Theorem concerning a RIGHT TRIANGLE

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The following elegant geometric result concerning a triangle is based on a problem that appeared in the Regional Mathematics Olympiad (RMO) of 2016.

Let ABC be a scalene triangle, and let D be the midpoint of BC. Draw median AD. Through D draw a line perpendicular to AD and let it meet the extended sides AB, AC at points K, L, respectively. Then points B, C, K, L lie on a circle if and only if angle BAC is a right angle. (See Figure 1.)

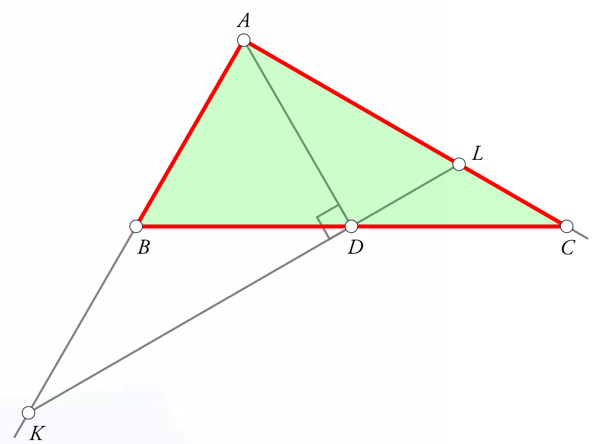

Figure 1

The implication in one direction is easy (if the triangle is right-angled, then the four points are concyclic); but the reverse implication seems more challenging. We shall give a geometric solution for the forward implication, followed by an algebraic solution in which both the implications are established at the same time.

Geometric proof that the four points are concyclic. We are given the fact that $\measuredangle BAC$ is a right angle, and we must prove that points B, K, L, C are concyclic.

Keywords: Cosine rule, intersecting chords theorem, crossed chords theorem, Apollonius, power of a point

The most obvious approach towards proving that four given points are concyclic is the angle-chasing route: prove that some two angles are equal. In the present instance, it suffices to prove that $\measuredangle AKD = \measuredangle ACD$, or that $\measuredangle ALD = \measuredangle ABD$. (These two statements are clearly equivalent to each other.) See Figure 2; we must prove that $x = y$.

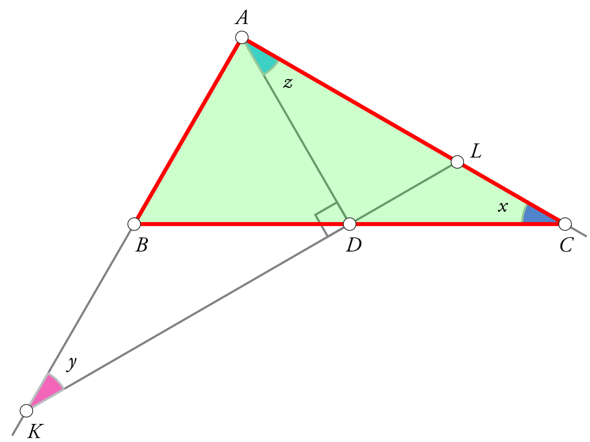

Figure 2

The desired equality follows when we notice that $x = z$ and $y = z$. To see why $y = z$, observe that both y and z are complementary to $\measuredangle ALK$ (and this follows because $\measuredangle LAK$ and $\measuredangle ADL$ are right angles). To see why $x = z$, note that since $\triangle ABC$ is right-angled at A, its circumcentre lies at the midpoint of the hypotenuse BC. This means that D is equidistant from vertices A, B, C. Hence $DA = DC$, which implies that $x = z$.

Thus the forward implication has been proved: if $\measuredangle BAC$ is a right angle, then points B, K, L, C are concyclic.

For the reverse implication, a geometric approach seems rather elusive; we opt for an algebraic approach. We shall need the following results:

(i) the cosine rule: in $\triangle ABC$, we have: $c^2 = a^2 + b^2 - 2ab \cos C$, etc;

(ii) the Intersecting Chords theorem, also called the Crossed Chords theorem, and the related notion of 'power of a point': given a circle with centre O and radius r, if two of its chords EF and GH intersect at a point P (which may lie inside or outside the circle), then we have the equality $PE \cdot PF = PO^2 - r^2 = PG \cdot PH$. (Note that the distances here are *signed*; so if PE and PF point in opposite directions, then $PE \cdot PF \leq 0$.) We need the **converse** of this theorem (which is also true): *if coplanar points E, F, G, H are placed such that the equality $PE \cdot PF = PG \cdot PH$ is true, where P is the point of intersection of lines EF and GH, then the points E, F, G, H are concyclic.*

(iii) the theorem of Apollonius which tells us that $AB^2 + AC^2 = 2AD^2 + 2BD^2$.

