How to discover the EXPONENTIAL FUNCTION e^x

GAURAV BHATNAGAR

If a function is such that its derivative is the function itself, then what would it be? Some interesting mathematical objects appear while trying to answer this question, including a power series, the irrational number e and the exponential function e^x. The article ends with a beautiful formula that connects e, π, the complex number $i = \sqrt{-1}$, 1 and 0.

A function whose derivative is the function itself

If a function is such that its derivative is the function itself, then what would it be? If it is anything like a polynomial, it will be of the form

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots.$$

Assume further that $f(0) = a_0 = 1$, and try to find a_1, a_2, a_3, ... Now

$$f(x) = 1 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots.$$

Its derivative is of the form

$$f'(x) = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + 5a_5 x^4 + \cdots.$$

Since the derivative is the same as the function itself, compare the two to obtain:

$$\begin{aligned}
 & & a_1 &= 1, \\
2a_2 = a_1 &\implies & a_2 &= \frac{1}{2}, \\
3a_3 = a_2 &\implies & a_3 &= \frac{1}{3 \times 2}, \\
4a_4 = a_3 &\implies & a_4 &= \frac{1}{4 \times 3 \times 2}, \\
5a_5 = a_4 &\implies & a_5 &= \frac{1}{5 \times 4 \times 3 \times 2}.
\end{aligned}$$

Keywords: Power series, derivative, polynomial, irrational number

Evidently,

$$a_n = \frac{1}{n \times (n-1) \times (n-2) \times \cdots \times 2} = \frac{1}{n!},$$

where we use the factorial notation

$$n! = n \times (n-1) \times \cdots \times 2 \times 1.$$

Note the extra factor of 1 to write the denominator as $n!$. Further, make a_1 fit the pattern by writing $a_1 = 1 = 1/1!$.

Thus

$$f(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots.$$

The function $f(x)$ appears to be a polynomial of 'infinite' degree. This is because the derivative of a polynomial of degree n has degree $n-1$. If the function and its derivative have to be equal, we have to make sure there is no 'largest' degree term. In other words, the polynomial never ends.

This kind of 'infinite polynomial' is called a **power series**.

The number *e*

For a given value of x, this power series contains an infinite number of additions. As you know, we can only perform a finite number of additions in this lifetime. So the idea is to sum only the first few terms, and use that as a guide to the real thing.

Try $f(1)$. Use a calculator (or a spreadsheet like MS-Excel) to find the following approximations to $f(1)$.

$$\begin{aligned}
1 &= 1\\
1 + \frac{1}{1} &= 2\\
1 + \frac{1}{1} + \frac{1}{2} &= 2.5\\
1 + \frac{1}{1} + \frac{1}{2} + \frac{1}{6} &= 2.67\\
1 + \frac{1}{1} + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} &= 2.71\\
&\vdots\\
1 + \frac{1}{1} + \frac{1}{2} + \frac{1}{6} + \cdots + \frac{1}{10!} &= 2.71828\ldots
\end{aligned}$$

The approximations seem to settle down to a number whose first few digits are $2.71828\ldots$. This number is called e.

Henceforth, we take $f(1) = e$.

The connection with exponents

Given that $f(x)$ is like a polynomial, you can expect many rules of the algebra of polynomials to apply. What happens, for instance, when you multiply $f(x)$ with $f(y)$? Here goes:

$$\begin{aligned} f(x)f(y) &= \left(1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots\right)\left(1+\frac{y}{1!}+\frac{y^2}{2!}+\frac{y^3}{3!}+\cdots\right) \\ &= 1+\left(\frac{x}{1!}+\frac{y}{1!}\right)+\left(\frac{x^2}{2!}+\frac{xy}{1!\,1!}+\frac{y^2}{2!}\right)+\left(\frac{x^3}{3!}+\frac{x^2y}{2!\,1!}+\frac{xy^2}{1!\,2!}+\frac{y^3}{3!}\right)+\cdots \\ &= 1+\frac{1}{1!}(x+y)+\frac{1}{2!}\left(x^2+2xy+y^2\right)+\frac{1}{3!}\left(x^3+3x^2y+3xy^2+y^3\right)+\cdots \\ &= 1+\frac{1}{1!}(x+y)+\frac{1}{2!}(x+y)^2+\frac{1}{3!}(x+y)^3+\cdots. \end{aligned}$$

From these first few terms, it looks like

$$f(x)f(y) = f(x+y),$$

which should remind you of the rule for exponents:

$$a^x a^y = a^{x+y}.$$

Indeed, the following calculations suggest that f is an exponential too.

From the relation $f(x+y) = f(x)f(y)$, it follows that

$$f(2) = f(1+1) = f(1)f(1) = (f(1))^2.$$

Thus $f(2) = e^2$, since $f(1) = e$. Similarly, you can show that $f(3) = e^3$. More generally, by induction, it follows that

$$f(n) = e^n \text{ for every natural number } n.$$

It seems reasonable to guess that $f(x) = e^x$, for any real number x.

This is the famous exponential function, and we have the power series representation:

$$e^x = 1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\frac{x^5}{5!}+\cdots.$$

We derived this from the two conditions $f(x)$ satisfies: Its derivative is the function itself; and its value at 0 is 1.

Notes

- The number e is named after the famed mathematician Euler (pronounced *Oiler*). It is similar to π in many ways. Both are irrational, i.e., cannot be written in the form p/q, where p and q are integers, $q \neq 0$. Moreover, both are transcendental numbers, which means that they are not solutions of any polynomial equations with rational coefficients. (**Note.** Like π and e, the number $\sqrt{2}$ is irrational, but it **is** a solution of a polynomial equation, namely $x^2 - 2 = 0$.)
- Just like π, the number e appears in a wide variety of contexts. Here are a few examples:
  - If you put 1 rupee in a bank account which pays 100% interest, *compounded continuously*, then at the end of one year, you will have Rs. e in the account!

