Octagon in a Square: Another Solution

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Problem Corner

In the November 2015 issue of AtRiA, the following geometrical puzzle had been posed. An octagon is constructed within a square by joining each vertex of the square to the midpoints of the two sides remote from that vertex. Eight line segments are thus drawn within the square, creating an octagon (shown shaded). The following two questions had been posed: (i) Is the octagon regular? (ii) What is the ratio of the area

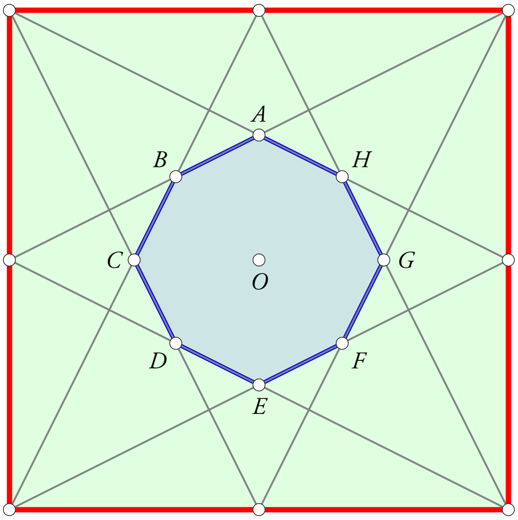

Figure 1

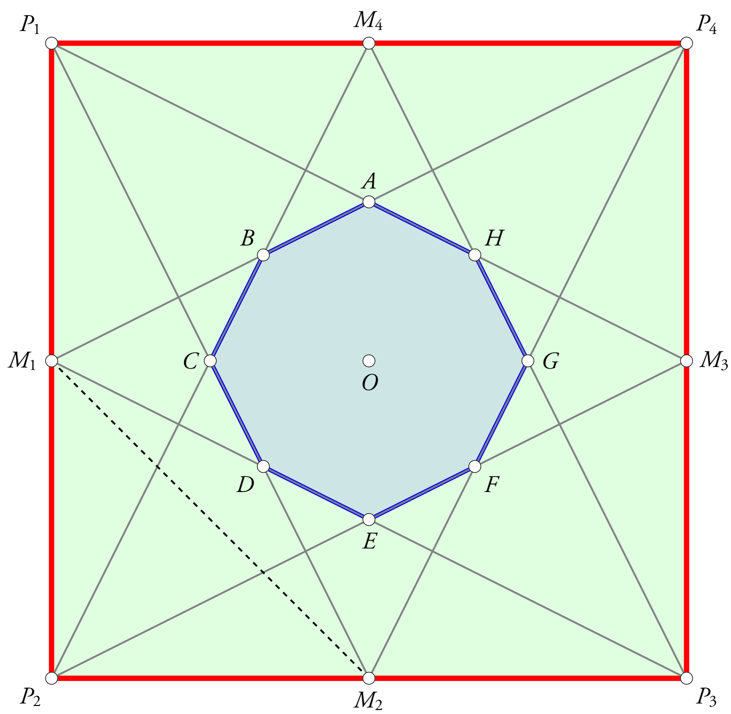

Figure 2

of the octagon to that of the square? We had given a solution in the March 2016 issue. Now we feature another solution to this problem, sent in by a reader: **R Desai**, of Gujarat.

Label the vertices of the octagon A, B, C, D, E, F, G, H as shown. Let O be the centre of the square. We shall show that the octagon is not regular by showing that $\measuredangle BAH < \measuredangle AHG$. (If the octagon were regular, these two angles would have had the same measure.)

Label the vertices of the square and the midpoints of the sides as shown in Figure 2. Join M_1M_2.

Observe that triangles $M_4P_2P_3$ and $P_4M_1M_2$ are both isosceles, and further have equal length for the congruent pairs of sides ($M_4P_2 = M_4P_3 = P_4M_1 = P_4M_2$). However, their bases are unequal; indeed, $P_2P_3 > M_1M_2$. It follows from this that their apex angles are unequal; indeed that $\measuredangle P_2M_4P_3 > \measuredangle M_1P_4M_2$.

Since $M_4P_2 \perp P_1M_3$ and $P_4M_2 \perp P_1M_3$, it follows that $\measuredangle AHG = 90^\circ + \measuredangle P_2M_4P_3$ and $\measuredangle BAH = 90^\circ + \measuredangle M_1P_4M_2$. Hence $\measuredangle AHG > \measuredangle BAH$. Thus the octagon is not regular, as claimed. $\square$

Computation of area. It remains to compute the area of the octagon relative to the area of the square. There are many ways of obtaining the result. Desai's solution avoids the use of trigonometry; it uses only pure geometry.

If all the vertices of the octagon are joined to the centre O, the octagon is divided into 8 congruent triangles. One of these triangles is $\triangle ODE$. Extend OD to P_2 and OE to M_2 as shown. Observe that $OE = OM_2/2$. Also, $\triangle ODE \sim \triangle P_2DM_1$, the ratio of similarity being $1/2$ since $OE = OM_2/2$. Hence $OD = OP_2/3$. It follows that

$$\frac{\text{Area of } \triangle ODE}{\text{Area of } \triangle OP_2M_2} = \frac{OD}{OP_2} \cdot \frac{OE}{OM_2} = \frac{1}{3} \cdot \frac{1}{2} = \frac{1}{6}.$$

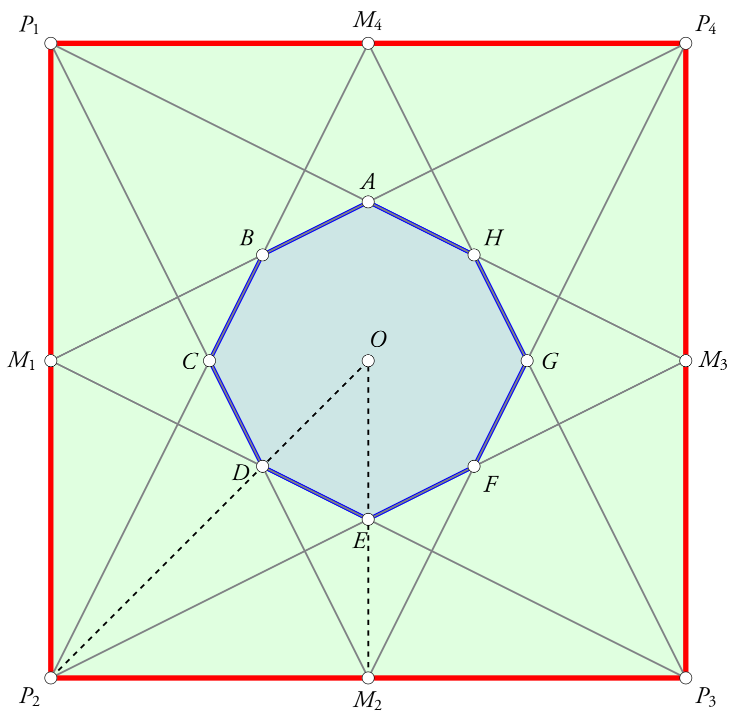

Figure 3

Since the area of $\triangle ODE$ is $1/8$ of the area of the octagon, and the area of $\triangle OP_2M_2$ is $1/8$ of the area of the square, it follows that the area of the octagon is $1/6$ of the area of the square. $\square$

The COMMUNITY MATHEMATICS CENTRE (CoMaC) is an outreach arm of Rishi Valley Education Centre (AP) and Sahyadri School (KFI). It holds workshops in the teaching of mathematics and undertakes preparation of teaching materials for State Governments and NGOs. CoMaC may be contacted at shailesh.shirali@gmail.com.