We reason as follows:

$$\begin{aligned} \text{Points } B, C, K, L \text{ concylic} &\iff AB \cdot AK = AC \cdot AL \\ &\iff c \cdot \frac{AD}{\cos \measuredangle KAD} = b \cdot \frac{AD}{\cos \measuredangle LAD} \\ &\iff \frac{c}{b} = \frac{\cos \measuredangle BAD}{\cos \measuredangle CAD}. \end{aligned}$$

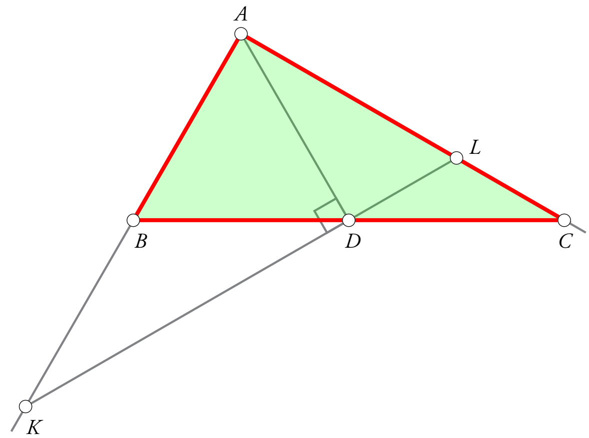

Figure 3

Next we have, by the cosine rule:

$$\cos\measuredangle BAD = \frac{AB^2 + AD^2 - BD^2}{2\,AB\cdot AD}, \qquad \cos\measuredangle CAD = \frac{AC^2 + AD^2 - CD^2}{2\,AC\cdot AD};$$

hence:

$$\frac{\cos\measuredangle BAD}{\cos\measuredangle CAD} = \frac{AB^2 + AD^2 - BD^2}{AC^2 + AD^2 - CD^2}\cdot\frac{AC}{AB} = \frac{c^2 + AD^2 - a^2/4}{b^2 + AD^2 - a^2/4}\cdot\frac{b}{c}.$$

The theorem of Apollonius implies that:

$$AD^2 = \frac{b^2}{2} + \frac{c^2}{2} - \frac{a^2}{4}.$$

Substituting this into the previous expression we get:

$$\frac{\cos\measuredangle BAD}{\cos\measuredangle CAD} = \frac{3c^2 + b^2 - a^2}{3b^2 + c^2 - a^2}\cdot\frac{b}{c}.$$

Hence we have:

$$\text{Points } B, C, K, L \text{ concylic} \iff \frac{c}{b} = \frac{3c^2 + b^2 - a^2}{3b^2 + c^2 - a^2}\cdot\frac{b}{c}.$$

That is:

$$\text{Points } B, C, K, L \text{ concylic} \iff c^2\left(3b^2 + c^2 - a^2\right) = b^2\left(3c^2 + b^2 - a^2\right).$$

Next we have:

$$\begin{aligned} c^2\left(3b^2 + c^2 - a^2\right) - b^2\left(3c^2 + b^2 - a^2\right) &= a^2\left(b^2 - c^2\right) - \left(b^4 - c^4\right) \\ &= \left(a^2 - b^2 - c^2\right)\left(b^2 - c^2\right). \end{aligned}$$

Hence:

$$\text{Points } B, C, K, L \text{ concylic} \iff \left(a^2 - b^2 - c^2\right)\left(b^2 - c^2\right) = 0.$$

Since $b^2 - c^2 \neq 0$ (we have specifically been told that the triangle is scalene), we deduce finally that:

$$\text{Points } B, C, K, L \text{ concylic} \iff a^2 - b^2 - c^2 = 0 \iff \measuredangle BAC = 90^\circ. \qquad \square$$

Some of you may like to take up the challenge of finding a purely geometric proof for the reverse implication.

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