ClassRoom

An Angle-in-a-Quadrilateral Problem

$\mathscr{C}\otimes\mathscr{M}\alpha\mathscr{C}$

Shown in Figure 1 is quadrilateral $ABCD$ with $DA = AB = BC$ and $\measuredangle DAB = 74°$, $\measuredangle ABC = 166°$. The problem is to find the measure of $\measuredangle BCD$.

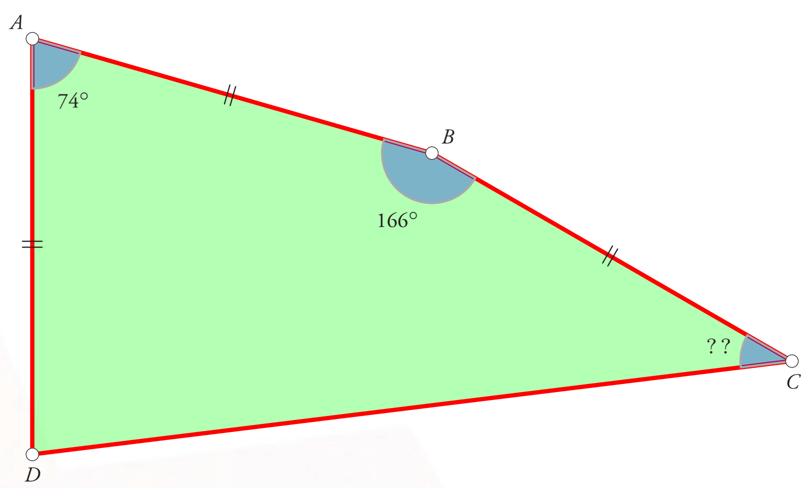

Figure 1.

The problem (taken from [1]) is challenging but admits many different solutions, some of which are very elegant (which is what makes this problem so interesting). We present some of these solutions here.

A solution using trigonometry.
Since $AD = AB$, $\measuredangle ABD = 53°$. Let $AB = a$. From the isosceles $\triangle ABD$ (Figure 2) we get $DB = 2a \sin 37°$. (To see why, imagine dropping a perpendicular from A to base BD.)

Keywords: Quadrilateral, angle

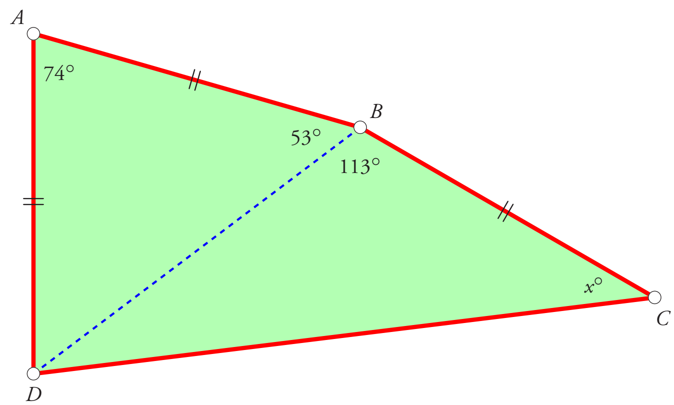

Figure 2.

Next, invoking the sine rule in $\triangle BCD$ and using the fact that supplementary angles have equal sines, we get:

$$\frac{DB}{\sin x^\circ} = \frac{BC}{\sin(x^\circ + 113^\circ)},$$

$$\therefore \quad \frac{2a \sin 37^\circ}{\sin x^\circ} = \frac{a}{\sin(x^\circ + 113^\circ)}.$$

This yields: $2 \sin 37^\circ \cdot \sin(x^\circ + 113^\circ) = \sin x^\circ$, and so:

$$2 \sin(67^\circ - x^\circ) = \frac{\sin x^\circ}{\sin 37^\circ}.$$

Now we resort to a clever argument based on the monotonic nature of the sine function over the interval from 0° to 90°. To start with, note that we obviously have $0 < x < 67$. (Else the quantities on the two sides of the above supposed equality have opposite signs.)

Now suppose that $x < 37$. Then the quantity on the right side is less than 1. On the other hand, the supposition that $x < 37$ leads to the following: $67 - x > 30$, hence

$$\sin(67^\circ - x^\circ) > \sin 30^\circ,$$

$$\therefore \quad 2 \sin(67^\circ - x^\circ) > 2 \sin 30^\circ,$$

$$\therefore \quad 2 \sin(67^\circ - x^\circ) > 1.$$

So if $x < 37$, the quantity on the left side is greater than 1, while the quantity on the right side is less than 1. We have arrived at a contradiction. Hence it cannot be that $x < 37$. The same reasoning works if we assume that $x > 37$; now we find that the quantity on the left side is less than 1, while the quantity on the right side is greater than 1. So this possibility does not work out either. Since x can neither be less than 37 nor greater than 37, it follows that $x = 37$. Hence $\measuredangle BCD = 37^\circ$.

