). In other words, we do not distinguish between two different permutations of the same triple. *In this article, we choose to study the problem concerning unordered triples.* We ask: *How many of these unordered triples possess the triangular property?* We devote Part II of the article to studying this question.

Keywords: *Integers, linear property, triangular property*

Notation

- $S_u(n)$ denotes the set of all unordered integer triples (a, b, c) with $1 \leq a, b, c \leq n$. As we do not distinguish between two different permutations of the same triple, we may as well insist that $1 \leq a \leq b \leq c \leq n$.
- $T_u(n)$ denotes the number of triples in $S_u(n)$ which possess the triangular property. This is equivalent to defining $T_u(n)$ as the number of integer triples (a, b, c) which satisfy the conditions $1 \leq a \leq b \leq c \leq n$ and $a + b > c$.

Counting the unordered triangular triples

To start with, let us enumerate by hand the values of $T_u(n)$ for a few small values of n.

$\mathbf{n = 1}$: Since $S_u(1)$ has just the one triple $(1, 1, 1)$, and this has the triangular property, $T_u(1) = 1$.

$\mathbf{n = 2}$: The triples in $S_u(2)$ which have the triangular property and are not included in the previous list are $(1, 2, 2)$ and $(2, 2, 2)$; hence $T_u(2) = 2 + 1 = 3$.

$\mathbf{n = 3}$: The triples in $S_u(3)$ which have the triangular property and are not included in the previous list are $(1, 3, 3)$, $(2, 2, 3)$, $(2, 3, 3)$ and $(3, 3, 3)$; hence $T_u(3) = 4 + 3 = 7$.

$\mathbf{n = 4}$: The triples in $S_u(4)$ which have the triangular property and are not included in the previous list are $(1, 4, 4)$, $(2, 3, 4)$, $(2, 4, 4)$, $(3, 3, 4)$, $(3, 4, 4)$ and $(4, 4, 4)$; hence $T_u(4) = 6 + 7 = 13$.

$\mathbf{n = 5}$: The triples in $S_u(5)$ which have the triangular property and are not included in the previous list are $(1, 5, 5)$, $(2, 4, 5)$, $(2, 5, 5)$, $(3, 3, 5)$, $(3, 4, 5)$, $(3, 5, 5)$, $(4, 4, 5)$, $(4, 5, 5)$ and $(5, 5, 5)$; hence $T_u(5) = 9 + 13 = 22$.

$\mathbf{n = 6}$: The triples in $S_u(6)$ which have the triangular property and are not included in the previous list are $(1, 6, 6)$, $(2, 5, 6)$, $(2, 6, 6)$, $(3, 4, 6)$, $(3, 5, 6)$, $(3, 6, 6)$, $(4, 4, 6)$, $(4, 5, 6)$, $(4, 6, 6)$, $(5, 5, 6)$, $(5, 6, 6)$ and $(6, 6, 6)$; hence $T_u(6) = 12 + 22 = 34$.

Proceeding thus, step by step, we construct by hand (or a computer) the following table of values of the T function:

n	1	2	3	4	5	6	7	8	9	10	11	12	13	$\cdots$
$T_u(n)$	1	3	7	13	22	34	50	70	95	125	161	203	252	$\cdots$

Do you see any obvious pattern in the sequence of values of $T_u(n)$? There is indeed a pattern, but it is somewhat of a challenge to find it.

A general comment on the analysis of sequences. When we are studying a sequence of numbers and feel that there is some underlying pattern which however we are unable to put a finger on, it often helps to study sub-sequences of the given sequence; for example, by listing every second element of the sequence; or by listing every third element of the sequence; and so on. Hidden patterns sometimes get revealed this way. Another well-known technique is to study the sequence of first differences of the given sequence, or the sequence of second differences.

Applying the idea to our sequence. We shall follow both these suggestions in the present case. We list below the values of $T_u(n)$ for odd n and for even n, separately.

n	1	3	5	7	9	11	13	15	17	19	$\cdots$
$T_u(n)$	1	7	22	50	95	161	252	372	525	715	$\cdots$

n	2	4	6	8	10	12	14	16	18	20	$\cdots$
$T_u(n)$	3	13	34	70	125	203	308	444	615	825	$\cdots$

Now we are able to spot a pattern. In the case of the **odd** values of n, the first differences of the sequence of values of $T_u(n)$ are the following:

$$6,\ 15,\ 28,\ 45,\ 66,\ 91,\ 120,\ 153,\ 190,\ \ldots,$$

and in the case of the **even** values of n, the first differences of the sequence of values of $T_u(n)$ are the following:

$$10,\ 21,\ 36,\ 55,\ 78,\ 105,\ 136,\ 171,\ 210,\ \ldots.$$

Weaving the two sequences of first differences together, we obtain the following:

$$6,\ 10,\ 15,\ 21,\ 28,\ 36,\ 45,\ 55,\ 66,\ 78,\ 91,\ 105,\ 120,\ 136,\ 153,\ 171,\ 190,\ 210,\ \ldots,$$

and this is exactly the sequence of triangular numbers (except that the first two terms, 1 and 3, are missing)!

We thus observe the following, where $t(n) = n(n+1)/2$ denotes the n-th triangular number:

$$\begin{aligned} T_u(1) &= t(1), \\ T_u(3) &= t(1) + t(3), \\ T_u(5) &= t(1) + t(3) + t(5), \end{aligned}$$

and so on. That is, for $n = 1, 2, 3, 4, \ldots$:

$$T_u(2n-1) = t(1) + t(3) + t(5) + \cdots + t(2n-1).$$

The sum on the right side is easily computed:

$$\begin{aligned} & t(1) + t(3) + t(5) + \cdots + t(2n-1) \\ = & \frac{1^2+1}{2} + \frac{3^2+3}{2} + \frac{5^2+5}{2} + \cdots + \frac{(2n-1)^2 + (2n-1)}{2} \\ = & \frac{1^2+3^2+5^2+\cdots+(2n-1)^2}{2} + \frac{1+3+5+\cdots+(2n-1)}{2} \\ = & \frac{n(2n-1)(2n+1)}{6} + \frac{n^2}{2} = \frac{n\left(4n^2+3n-1\right)}{6}. \end{aligned}$$

So, our conjecture **on the basis of numerical evidence** is that for $n = 1, 2, 3, 4, \ldots$:

$$T_u(2n-1) = \frac{n\left(4n^2+3n-1\right)}{6}.$$

For example, take the values $n = 5$ and $n = 7$. We have:

$$\left.\frac{n\left(4n^2+3n-1\right)}{6}\right|_{n=5} = 95 = T_u(9),$$

$$\left.\frac{n\left(4n^2+3n-1\right)}{6}\right|_{n=7} = 252 = T_u(13),$$

thus confirming the conjecture for these two values of n. Our conjecture is therefore the following: for **odd** values of n, i.e., $n = 1, 3, 5, \ldots$,

$$T_u(n) = \frac{(n+1)\left(2n^2+7n+3\right)}{24}.$$

We similarly conjecture that for $n = 1, 2, 3, 4, \ldots$:

$$T_u(2n) = t(2) + t(4) + t(6) + \cdots + t(2n).$$

The sum on the right side is easily computed:

$$\begin{aligned}
&t(2) + t(4) + t(6) + \cdots + t(2n) \\
&= \frac{2^2 + 2}{2} + \frac{4^2 + 4}{2} + \frac{6^2 + 6}{2} + \cdots + \frac{(2n)^2 + 2n}{2} \\
&= \frac{2^2 + 4^2 + 6^2 + \cdots + (2n)^2}{2} + \frac{2 + 4 + 6 + \cdots + 2n}{2} \\
&= \frac{n(n+1)(2n+1)}{3} + \frac{n(n+1)}{2} = \frac{n(n+1)(4n+5)}{6}.
\end{aligned}$$

So, our conjecture **on the basis of numerical evidence** is that for $n = 1, 2, 3, 4, \ldots$:

$$T_u(2n) = \frac{n(n+1)(4n+5)}{6}.$$

For example, take the values $n = 4$ and $n = 6$. We have:

$$\left.\frac{n(n+1)(4n+5)}{6}\right|_{n=4} = 70 = T_u(8),$$

$$\left.\frac{n(n+1)(4n+5)}{6}\right|_{n=6} = 203 = T_u(12),$$

thus confirming the conjecture for these two values of n. Our conjecture is therefore the following: for **even** values of n, i.e., $n = 2, 4, 6, \ldots$,

$$T_u(n) = \frac{n(n+2)(2n+5)}{24}.$$

So our conjecture on the basis of numerical evidence is the following:

$$T_u(n) = \begin{cases} \dfrac{(n+1)\left(2n^2 + 7n + 3\right)}{24} & \text{if } n \text{ is odd,} \\[2ex] \dfrac{n(n+2)(2n+5)}{24} & \text{if } n \text{ is even.} \end{cases}$$

Note that all this is on the basis of numerical evidence. Now we must prove the formula.

