Problem Corner

Solution to 'A Circular Challenge'

C⊗MαC

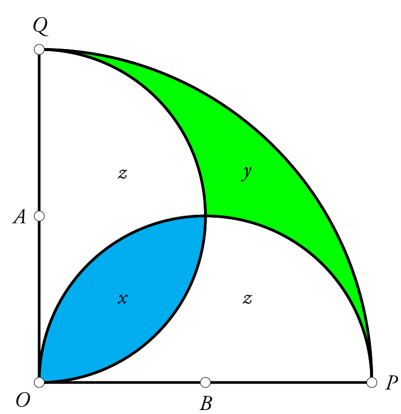

Figure 1

Shown in Figure 1 is a portion of Figure 1 in the original article (July 2016 issue): two semi-circles centred at A and B, with radii 1 unit each, and a quarter-circle OPQ, with radius 2 units. The portions of the semi-circles minus the blue region (with area x) have area z each. (They obviously have equal area.) Clearly:

$$x+z=\frac{1}{2}\,\pi\times 1^2=\frac{\pi}{2},\qquad x+2z+y=\frac{1}{4}\,\pi\times 2^2=\pi,$$

hence $x+2z+y=2(x+z)$, therefore $x+y=2x$, therefore $x=y$. Hence $x:y=1:1$. $\square$

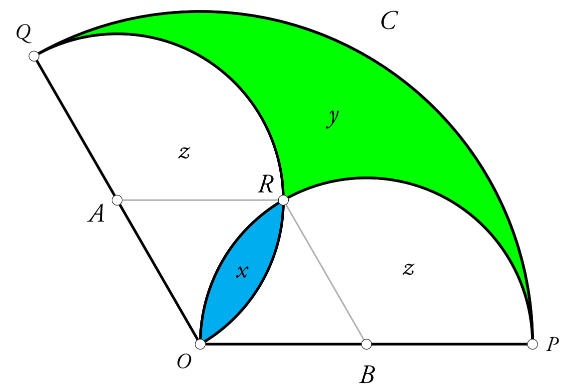

Figure 2

Remark. We did not bother to find x and y individually, as we had only been asked to find $x : y$. But if we need these as well, then:

$$x = 2\left(\frac{1}{2}\,1^2 \times \frac{\pi}{2} - \frac{1}{2}\,1^2 \times \sin\frac{\pi}{2}\right) = \frac{\pi}{2} - 1,$$

and of course, y has the same value. □

Generalization

Now suppose that $\measuredangle POQ = t$, where $0 \le t \le \pi$; see Figure 2. As earlier, let z denote the areas of the semi-circles minus the blue region. Let A and B denote the centres of the two circles, and R the point of intersection of the two small circles other than O (the center of the large circle). Since $OARB$ is a rhombus, $BR \parallel OA$, hence $\measuredangle PBR = t$, hence $\measuredangle OBR = \pi - t$. Hence:

$$\begin{aligned} x &= 2\,(\text{Area of sector } OBR - \text{Area of } \triangle OBR) \\ &= 2\left(\frac{1^2 \cdot (\pi - t)}{2} - \frac{1^2 \cdot \sin(\pi - t)}{2}\right) = \pi - t - \sin t. \end{aligned}$$

Again, we have:

$$x + z = \frac{1}{2}\,1^2 \times \pi = \frac{\pi}{2}, \qquad x + 2z + y = \frac{1}{2}\,2^2 \times t = 2t.$$

Hence by subtraction $y - x = 2t - \pi$, and therefore $y = x + 2t - \pi$. Hence the desired ratio $x : y$ is:

$$\frac{x}{y} = \frac{\pi - t - \sin t}{t - \sin t}.$$

For the particular value $t = \pi/2$ we have $x : y = (\pi/2 - 1) : (\pi/2 - 1) = 1 : 1$, as earlier. □

Note. Reader **Tejash Patel** of Gujarat sent in correct solutions to both parts of the problem.

The COMMUNITY MATHEMATICS CENTRE (CoMaC) is an outreach arm of Rishi Valley Education Centre (AP) and Sahyadri School (KFI). It holds workshops in the teaching of mathematics and undertakes preparation of teaching materials for State Governments and NGOs. CoMaC may be contacted at shailesh.shirali@gmail.com.