A pretty solution combining trigonometry and geometry.

Here is an elegant and pleasing solution that combines geometry and trigonometry and makes effective use of the identity $\sin\theta = \sin(180^\circ - \theta)$. Let the diagonals AC, BD of the quadrilateral intersect at E (Figure 3). An easy angle computation shows that $\measuredangle DEC = 120^\circ$.

We have now, applying the sine rule to $\triangle ADC$ and $\triangle BDC$ respectively:

$$\frac{AD}{\sin \measuredangle ACD} = \frac{CD}{\sin 67^\circ},$$

$$\frac{BC}{\sin \measuredangle BDC} = \frac{CD}{\sin 113^\circ}.$$

Since $\sin 67^\circ = \sin 113^\circ$, the quantities on the right-hand sides of the two equalities are equal. We also have $AD = BC$. It follows that $\sin \measuredangle ACD = \sin \measuredangle BDC$, and therefore that $\measuredangle ACD = \measuredangle BDC$, as both angles are acute (indeed, $\measuredangle BDC + \measuredangle ACD = 60^\circ$).

Hence $\measuredangle ECD = 30^\circ$ and, therefore, $\measuredangle BCD = 37^\circ$.

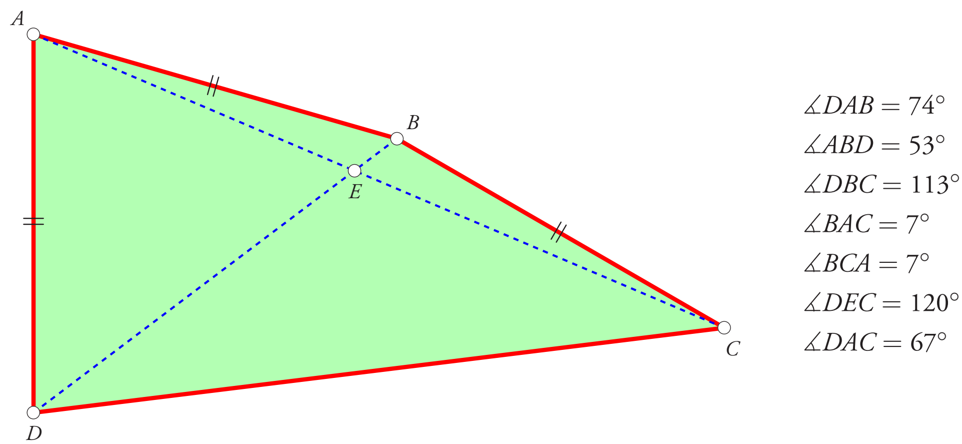

Figure 3.

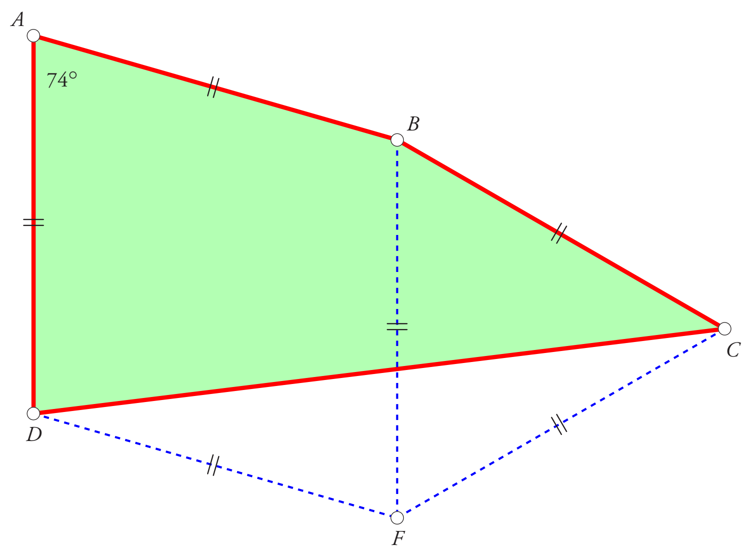

Figure 4.