Formal proof of the conjecture. Let $f(k)$ denote the number of integer triples (a, b, k) where $1 \le a \le b \le k$ and $a + b > k$. Thus, $f(k)$ counts the number of unordered triangular triples whose largest number is k. It should be evident that

$$T_u(n) = f(1) + f(2) + f(3) + \cdots + f(n).$$

So if we are able to find a formula for $f(k)$, then we should be able to find, using summation, a formula for $T_u(n)$. To find a formula for $f(k)$, it turns out to be convenient to consider separately the cases when k is even and when k is odd.

The case when k is even. Let $k = 2j$. We list all possible triangular triples $(a, b, 2j)$ in a systematic way as follows:

b	List of triangular triples	Number of triples
$2j$	$(1, 2j, 2j),\ (2, 2j, 2j),\ (3, 2j, 2j),\ \ldots, (2j, 2j, 2j)$	$2j$
$2j-1$	$(2, 2j-1, 2j),\ (3, 2j-1, 2j),\ \ldots, (2j-1, 2j-1, 2j)$	$2j-2$
$2j-2$	$(3, 2j-2, 2j),\ (4, 2j-2, 2j),\ \ldots, (2j-2, 2j-2, 2j)$	$2j-4$
$\cdots$	$\cdots\ \cdots\ \cdots\ \cdots$	$\cdots$
$j+1$	$(j, j+1, 2j),\ (j+1, j+1, 2j)$	2

Hence the total number of triangular triples in the case when k is even is equal to:

$$2 + 4 + 6 + \cdots + 2j = j(j+1) = \frac{k}{2}\left(\frac{k}{2}+1\right) = \frac{k(k+2)}{4}.$$

The case when k is odd. Let $k = 2j+1$. We list all possible triangular triples $(a, b, 2j+1)$ in a systematic way as follows:

b	List of triangular triples	Number of triples
$2j+1$	$(1, 2j+1, 2j+1),\ (2, 2j+1, 2j+1),\ \ldots, (2j+1, 2j+1, 2j)$	$2j+1$
$2j$	$(2, 2j, 2j+1),\ (3, 2j, 2j+1),\ \ldots, (2j, 2j, 2j+1)$	$2j-1$
$2j-1$	$(3, 2j-1, 2j+1),\ (4, 2j-1, 2j+1),\ \ldots, (2j-1, 2j-1, 2j+1)$	$2j-3$
$\cdots$	$\cdots\ \cdots\ \cdots\ \cdots$	$\cdots$
$j+1$	$(j+1, j+1, 2j+1)$	1

Hence the total number of triangular triples in the case when k is odd is equal to:

$$1 + 3 + 5 + \cdots + (2j+1) = (j+1)^2 = \left(\frac{k+1}{2}\right)^2.$$

It follows that the function $f(k)$ has the following formula:

$$f(k) = \begin{cases} \dfrac{k(k+2)}{4} & \text{if } k \text{ is even,} \\[2ex] \left(\dfrac{k+1}{2}\right)^2 & \text{if } k \text{ is odd.} \end{cases}$$

We are now in a position to derive the formula we need. Suppose that n is even, say $n = 2r$; then:

$$\begin{aligned}
T_u(n) &= \sum_{\substack{i=1\\ i\text{ odd}}}^{2r-1} f(i) + \sum_{\substack{i=2\\ i\text{ even}}}^{2r} f(i)\\
&= \sum_{\substack{i=1\\ i\text{ odd}}}^{2r-1} \left(\frac{i+1}{2}\right)^2 + \sum_{\substack{i=2\\ i\text{ even}}}^{2r} \frac{i(i+2)}{4}\\
&= \frac{r(r+1)(2r+1)}{6} + \frac{r(r+1)(r+2)}{3} \qquad \text{(steps omitted)}\\
&= \frac{r(r+1)(4r+5)}{6} = \frac{n(n+2)(2n+5)}{24}, \quad \text{since } n = 2r.
\end{aligned}$$

We have recovered the formula which we had empirically derived earlier, purely on the basis of numerical evidence.

Next, suppose that n is odd, say $n = 2r - 1$; then:

$$\begin{aligned}
T_u(n) &= \sum_{\substack{i=1\\ i\text{ odd}}}^{2r-1} f(i) + \sum_{\substack{i=2\\ i\text{ even}}}^{2r-2} f(i)\\
&= \sum_{\substack{i=1\\ i\text{ odd}}}^{2r-1} \left(\frac{i+1}{2}\right)^2 + \sum_{\substack{i=2\\ i\text{ even}}}^{2r-2} \frac{i(i+2)}{4}\\
&= \frac{r(r+1)(2r+1)}{6} + \frac{r(r+1)(r-1)}{3} \qquad \text{(steps omitted)}\\
&= \frac{r\,(4r^2+3r-1)}{6} = \frac{(n+1)\,(2n^2+7n+3)}{24}, \quad \text{since } n = 2r-1.
\end{aligned}$$

Once again, we have recovered the formula which we had empirically derived earlier, purely on the basis of numerical evidence.

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A Review of

'My search for Ramanujan: How I Learned to Count'

By Ken Ono & Amir D. Aczel

Ramesh Sreekantan

Review

Ken Ono is the Asa Griggs Candler Professor of Mathematics at Emory University. Over the last twenty years he has proved many beautiful theorems, many of which explain and generalize the work of Ramanujan which were hinted at in Ramanujan's notebooks a hundred years ago. This book, which is largely an autobiography as told to Amir Aczel, is an attempt on Ken's part to explain, at some level, how it was not only the Mathematics of Ramanujan, but his life's story which influenced Ken and, to a certain extent, made him what he is today.

The book is in three parts, The first part is about his growing up in a suburb of Baltimore, Maryland. Ono's father, Takashi Ono, was a Professor of Mathematics at Johns Hopkins University. He and his wife had moved to the United States in the mid fifties with the intention of being there for a few years and then returning, but at that point the situation for academics in post-war Japan was not so great, so they ended up staying. They had three sons, Momoro, who is an accomplished musician, Santa, who is a bio-chemist and now the president of University of British Columbia, and Ken. Ken was always expected to

Keywords: *Ken Ono, Ramanujan, Bruce Berndt, parental pressure, teenage rebellion, criticism*

follow in his father's footsteps and become a mathematician - something he was not so keen on at that point. This led to some amount of tension between him and his parents - they were what are now called 'Tiger Parents' - Asian American parents pressuring their children to succeed to a fault. This eventually lead to him dropping out of high school at the age of 16 and going to live with his elder brother Santa in Montreal. Around the same time he developed a strong interest in bicycling, and going on long bike rides gave him the peace of mind that he was craving. It helped him escape the pressures he felt his parents put on him. He was also a talented violinist - but as an act of rebellion against his parents he gave it up.

However, before leaving for Montreal, an event took place which turned out to have a profound influence on Ken's life. In early 1984 his father received a letter from India. It was a form letter sent by Janakiammal, the wife of S. Ramanujan, thanking him for his contribution towards the making of a bust of Ramanujan. Takashi Ono was moved to tears by this letter - a fact that surprised Ken as he had not seen his father show such emotion before. His father then told him the remarkable story of Ramanujan's life. This had a great impact on him; and Ken feels that it was this story - of a brilliant Mathematician who had problems with the conventional school and college education - that made his parents allow him to leave school and go spend time with his brother.

The second part of the book is the remarkable story of Ramanujan. Perhaps most readers will be familiar with the fact that Ramanujan was born into a poor family in 1887 in Kumbakonam, now in Tamil Nadu. While he showed exceptional talent in Mathematics, there were no opportunities for a person like him to pursue it. He struggled with the conventional schooling system, though finally, thanks to the patronage of someone who had some understanding of his mathematical genius, he secured the position of a clerk in the Madras Port Trust. While he was there, he wrote letters to several mathematicians in Europe stating his results. All of them ignored him, with the exception of the Cambridge mathematician G.H. Hardy, who, on receiving the letter, thought that the formulas which Ramanujan had written without proof "must be true as no one would have the imagination to invent them". This led to a correspondence between them, and Hardy then arranged for Ramanujan to come to Cambridge. This resulted in some fruitful collaboration, leading, among other things, to what is known as the *Hardy-Ramanujan Partition Formula and the Circle Method.*

In Cambridge Ramanujan struggled - not only because of the blatant racism that was the norm then - but because he was a strict vegetarian and so did not eat in the dining halls. He cooked his own food, and in those days nutritious vegetarian food was hard to come by. Added to this, the First World War broke out, which led to shortages. His family was far away and apparently his mother did not post the letters his wife had written to him, adding to his isolation. All this took a severe toll on Ramanujan's health; he became ill with what they thought was tuberculosis, though now they suspect it was simply amoebiasis, and this led to his return to India and eventual death, a year later, at the young age of 32.