- Shorn of all the financial-ese, the above statement simply says that:

$$e = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n.$$

- Suppose n letters are written to n people, and are inserted randomly into n (previously addressed) envelopes. The probability that **no** letter is placed in the correct envelope approaches $1/e$, as $n \to \infty$.

- There are many mathematical rules I have broken in this article. I argued on the basis of a few examples, and jumped from the finite to the infinite case with abandon. All our computations can be justified, but they require the development of more theory, and clarifications of ideas of convergence, power series, etc.

  A lot of math is discovered this way. Mathematicians regularly guess, or conjecture, properties of objects they study on the basis of calculations (done by hand or computer). The computations serve to build their intuition which is useful when it comes to knowing which rules are possible to break or extend. Then they prove their conjectures, or perhaps, develop a new theory to allow for the extended rules.

  Progress in mathematics (and indeed in almost any field) often happens when old rules are broken and new ones are made.

- We didn't even bother explaining what it means to raise e to the power of a real number. The expression e^n is easy to understand—here e is multiplied with itself n times. So is $e^{p/q}$, where p/q is a rational number, where we take the qth root, and then the pth power. But e^x for x an irrational number? Think of taking a rational number r very 'close' to x, and taking e^r as an approximation to e^x. Indeed, one can take a sequence of rational numbers $x_1, x_2, x_3, \ldots$ that come closer and closer to x, in other words, converge to x, and define e^x as the limit of e^{x_n} as n goes to infinity.

- You can use the approach of this article to find the power series expansion of $\sin x$. Recall that the derivative of $\sin x$ is $\cos x$, and its derivative is $-\sin x$. Further, $\sin 0 = 0$ and $\cos 0 = 1$. Now assume that

  $$\sin x = a_0 + a_1 x + a_2 x^2 + \cdots,$$

  differentiate it twice and figure out a_0, a_1, a_2, and so on. In this manner show that:

  $$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots.$$

  Differentiate both sides (again, by reinterpreting the 'derivative of sum is sum of derivatives' rule to allow for the sum of an infinite number of functions!) to find the power series corresponding to $\cos x$:

  $$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots.$$

  Using these power series, prove that

  $$e^{i\theta} = \cos\theta + i\sin\theta,$$

  where $i = \sqrt{-1}$, and we assume the power series expansion holds even when x is a complex number. Finally, set $\theta = \pi$ to obtain the sublime formula

  $$e^{\pi i} + 1 = 0.$$

  This formula connects e with π, i, 1 and 0, the other fundamental numbers of math!

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NUMBER CROSSWORD

Solution on Page 36

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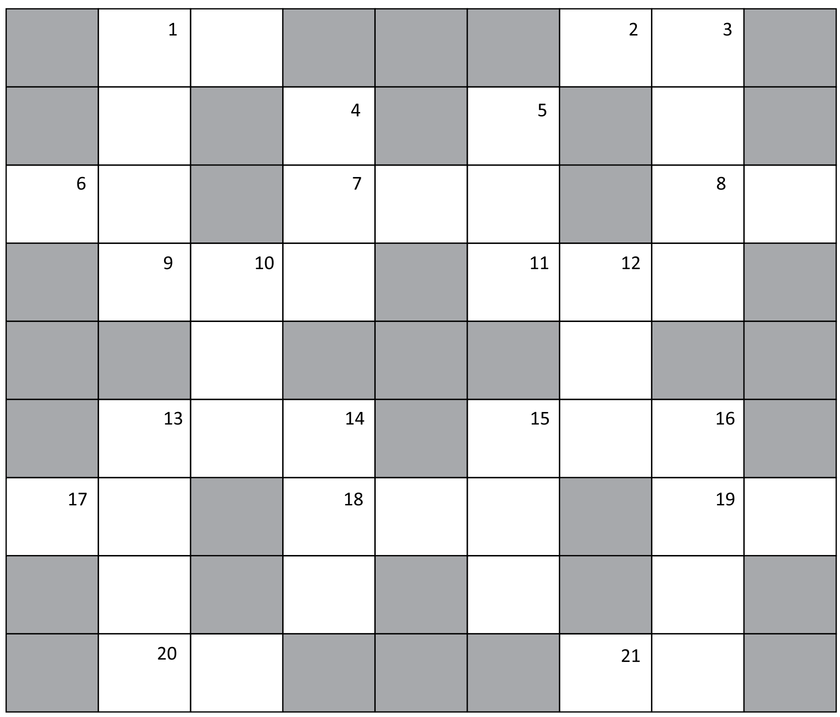

CLUES ACROSS

1 Four score and ten
2 Quarter of 7A minus 2
6 Three dozen and three
7 Number of zeroes in a googol
8 Second in a twin prime pair with its reverse also a prime
9 Multiple of 11
11 9A plus 1A
13 Digits in arithmetic progression
15 Product of 1 more than 10 and 1 less than 30
17 Fourth power of a prime number
18 Hundreds digit is the sum of the units and tens digit
19 Six times the unlucky number
20 Double of 2A
21 A number which is a square, a cube and a sixth power

CLUES DOWN

1 The highest two digit number times the first three digit prime
3 Cube of 6 times 17
4 Two raised to (1A divided by 10)
5 17A times 6 plus 2A
10 James Bond's number
12 Medical emergency call number in the US
13 Kaprekar Constant
14 13A in reverse
15 Third term of a GP in which the first number and common ratio are both the highest single digit prime
16 Anagram of year of Indian independence