An elegant pure geometry solution.
Next, we present an extremely elegant solution that draws on basic geometrical ideas about parallelograms (Figure 4). Draw $\overrightarrow{BF} = \overrightarrow{AD}$; then $ABFD$ is a parallelogram, and since $AB = AD$, it is a rhombus (as shown).

From this we deduce that $\measuredangle ABF = 106^\circ$, and therefore that $\measuredangle FBC = 60^\circ$. Since $BF = BC$, this makes BFC an equilateral triangle, so $\measuredangle BCF = 60^\circ$. Again, in the isosceles $\triangle FCD$, $\measuredangle CFD = 74^\circ + 60^\circ = 134^\circ$, hence $\measuredangle FCD = 23^\circ$. It follows that $\measuredangle BCD = 60^\circ - 23^\circ = 37^\circ$.

Another elegant pure geometry solution.
We conclude by presenting yet one more extremely elegant pure geometry solution. Locate point K such that $\triangle AKB$ is equilateral (Figure 5). Then we also have $AK = AD$ and $BK = BC$. And since $\measuredangle KAB = \measuredangle KBA = 60^\circ$, we have $\measuredangle KAD = 14^\circ$ and $\measuredangle KBC = 106^\circ$. These in turn imply that $\measuredangle AKD = 83^\circ$ and $\measuredangle BKC = 37^\circ$. Also, obviously, $\measuredangle AKB = 60^\circ$.

But now note that $83^\circ + 60^\circ + 37^\circ = 180^\circ$. This means that points D, K, C lie in a straight line! So the picture shown is not accurate (it was deliberately shown that way; note that we chose to

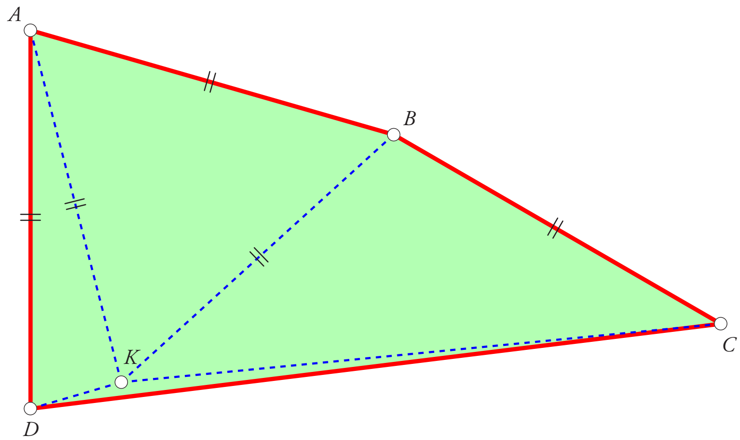

Figure 5.

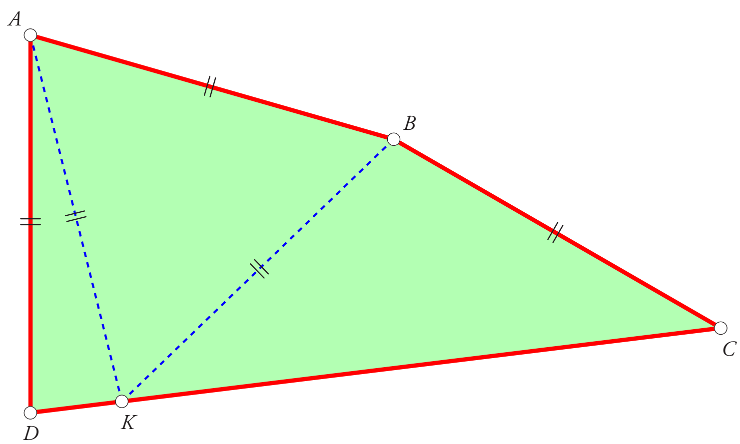

Figure 6.

locate K inside the quadrilateral, but we could as well have shown K outside the quadrilateral); the actual picture is as shown in Figure 6.

It follows immediately from the above that $\measuredangle BCK = 37^\circ$, i.e., $\measuredangle BCD = 37^\circ$.

Remark. It is noteworthy that the pure geometry solutions (the last two solutions presented above) featured the use of an equilateral triangle. This is a theme which occurs very often in the solutions of such problems.

References

1. Mathematics Stack Exchange, "Is there a way to solve for the missing angle?" https://math.stackexchange.com/questions/2564493/is-there-a-way-to-solve-for-the-missing-angle?newsletter=1&nlcode=838029%7c198e

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