The third part of the book concerns the last 25 years or so of Ken's life. After spending some time in Montreal, he secured admission to the University of Chicago thanks to the recommendation of a psychologist who was involved in a programme for gifted and talented youth that Ken was in earlier. The University was willing to take a chance on him, notwithstanding that he was a high school dropout. In Chicago he initially took things easy and found the course work hard. It was here that another incident took place which had a singular impact on his later life. A certain visiting faculty member, who had taught Ken *Complex Analysis,* told him that the career of a research Mathematician was not for him; he would be better off being a banker or something else. This upset Ken, as till that point, while he did not care and did not really want to be a Mathematician, the fact that someone said that he was not capable of being one was like a slap in the face. This inspired Ken to take things more seriously and work harder. Around the same time, while flipping channels on TV, he came across a documentary on

Ramanujan, which reminded him of the letter his father had received.

With his new found motivation to prove that professor wrong, Ken did much better in his classes. Paul Sally, a well known professor in Chicago, took interest in Ken's education and helped him secure admission into graduate school at the University of California, Los Angeles. During the time he was in Chicago another important event happened: he met his future wife Erica, who was also a student in Chicago, and they got married shortly before he left for graduate school.

The change in life, from gloomy, cold Chicago to bright, sunny Los Angeles once again made Ken take things easy. He spent time goofing off and cycling till another incident gave him a wake up call. He failed his Algebra qualifier. Though he managed to convince a professor that one of his answers was correct even though it had been marked wrong - and hence he would pass - the professor said that merely passing was not enough for someone who intended to do a Ph.D. in a subject related to Algebra. Once again, this shock was enough to make him take things seriously.

It was around this time that he took an Algebraic Number Theory class with Basil Gordon, whom he managed to impress with a new proof of a known result. Shortly after that he started working with Gordon towards a Ph.D. This was a turning point in his life as the time spent discussing work with Gordon made him discover his love for Mathematics. In a short couple of years he finished his Ph.D. in the area of modular forms - a subject close to Ramanujan's heart.

Towards the end of his Ph.D., a third incident took place, which apparently nobody, not even his wife, was aware of till this book was written. Ken's wife is from Montana and Ken was hoping to move there and get a job in the University of Montana at Missoula. He was invited to a conference close by and took that opportunity to give a talk at the University in the hope that that would help him later on when he applied. Not being aware of his audience, he gave a technical talk, which did not go down well with some of the faculty members. Some old curmudgeon told him that he had wasted his time. This upset Ken greatly; it made him feel that he was a failure who had blown his chances of a job in Missoula. A couple of days later he attempted to end his life by driving onto the lane of an oncoming logging truck. Fortunately he came to his senses and swerved away in the last minute.

Not getting the job in Missoula was a blessing in disguise as shortly after that he received a post doctoral offer from Andrew Granville in Georgia. Granville proved to be an excellent mentor and under his guidance Ken was able to write a very nice paper on Ramanujan graphs. This led to a post-doctoral offer from the University of Illinois at Urbana-Champaign, where he met Bruce Berndt - one of the people in the documentary on Ramanujan, and a person who has spent a large amount of time understanding Ramanujan's *Notebooks*.

It was when he was here that he got an invitation to spend a couple of years at the Institute for Advanced Study (IAS) in Princeton. The IAS is a prestigious institute most famous for the fact that Einstein worked there. It was during this period that Ken's career took off: he wrote a couple of important papers with Kannan Soundararajan and Chris Skinner.

After that, he has not looked back and has done a tremendous amount of work in the areas of elliptic curves and modular forms. More recently he has been working in the area of 'mock modular forms', a field that grew out of Ramanujan's last letter to Hardy, a month before he died. Ken, along with others, has discovered remarkable properties of these forms and they seem to appear in several seemingly disparate areas of Mathematics and even Physics. The book ends with some description of Ramanujan's work as well as some of his own.

As is the case with any autobiography or biography of a successful person - one should always look to see what are the lessons one can take from his or her life. In Ken's case, I think the lesson is how to deal with setbacks. Everyone has them - life never goes as smoothly as one expects - and perhaps what separates exceptional people from ordinary ones is how one deals with it. Ken used the negative comments of others as inspiration to prove them wrong - and has done so in a spectacular manner. In spite of all these tribulations, he finished his Ph.D. by the age of 24 and was an established researcher, winning a Presidential Early Career grant, within a few years after that. Over the last twenty years he has written over 150 papers! Most mathematicians write about 30 - so this is quite an achievement in itself. He continues to discover remarkable new results in Mathematics - especially related to the Mathematics of Ramanujan.

Another lesson one can take from this book is to be careful with one's words, especially for those who are in a position of authority. The professor who said that Ken was not cut out for research perhaps had not given it a second thought. He perhaps did not realise, till the publication of this book, the impact of his off hand comment on the young undergraduate. In this case it worked out for the best, but that is not always the case. It is often the case for graduate students that their thesis is their whole life, and so feeling that they have failed in some way can have disastrous consequences.

This book, which is very open and honest, does give you the example of someone who struggled initially but was able to overcome those difficulties to succeed spectacularly. I think it would be beneficial for an undergraduate or early graduate student who is plagued with doubts as to whether he or she is 'good enough' to pursue the path they have chosen, to read this book.

In my personal opinion Ken was perhaps an exceptional person all along - his father perhaps recognised his mathematical talent before he himself did - and much of his fear of failure was caused by self-doubt rather than reality. Then again, at the end of the day, it is only one's perception of oneself that matters and being able to control this is perhaps the secret of success.

Another point about Ken's life which is perhaps relevant in the Indian context is the importance of physical exercise. Many Indian students, especially in their late teenage years, give up any sort of physical activity because they think they have to study and have no time for it. In fact, studies show that exercising regularly is very good for the brain, and Ken - who is almost a professional triathlete at the age of 48, is living proof of it.

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A word about partitions

Much of the work of Ramanujan and Ken Ono is perhaps too difficult to describe. However, some of their work centres around the study of ***partitions of numbers.*** The partition of a natural number n is an expression of n as a sum of non-decreasing sequence of natural numbers. The function $p(n)$ denotes the number of partitions of n.

For example:

$$\begin{aligned} 5 &= 1 + 1 + 1 + 1 + 1 \\ &= 1 + 1 + 1 + 2 \\ &= 1 + 2 + 2 \\ &= 1 + 1 + 3 \\ &= 1 + 4 \\ &= 2 + 3 \\ &= 5 \end{aligned}$$

So $p(5) = 7$. While the first few values are easy to compute, the number $p(n)$ grows rapidly. For example $p(100) = 190, 569, 292$. A natural question, then, is whether there is a formula for $p(n)$. That may be too much to expect, but Hardy and Ramanujan discovered a remarkable asymptotic formula for $p(n)$. Ken Ono and his collaborators discovered another arithmetic formula for $p(n)$.

Another set of results which Ramanujan had arrived at concerned congruences satisfied by $p(n)$. He proved, for instance, that for any non-negative integer k, $p(5k + 4) \cong 0 \bmod 5$; i.e., $p(5k + 4)$ is always divisible by 5. He also discovered that there were similar properties for the primes 7 and 11, but also made the remark that there were no 'easy' congruences for higher primes - namely of the form $p(mk + r) \cong 0 \bmod m$. In the 1960s Atkin discovered that $p(11^3 \cdot 13 \cdot k + 237) \cong 0 \bmod 13$. One of the remarkable results of Ono concerned showing that there exist congruences modulo every prime. Further, along with Scott Algren, he showed that there exist congruences modulo every integer co-prime to 6.

SEVEN MUST-SEE

MATHEMATICS MOVIES INSPIRED FROM TRUE-LIFE EVENTS

This list has been compiled by **Guillermo Bautista**, a Filipino math blogger, who works at a university in the Philippines and maintains the blog Mathematics and Multimedia http://mathandmultimedia.com/view-posts-by-topics/

His list is interesting; how many of these movies have you seen?

1. **Stand and Deliver (1988)** *Starring* ***Edward James Olmos***
2. **A Beautiful Mind (2001)** *Starring* ***Russell Crowe***
3. **Moneyball (2011)** *Starring* ***Brad Pitt***
4. **The Imitation Game (2014)** *Starring* ***Benedict Cumberbatch***
5. **x + y (2014)** *Starring* ***Asa Butterfield***
6. **Theory of Everything (2014)** *Starring* ***Eddie Redmayne***
7. **The Man Who Knew Infinity (2016)** *Starring* ***Dev Patel***

Retrieved from http://mathandmultimedia.com/2016/08/21/mathematics-movies/?utm_source=feedburner&utm_medium=email&utm_campaign=Feed%3A+MathematicsAndMultimedia+%28Mathematics+and+Multimedia%29 on Sep 15, 2016

Any math movies that you would recommend? Post your favourites on AtRiUM – our FaceBook page.

The Closing Bracket . . .

One of the most pressing issues in mathematics education in our country is perhaps that of teacher preparation, an issue which needs to be revisited in the light of the changing demands of mathematics education. This point has been highlighted in the position paper on the *Teaching of Mathematics* of the National Curriculum Framework 2005. The document states: "More so than any other content discipline, mathematics education relies heavily on the preparation that the teacher has, in her own understanding of mathematics, of the nature of mathematics, and in her bag of pedagogic techniques."

Typically, secondary and senior secondary school mathematics teachers enter the teaching profession with a Master's degree in mathematics followed by a Bachelor of Education (B.Ed.). However, once they join a school they get very few opportunities to further their learning. Schools sometimes require teachers to attend workshops and training programmes organised by various agencies, but these are very sporadic and do not really contribute to the professional growth of teachers. Also, there is no follow-up mechanism to assess the impact of these programmes on either the teacher's classroom practices or their content knowledge.

Often, it is the case that teachers lack strong fundamentals in the subject and are unable to enhance their learning owing to lack of professional development opportunities or access to good resource materials. Also, having spent several years in the profession without adequate professional growth, they begin to find it monotonous and lose their motivation. Their inability to make connections between the different topics in mathematics as well as between mathematics and other subject areas reflects negatively on the way they teach mathematics. The overall effect is that students develop a 'blinkered approach' to the subject.

Thus there is a tremendous need for sustainable in-service teacher education programmes or 'nurture programmes' for all levels of schooling which not only help teachers to enhance their content knowledge in the subject but also develop their pedagogical skills. Given the changing demands of mathematics education, this needs to reflect in the way the teacher preparation programmes are envisioned and designed. In-service teacher education programmes have to be implemented on a large scale, uniformly across the country. Further, these need to be sensitive to the conditions under which teachers work in schools. Constraint of time, large syllabus, inflexible modes of assessment, preparing students for the board examinations are some of the problems faced by teachers, especially at the secondary and higher secondary stage. Thus, in-service programmes should be flexible, and teachers should have the opportunity to complete the programme within the prescribed span of time. Such programmes may have an online component (which may be provided in a distance mode in the form of study material for teachers who do not have access to the internet) as well as actual contact classes and workshops. The focus of nurture programmes should be to motivate the teacher with a sense of pride and confidence, deepen her subject knowledge, enhance her pedagogical skills and also provide scope for innovation. The courses should be evolved by taking adequate feedback from teachers at regular intervals. While the thrust of these courses should be on the topics of secondary and higher secondary school mathematics, they should be dealt with from an advanced standpoint so as to enable the teacher to understand how the topics develop at the undergraduate level and beyond. For example, a course in linear algebra (which may be a part of such a course) should not only emphasise the concepts covered in the topic matrices and determinants (usually taught in classes 11 and 12) providing adequate resources and ideas to teach it better, but also familiarise the teacher with a glimpse of the subject at a higher level. An attempt should be made to provide a blend of content knowledge as well as pedagogic approaches in transacting the content of various topics of the curriculum. Care should be taken to provide linkages across topics. Applications, mathematical models and technology-enabled explorations may be integrated into the topics wherever possible. A course based on the nature of mathematical thinking and historical background of concepts should form an integral part of the nurture programme. Further, teachers should be familiarised with learning theories and frameworks which impact mathematical learning. Finally, and above all, the programme should lead to certification and meritorious teachers should be given due recognition.

The professional growth of mathematics teachers is strongly linked to their mathematical knowledge, and this has been an issue of concern in mathematics education in most countries. But what is the nature of mathematical knowledge required by a teacher? It has long been established through several large-scale research studies across the world that a strong content knowledge in mathematics is necessary but not sufficient for good mathematics teaching. Results from such studies have shown the need to reinforce the school teachers' mathematical content knowledge (MCK), as well as their pedagogical content knowledge (PCK). So how is mathematics used in teaching, and what is the professional knowledge required by a mathematics teacher? What is the mathematical work done by a teacher, and how is this different from the mathematical work done by people who use mathematics in their profession? What is the nature of PCK required by teachers in an ever evolving world of technological tools? Above all, how can in-service continuing professional development programmes or nurture programmes be created which promote a deeper understanding of mathematics among teachers? These are critical questions pertaining to the professional growth of mathematics teachers that need to be jointly addressed by mathematics education researchers, mathematics educators, mathematics teachers and mathematicians alike.

Jonaki Ghosh

Specific Guidelines for Authors

Prospective authors are asked to observe the following guidelines.

1. Use a readable and inviting style of writing which attempts to capture the reader's attention at the start. The first paragraph of the article should convey clearly what the article is about. For example, the opening paragraph could be a surprising conclusion, a challenge, figure with an interesting question or a relevant anecdote. Importantly, it should carry an invitation to continue reading.
2. Title the article with an appropriate and catchy phrase that captures the spirit and substance of the article.
3. Avoid a 'theorem-proof' format. Instead, integrate proofs into the article in an informal way.
4. Refrain from displaying long calculations. Strike a balance between providing too many details and making sudden jumps which depend on hidden calculations.
5. Avoid specialized jargon and notation — terms that will be familiar only to specialists. If technical terms are needed, please define them.
6. Where possible, provide a diagram or a photograph that captures the essence of a mathematical idea. Never omit a diagram if it can help clarify a concept.
7. Provide a compact list of references, with short recommendations.
8. Make available a few exercises, and some questions to ponder either in the beginning or at the end of the article.
9. Cite sources and references in their order of occurrence, at the end of the article. Avoid footnotes. If footnotes are needed, number and place them separately.
10. Explain all abbreviations and acronyms the first time they occur in an article. Make a glossary of all such terms and place it at the end of the article.
11. Number all diagrams, photos and figures included in the article. Attach them separately with the e-mail, with clear directions. (Please note, the minimum resolution for photos or scanned images should be 300dpi).
12. Refer to diagrams, photos, and figures by their numbers and avoid using references like 'here' or 'there' or 'above' or 'below'.
13. Include a high resolution photograph (author photo) and a brief bio (not more than 50 words) that gives readers an idea of your experience and areas of expertise.
14. Adhere to British spellings – organise, not organize; colour not color, neighbour not neighbor, etc.
15. Submit articles in MS Word format or in LaTeX.

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Also welcome are short pieces featuring: reviews of books or math software or a YouTube clip about some theme in mathematics; proofs without words; mathematical paradoxes; 'false proofs'; poetry, cartoons or photographs with a mathematical theme; anecdotes about a mathematician; 'math from the movies'.

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TEACHING Data Handling

PADMAPRIYA SHIRALI

At Right Angles
A Resource for School Mathematics

Data Handling

INTRODUCTION

All mathematical topics that are taught in the primary school are born out of the need of children to understand their world. In the process of acquiring and growing in this understanding children encounter various types of numerical data. Often this data acquires significant meaning when seen in relation to other numerical data of a similar kind. When this data is organised into a meaningful form and presented graphically it highlights similarities and differences. Comparisons can be made and patterns observed leading to useful conclusions.

In today's world there is an explosion of data and much of this data is presented to us in the form of graphs. Graphs have become a form of communication. The skill of representing data graphically, reading and interpreting graphs accurately, and learning their limitations has acquired significance.

While teaching this topic it is important to ensure that representation of data in the form of a graph is not perceived as an end in itself. Ideally a graph should lead to important observations. Once the graph is made the crucial question to be raised and discussed in the classes is 'what does the graph say?' It has an open ended aspect to it. Also data used needs to be realistic, meaningful and should provide an opportunity for useful graph work.

What does data handling at the primary level involve?

It involves raising questions (preferably organically), collection of data or working with a given data set, representation (graphs ranging from physical object graphs, pictographs to bar graphs and Venn diagrams) and interpretation. Interpretation at this level would take the form of reading of facts from the graph, making comparative statements, pattern recognition and understanding causes and implications of the data.

Data handling and graphing skills progress through various levels of increasing complexity in terms of collection, organisation, representation, interpretation finally leading to conjecture and analysis. It is the context where children apply their numerical skills.

What should be the order or the progression of graphic skills?

In all areas of mathematics, while teaching young children one introduces concepts through concrete materials followed by semi-concrete, i.e., drawings, and then we make the transition to the abstract. In a similar manner, even in the teaching of data handling there should be a developmental sequence starting with physical object graphs, pictographs, bar graphs, leading finally to the abstract graphical forms.

Keywords: *data collection, representation, visualization, survey, graphs, pictographs, bar graphs*

What is the place of a data survey or enquiry?

A survey project needs to emerge organically from the questions that children ask. What is the most favourite game of the class? Which is the month with maximum number of birthdays in our class?

Once the question is framed, children can figure out the kind of data they need to collect to seek the answer to that question. Identifying the data needed to answer a question is one of the learning objectives of data handling. Children can now go through the process of collecting data. Collection of data can happen through various forms. It may be as simple as asking their classmates to raise their hands or making a questionnaire to be filled in. Recording of this data can also take various forms from making pictures to tally marks to summing and recording.

The next challenge is organising it in a readable form and representing the data through various means.

Interpretation of the data can be done at three levels. Firstly, reading the numerical values of the graph, answering the 'which is the highest/lowest?' kind of question. Secondly, reading relationships, that is, comparative statements, how much less, how many times more, etc. Thirdly, noticing patterns and seeing connections, understanding causes and implications of the data.

So the whole task of conducting a survey involves several objectives: identifying the data needed, working out the format of collecting data, finding a good way of recording the data, choosing an appropriate form for presenting the data, answering the basic questions as well as noticing other patterns in the data and interpreting the data.

However, carrying out a survey is a very time-consuming process, and sometimes it is necessary to present the children with structured pre-determined data sets. These need to be connected to their life and their immediate experiences, and they must be meaningful to the children.

LEVEL 1

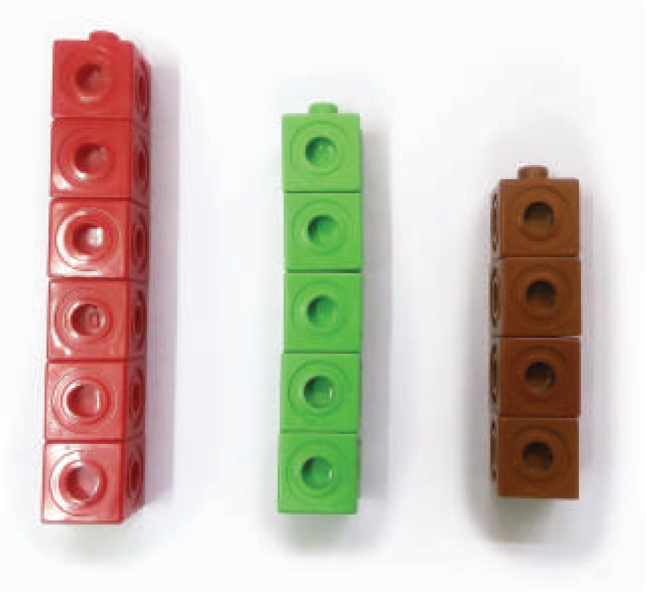
CHILDREN OF AGES 3 TO 5

Expectations: Children build graphs of real objects.

Introduction to graphs is generally initiated by the teacher through play activities which children enjoy. At this stage teacher can help the children to build graphs of real objects. They can then count and compare quantities using number words. Since the graphs are made of real objects which can be placed on the floor or affixed on a bulletin board, the floor or the base of the board acts as the x-axis. Labelling and uniformity of scale arise naturally from the situations. Children should not be obviously exposed to terms like axis, base and scale.

ACTIVITY 1

Material: Unifix cubes or big wooden blocks of different colours

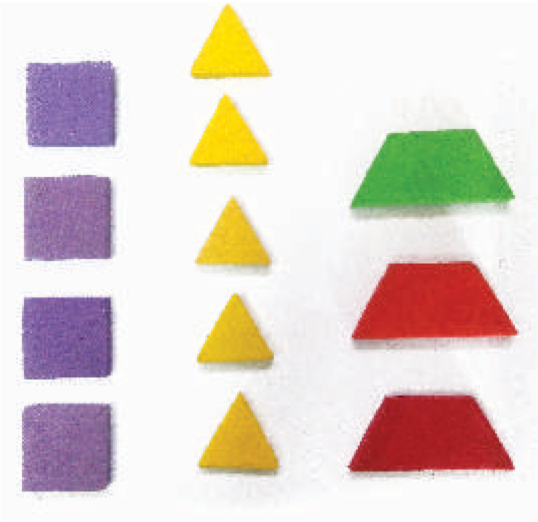


Children love to build trains. They would be quite excited to build coloured cube trains with Unifix cubes (cubes which fit together). The cube train becomes a graph of real objects. Once the graph is built children can talk about the graph. The number of cubes in the red train is more than the number of cubes in the brown train. If we have one more cube in the green train it will be the same as the number of cubes in the red train. The teacher can raise further questions about the graph and the different possibilities. What will happen if three cubes of the red train are left behind at the station? Which train would be the smallest? What if we attach 4 more cubes to the brown train?

If cubes are not available, children can make towers of wooden blocks and compare these towers. However, the blocks should be of uniform size.

ACTIVITY 2

Material: Shapes of different colours



Give each group of children around 20 shapes (different shapes in different colours). Let children sort the shapes into different sets (say by shape) and build a graph with them.

Raise questions: Why do these belong together? Which has the most? Which has the least? How many more? Can these shapes be sorted in another way?

Children can also sort the same set by colour. They may sort them by the number of edges or on the basis of number of corners.They may sort them as shapes with edges of same length, edges of different lengths and shapes with no straight edges.

3-D shapes can also be sorted in different ways (by number of faces, shapes of the faces, number of edges).

ACTIVITY 3

Material: Chess Kit

Once upon a time there was a king and a queen. … The story can be the starting point for a graph built with chess pieces.

Children can sort and line up the pieces. The teacher can label the columns to familiarise the children with the names of the various pieces.

LEVEL 2

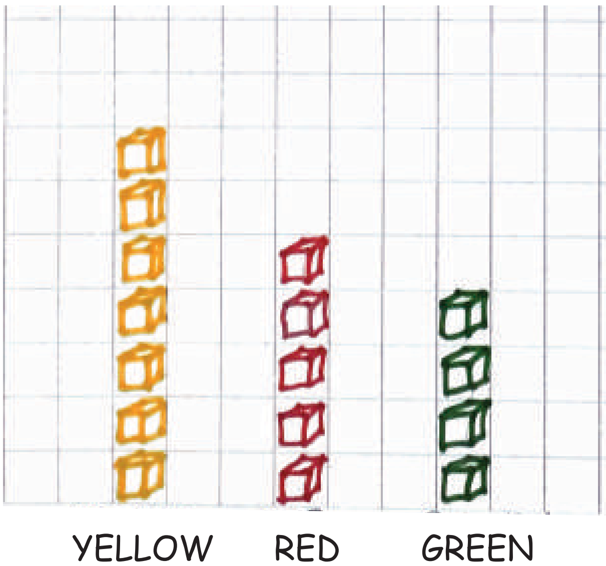
CHILDREN OF AGES 5 TO 6

Expectations: Children will be able to construct pictographs.

They will begin to learn to collect, display and interpret data for seeking an answer to their question. They will also be able to read and interpret teacher generated pictographs. They will be able to use simple Venn diagrams.

Teachers can use questions which arise naturally in a classroom situation to initiate graph making. Often children express their likes and dislikes about food items. It may be about vegetables, fruits, biscuits or chocolates. They have their favourite games. 'I wonder what the favourite fruit of our class is!' This can become the starting point for a lot of chatter which can be skilfully used to create a pictograph.

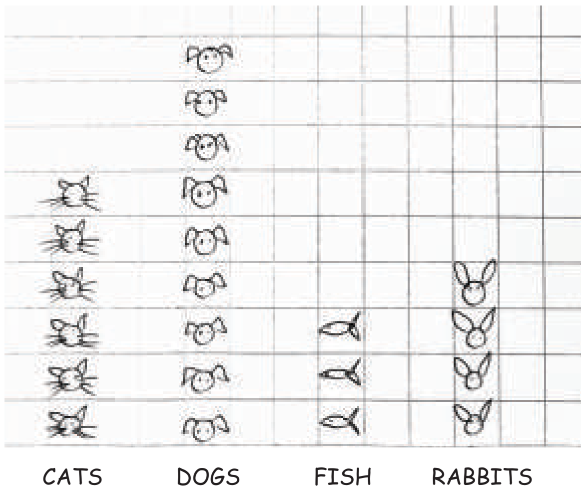
ACTIVITY 4



Children can now draw pictographs by mapping real objects to pictures. They are also in a position to make pictographs for given data. Square grid paper can be used to facilitate the drawing. It serves the need for uniform scaling. Vertical columns can be easily filled in by children. The square acts as a frame for children to fill in. Labelling can be done at the base. Since each unit square stands for one entry it facilitates counting.

Drawings made in these squares need not be realistic and are generally symbolic. Children can now interpret this graph by comparing quantities: more, fewer, less than, greater than, together, etc.

ACTIVITY 5



Pets or favourite animals are an attractive theme for children. 'Which is the most popular pet?' may be the starting point.

Once the graph is built, children can be encouraged to make their observations:

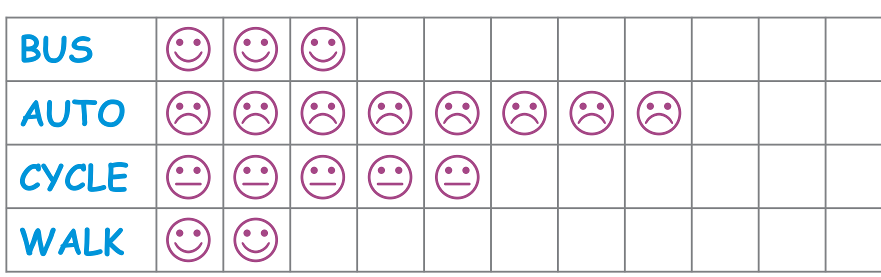
- Which pet is the least popular?
- Which animals are equally popular in our class?
- If ____________ more students choose 'cat' then it will be as popular as 'dog'.
- _________ and __________ together are more popular than ___________.

Teacher can draw attention to other aspects: If we add up all the numbers in the graph, does the total match the total number of students in our class? Why? Why not?

It is important to present graphs both in vertical and horizontal form. Children should be able to read data from different forms.

ACTIVITY 6: Horizontal graph

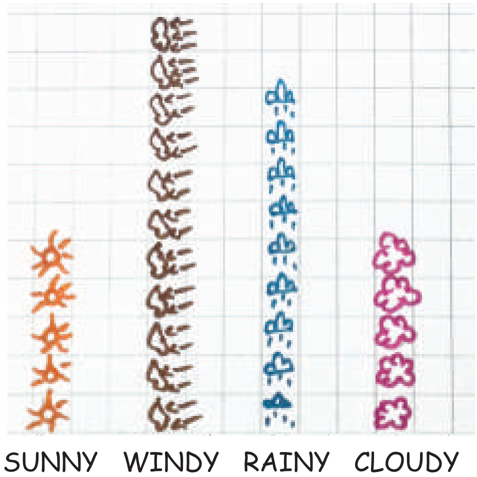
Modes of travel to school are closely linked with children's lives. This topic provides opportunities for discussion about traffic, safety, overcrowding, etc.



The teacher can create a chart as shown in the picture for children to fill in. Once the chart is ready, the teacher can at first encourage children to make their observations. Numerical figures can be read and recorded. Comparative statements can be made.

Some possible questions which can lead to general discussion: Why are more students coming by the auto than by the bus? How are some students able to come walking? Why don't we have students come by train? Are there some students who use a vehicle as well as walk? Why does that happen?

ACTIVITY 7



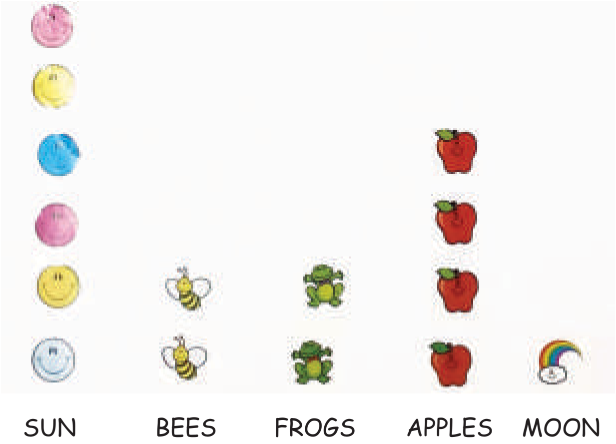
Weather calendar and chart: Weather can be categorised as 'sunny', 'windy', 'rainy', and 'cloudy'. Children can create symbols to depict these four categories of weather. Each day an entry is made into the calendar according to the weather. At the end of the month, a pictograph can be prepared based on the information from the calendar.

Why do we have so many rainy days this month? Is it possible to guess what kind of a day tomorrow will be? If we make a weather chart for March, will it look like this? What would be different? Are there some days when it is both rainy and windy? How did we mark those days?

ACTIVITY 8

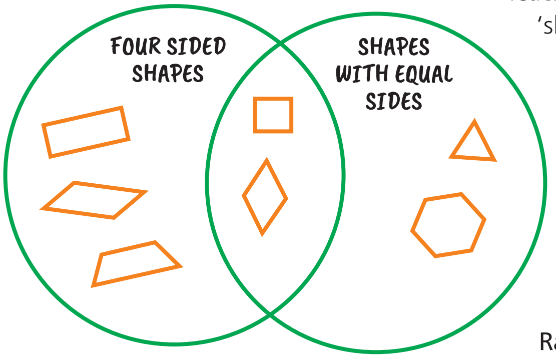
Stickers are a great favourite with children. As homework they can be asked to create a graph using stickers. They can bring the graph to the class room and talk about it.

The question "which is the favourite cartoon programme of the class?" will interest the children immensely!



ACTIVITY 9: Venn diagrams with two circles

Expectations: Children build graphs of real objects.



Teacher can draw two separate circles labelled as 'four sided shapes' and 'shapes with equal sides' on the floor. Let children pick up the various shapes one at a time and work out where the different pieces go. However when they pick up a square or a rhombus they will be unclear as to where to place them. Through discussion between students and teacher, students will see that square belongs to both the sets. Teacher can now redraw the two circles as intersecting circles and shapes can be placed in the right places. Children will need help in learning to describe a shape in terms of two different attributes. At this stage, they do not need to know the names of all the shapes.

Raise questions: Why is this shape part of both the sets? Why are these shapes not part of any set? When will a shape be only in one of the circles?

Expectations: At this stage children are able to create data tables and make tally charts.

They will be able to draw bar graphs using square paper. They will be able to read and interpret teacher generated bar graphs. They will be able to use pictographs where each picture represents more than one (e.g., 1 face may stand for 5 people). However, such pictographs have a built-in limitation. For instance, if a face represents 5 people, it is possible to represent all multiples of 5. But there would be a difficulty in representing in-between numbers, e.g., from 21 to 24. Half-multiples may be possible, say by drawing half of the symbol; but fractions such as a third or a quarter would clearly be difficult to represent. The teacher should help children understand the limitations of such pictographs. Also, the teacher can show students that it is easier to represent larger numbers with a tally chart or a data table than to make a pictograph. They will be able to use simple Venn diagrams (two circles) with subsets.

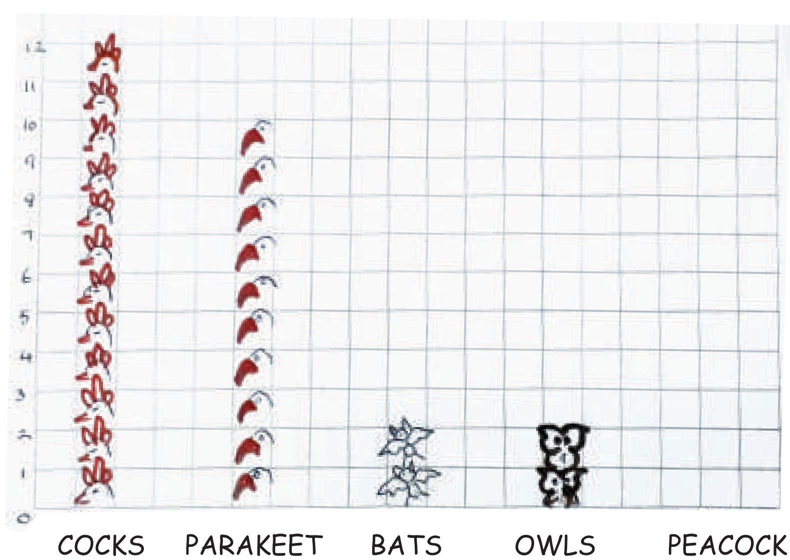
Transition from pictographs to bar graphs: This is the next challenge that children face. Usage of the square grid as a framework facilitates this process. Instead of drawing the pictures in the grid, they will now fill the squares with colour and outline the bars.

ACTIVITY 10

Material: Chess Kit

Use the tally chart to answer the questions.

BIRD	TALLY	TOTAL
COCKS	卌 卌 II	12
PARAKEET	卌 卌	10
BATS	IIII	4
OWLS	II	2
PEACOCK		0



Data to be represented in these graphs is taken from tables or tally charts. Children will need demonstration of the usage of tally charts and data tables as a means of recording information. As a second step, they need to be shown the transference of this data from the tally chart or the data table to the graph. Also at this stage they number the vertical line (Y- axis). Special attention needs to be given to the placement of 0 and 1. They need to understand the need for placing 0 at the base line or the lower end of the first square and 1 at the upper end of the first square.

ACTIVITY 11

DATA ON WEIGHT	
NAME	WEIGHT(KILOGRAMS)
1. SAMRUDDHI	26
2. SHREYA	24
3. AMUDHA	27
4. ANIKET	30
5.	
6.	
7.	

WEIGHT CHART		
KILOGRAMS	TALLY	TOTAL
20 - 22	IIII	4
22 - 24	卌 I	6
24 - 26		
26 - 28		
28 - 30		
30 - 32		
32 - 34		

Children at this level are aware of concepts like 'height' and 'weight'. These can be measured in the class. They also like to measure their capabilities? – number of times they can bounce a ball, time taken to run 100 metres, etc. Plenty of activities can be conducted to generate data. Children can be actively involved in measuring and recording data. Tally charts can be designed by them with the help of the teacher. Such data carry a personal touch and interest them greatly.

ACTIVITY 12

TIME SPENT

NAME:

ACTIVITY	FROM	HOURS	MINS
SLEEP	-TO-	—	—
EAT	-TO-	—	—
PLAY	-TO-	—	—
WATCH TV	-TO-	—	—
READ	-TO-	—	—

At this stage children learn to read time in hours and minutes. They can make a study of how they spend their time on some major activities in the day.

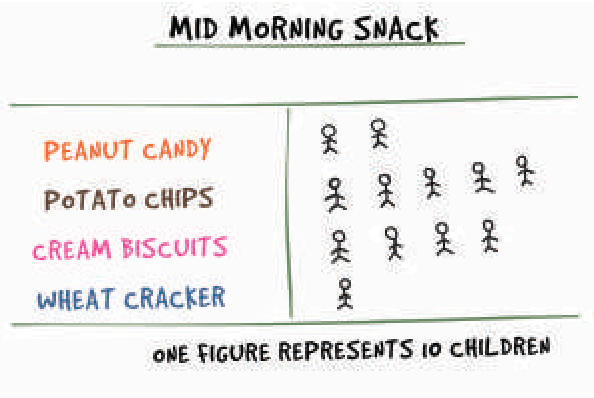
How long do I sleep? Eat? Play? Watch TV? Read books with my parents' help?

There should be a discussion on such questions before plunging into the task of collecting data. What do I need to draw this graph? What type of graph should I make (bar graph or pictograph)? After making their individual time graphs, children can sit together in pairs and compare their graphs.

Raise questions: Are there differences in the number of hours you sleep? Do some of you feel sleepy during class? Are you sleeping for less time than the others? Discuss the number of hours of sleep children need. Discuss the time at which they should go to bed. Discuss the need for having a reading time towards the end of the day. This can be followed up by creating a time planner for a Sunday.

ACTIVITY 13

The teacher can provide students with a graph which increases awareness of healthy foods and junk foods. Initially, questions can be raised about the information that can be obtained from the graph. Subsequently, the discussion can be about foods eaten at snack time in the school. What do you have snack time? Sandwiches, chaklis, biscuits, chips, milk, cold drinks, fruits, etc. Which of the items that we eat are healthy? If the class were to make a graph of the items that are consumed at snack time, how would the graph look? Will the healthy items be more or less than the junk food?

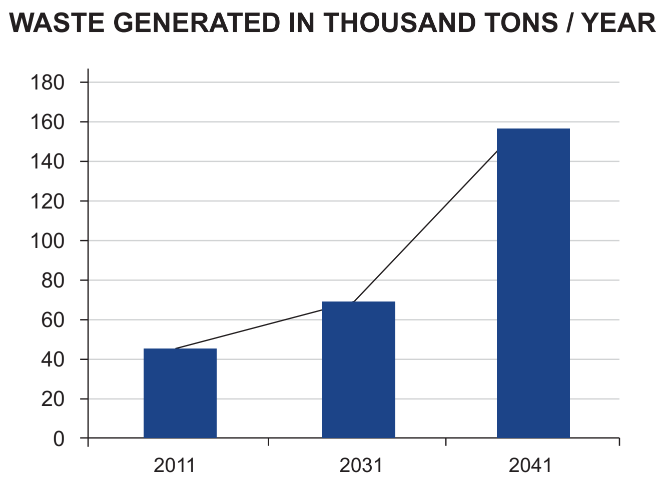


Children will now be able to read and create graphs where one unit stands for many objects. Discuss the usage of one unit square representing 2, 5 or 10. How should the unit squares be shaded if we have to represent numbers like 3 or 5? Would the graph be easily readable if we do this?

If each unit represents 5, is there a way of showing 6, 7, 8 or 9? Can we devise a way of showing such numbers on such a graph? (Children may suggest dividing the unit square into 5 equal parts and shading the required number.) Does the graph look easily readable now?

Children can be taken to the primary section of the library. They can categorise books as picture books, fairy tales, animal stories, comics, etc. They can count and record the total numbers in each category. Now pose the question: How do we show this information in a graph? Will it be possible to show numbers like 20 and 25 on this square paper? Slowly lead them to the idea of one unit representing 5 or 10 units. They can now draw a graph to represent books in the library.

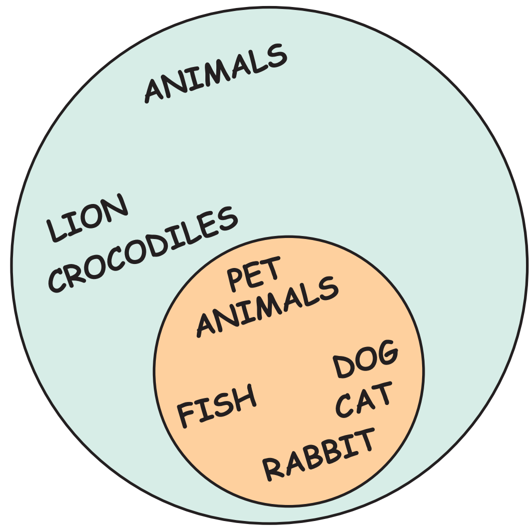
ACTIVITY 13A



The teacher can expose children to simple bar graphs or tables that are found in newspapers or magazines. Children can share their observations and discuss the data presented in these graphs.

At this point, help the children to evaluate the use of graphs, their readability, etc.
Ask the questions: Which graph is easy to read? Which graph is easy to count?

ACTIVITY 14: Venn diagrams with subsets



Children can use Venn diagrams which depict relationships between sets that contain other sets. Let them come up with more such examples, where one set is contained in another. They can be asked to show multiples of 2 and 4 in such a form.

Questions for discussion: What is the relationship between these two sets? What smaller set does the larger set contain? What else does the larger set contain which is not in the smaller set?

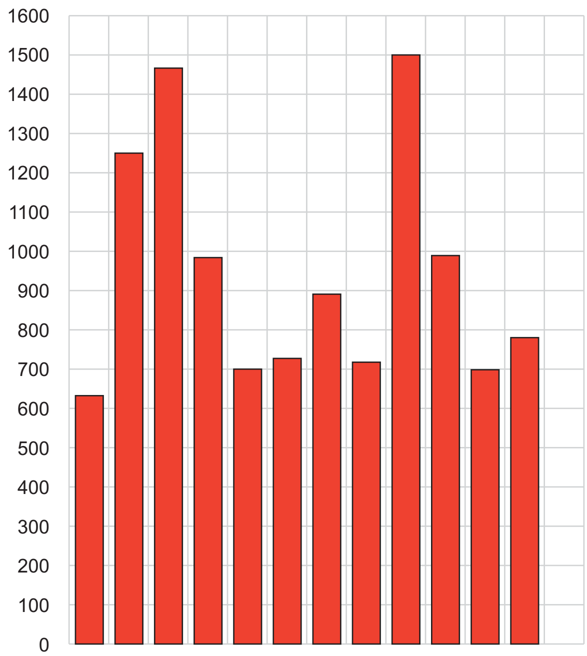
Expectations: Children can design a survey with the support of the teacher. They can collect data, process it, display it as a bar graph and interpret it.

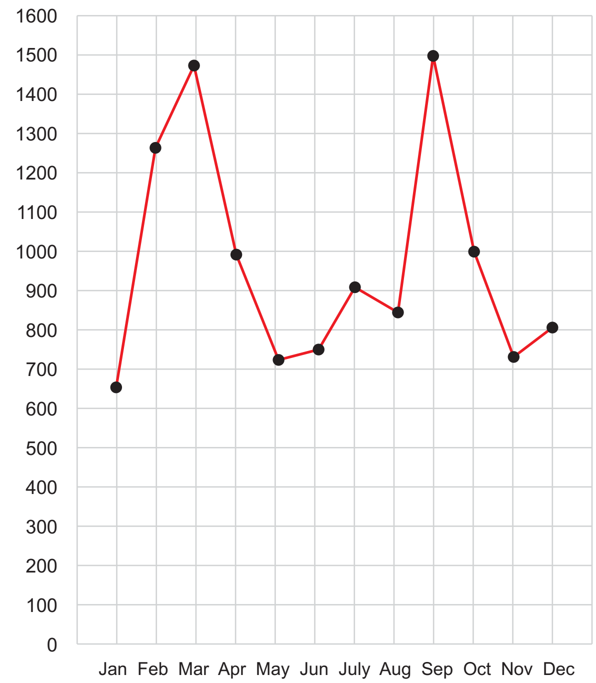
They can create bar graphs on plain paper using an L-shaped tool or a set square, and make the necessary markings to create a uniform scale. They can represent large data. They can draw a line graph for non-discrete data. They can create a simple survey which involves ratings. They can create and read Venn diagrams with three sets.

It is good to integrate graphs with different subject areas at this level. EVS which includes sciences and social studies and other topics in mathematics can become the source of the data.

ACTIVITY 15

ELECTRICITY CONSUMPTION (KWH) 2015





Large data: The teacher can provide students with data tables containing large numbers. It could be the size of the classes in the school. Data can be restricted to numbers up to 10,000.

Discuss the challenge of presenting this data in the form of the graph. Slowly lead them to the usage of scale and the selection of a suitable scale.

Children can now be exposed to the construction of line graphs which lend themselves well to certain kinds of non-discrete data.

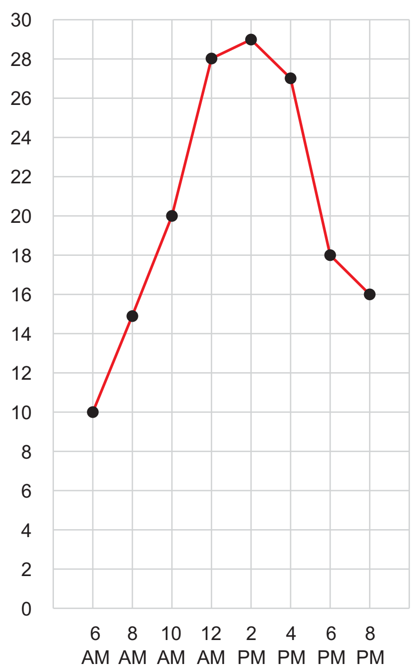
ACTIVITY 16

TEMPERATURE								
Time	6 AM	8 AM	10 AM	12 PM	2 PM	4 PM	6 PM	8 PM
Temperature (degrees)	10°C	15°C	20°C	28°C	29°C	27°C	18°C	16°C

Comparing different representations: Make 4 groups in the class. Give all the groups the same structured data; e.g., temperatures at different times of the day. Tell them to discuss and come up with a way of representing that data. Different groups may use different scales for the vertical axis as well as different forms (tally charts, bar graph, line graph).

ACTIVITY 16A

TEMPERATURE ON 12.6.2016



Raise questions about the information which requires a deeper study of the graph:

- What is the change in temperature from ____________ AM to ____________ PM?
- Was the temperature between 2 PM and 4 PM rising or falling or did it stay the same?
- During which interval was the change in temperature the largest?
- During which interval was the change in temperature the least? Why?
- How high can the temperature get? In which season does that happen?
- How low does the temperature get?
- What do you think will happen on a rainy day?

Raise questions on the organisational aspects:

- How do we know that these diagrams represent the same data?
- Which is a better way of representing the data?
- How is the graph different from a tally chart?
- How is the bar graph like a pictograph? How are they different?

Trees: As part of science children learn about different trees. They can measure the girth of different trees in their surroundings. They can make a graph based on their girth: trees with girth between 50 and 75 cm, trees with girth between 75 and 100 cm, etc. The teacher must observe how far up the tree trunk the children measure the girth. Should it always be in the same place on each tree? (Typically girth is measured about one metre up.)

Graphs can also be made on the type of leaves a tree has: simple or compound leaves, leaves with smooth edges or jagged edges, etc.

ACTIVITY 17

TELEVISION SURVEY

How do you rate the ___________ ?

RATING SCALE

5 EXCELLENT   4 VERY GOOD

3 GOOD   2 FAIR

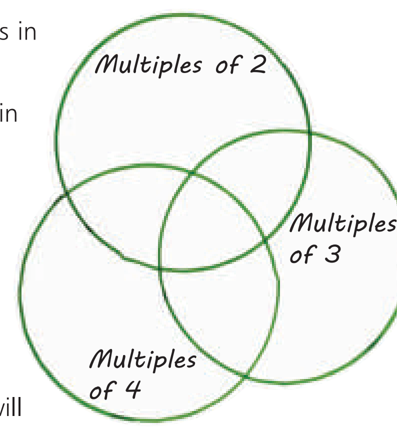
1 POOR

NO	PERSON	RATING

Ratings in surveys: Many surveys use rating on a scale from 1 to 5. This is a way of evaluating the success or failure of a product or idea. Explain to the children how each number on the rating scale stands for a different response (poor, fair, good, very good, excellent).

Home work: Family survey – help children to create a questionnaire to find information from their family members. 'How do they rate the quality of a particular show (say a music channel or a film channel) on television?' The rating scale can be prepared ranging from 1 to 5.

ACTIVITY 18: Venn diagrams with three intersecting sets



Let children write multiples of 2, 3 and 4 (up to 36) in the labelled circles in appropriate places.

What numbers are part of all the three circles? Why are there no numbers in some parts? They should be able to justify their answer.

In a similar way, they can write factors of 36, 48 and 64.

Some interesting themes for graphs:

- Monitor the growth of a suitable plant (example: sunflower) on a graph for a month.

  What is the height at the end of the first week? Did the plant double its height at the end of the second week? What do you think will happen in the third week?

- Litter found in the school: This is a real problem everywhere in India. The teacher can go with the children for a litter picking session along a path. Record the type of litter (toffee wrappers, biscuit wrappers, chips packet wrappers, juice cartons) found and the amount found in a tally chart. Litter can also be analysed as bio-degradable and non-bio-degradable.

- Sports day data: Method of scoring and ways of scoring data, timings of events

- Put up some graphs from magazines and raise questions.

  How will you describe this graph? Is it at the same level every day? Is it increasing? Is it decreasing? Is it going up and down?

- Graph detectives: Draw a graph with numbers on the Y-axis but without any labels on the X-axis.

  What could this graph be about?

  What numbers are seen? What is the maximum number? What is the minimum number? How are they changing?

  Could this graph be about _______? or _______?

Padmapriya Shirali

Padmapriya Shirali is part of the Community Math Centre based in Sahyadri School (Pune) and Rishi Valley (AP), where she has worked since 1983, teaching a variety of subjects – mathematics, computer applications, geography, economics, environmental studies and Telugu. For the past few years she has been involved in teacher outreach work. At present she is working with the SCERT (AP) on curricular reform and primary level math textbooks. In the 1990s, she worked closely with the late Shri P K Srinivasan, famed mathematics educator from Chennai. She was part of the team that created the multigrade elementary learning programme of the Rishi Valley Rural Centre, known as 'School in a Box'. Padmapriya may be contacted at padmapriya.shirali@gmail.com